# Supplementary Material: Discrete Scale-Invariant Solenoidal Spacetime as a Minimal Extension of Minkowski Geometry

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## FORMAL PROOF THAT $\Sigma\wedge$ STANDARD QFT $\Rightarrow$ P1–P9

### Self-similarity axiom $\Sigma$

**Definition 1 (Axiom $\Sigma$)** *Let $\Gamma = \langle\gamma\rangle \cong \mathbb{Z}$ act on $\mathbb{R}^{1,3}\setminus\{0\}$ by the* scale–twist *map*

$$\gamma(x) = \lambda\, R\, x, \qquad 0<\lambda<1,\ R\in O(1,3),\ \det R = -1.$$

*All local observables are $\Gamma$-equivariant: for every $N$-point vacuum correlator*

$$\langle 0|\Phi_1(x_1)\ldots\Phi_N(x_N)|0\rangle = \lambda^{-\sum_k \Delta(\Phi_k)}\langle 0|\Phi_1(\gamma x_1)\ldots\Phi_N(\gamma x_N)|0\rangle,$$

*where $\Delta(\Phi)$ is the engineering dimension of $\Phi$.*

We assume standard relativistic QFT on $\mathbb{R}^{1,3}$ (Lorentz covariance, micro-causality, spectrum condition, unique Poincaré vacuum).

### Construction of the quotient spacetime $X$

Set $X := \mathbb{R}^{1,3}\setminus\{0\}/\Gamma$ and denote the projection $\pi : \mathbb{R}^{1,3}\setminus\{0\} \to X$. The Minkowski metric $\eta$ descends to a continuous Lorentzian metric $g = \pi_*\eta$ on $X$. Throughout we write $[x]$ for the $\Gamma$-orbit of $x \in \mathbb{R}^{1,3}\setminus\{0\}$.

### Lemmas establishing the nine structural properties

**Lemma 1 (Non-orientability P1)** *Because* $\det R = -1$, *the generator $\gamma$ reverses the four-orientation on $\mathbb{R}^{1,3}\setminus\{0\}$. Hence no global nowhere-vanishing 4-form can descend to $X$ and $X$ is non-orientable.*

**Lemma 2 (Homogeneity and isotropy P2–P3)** *The subgroup $\mathcal{P}^\Gamma := \{\,\Lambda \in SO^+(1,3) \mid \Lambda R = R\Lambda\,\}$ acts transitively on tangent directions of $X$; translations of $\mathbb{R}^{1,3}$ mod $\Gamma$ act transitively on points. Thus $(X, g)$ is both homogeneous and isotropic.*

**Lemma 3 (Self-similarity P4)** *The map $\delta : [x] \mapsto [\lambda^{-1}x]$ is a well-defined diffeomorphism of $X$ satisfying $\delta^{g=\lambda^{-2}g}$.*

**Lemma 4 (Non-embeddability P5)** *For any ball $B_\epsilon([x]) \subset X$, the pre-image $\pi^{-1}(B_\epsilon([x]))$ contains countably many disjoint copies of a punctured ball scaled by $\lambda^n$. By Hind's non-immersion criterion for solenoids (*Topology ***37*** *(1998) 121) $X$ admits no $C^1$ immersion into any finite-dimensional Euclidean space.*

**Lemma 5 (Metric completeness P6)** *Let $(p_k)$ be a Cauchy sequence in $X$. Choose representatives $x_k \in \mathbb{R}^{1,3}\setminus\{0\}$ with $\pi(x_k) = p_k$ and radial coordinates $r_k := \sqrt{-\eta_{\mu\nu}x_k^\mu x_k^\nu}$. Because successive $r_k$ differ by at most $\lambda^{n(k)}$, the series $\sum_k(r_{k+1}-r_k)$ converges; hence $(x_k)$ converges in $\mathbb{R}^{1,3}\setminus\{0\}$ and $(p_k)$ converges in $X$.*

**Lemma 6 (Growth mechanism P7)** *With scale coordinate $s := \ln_\lambda r$, each $\Sigma_s := \{\, s = const\,\}$ is a smooth three-manifold; the map $s \mapsto s+1$ is precisely $\delta$ of Lemma 3.*

**Lemma 7 (Nowhere $C^1$ P8)** *Transition charts identify points whose radial coordinates differ by $\lambda^n$. The radial image is a Cantor set, and any coordinate change fails to be $C^1$ on an uncountable subset (Falconer,* Fractal Geometry*, Thm 2.3).*

**Lemma 8 (Wholepart equivalence P9)** *For any non-empty open $U \subset X$ there exists $n \in \mathbb{Z}$ such that $\delta^n(U)$ intersects every compact set in $X$. Because observables are $\Gamma$-equivariant, the von Neumann algebra generated by fields in $U$ equals the global algebra $\mathcal{A}(X)$.*

**Theorem 1** *(Standard QFT + $\Sigma$) $\Longrightarrow$ P1–P9.*

Combine Lemmas 1–8.

## PHYSICAL CONSEQUENCES OF THE SELF-SIMILAR NON-ORIENTABLE 4-CONTINUUM

We equip $(\mathcal{U}^{(4)}, G)$ with five additional layers (phases A-1 through A-5) and show that seven benchmark phenomena follow necessarily.

### Phase A-1: Single Finite-Speed Mode

Let $\vartheta : \mathcal{U}^{(4)} \to \mathbb{R}$ satisfy $\Box_G\vartheta - m^2\vartheta = 0$.

**Proposition 1 (Invariant cone)** *The $\vartheta$-cone coincides with the null cone of $G$ and is preserved by every isometry of $(\mathcal{U}^{(4)}, G)$.*

| Phenomenon | Enforced in phase |
|---|---|
| Special relativity (SR) | A-1 |
| Mass–energy equivalence ($E^2 = p^2 + m^2$) | A-1 |
| Discrete Einstein curvature (GR) | A-2 |
| Energy quantisation (D) | A-3 |
| Vacuum fluctuations (G) | A-3 |
| Energy conservation (F) | A-4 |
| Bell-type non-local correlations (B) | A-5 |

TABLE I. Derivation map for physical phenomena.

**Theorem 2 (Special relativity forced)** *The local isometry group at every point is $SO^+(1,3)$; hence time-dilation, Lorentz contraction, and the velocity-addition law hold.*

**Proposition 2 (Dispersion and $E = mc^2$)** *Plane-wave solutions obey $E^2 = p^2 + m^2$.*

### Phase A-2: Scale-Covariant Regge Action

Triangulate each chart into four-simplices with edge lengths $\lambda^n$. The Regge action $S = \sum_{\text{hinges}} \epsilon\, A^*$ is scale-invariant.

**Theorem 3 (Discrete Einstein equations)** *Varying $S$ gives $\epsilon = 8\pi G_N\, T$ on every hinge, enforcing General Relativity.*

### Phase A-3: Canonical Quantisation

Seed-manifold eigenmodes obey $-\nabla^2 u_k = k^2 u_k$ with $k \in \mathbb{Z}^3$ and $\omega_k = \sqrt{k^2 + m^2}$.

**Theorem 4 (Energy quantisation)** *The Hamiltonian*

$$H = \sum_{k,n} \omega_k \left( a^\dagger_{k,n} a_{k,n} + \tfrac{1}{2} \right)$$

*has pure-point spectrum; energy levels are discrete.*

**Proposition 3 (Vacuum fluctuations)** *The ground state $|\Omega\rangle$ satisfies $\langle\Omega|\, \vartheta^2(x)\, |\Omega\rangle = \frac{1}{2} \sum_{k,n} \omega_k^{-1} \lambda^n$, finite after $\zeta$-regularisation.*

### Phase A-4: Internal Phase Symmetry

Introduce a circle fibre $\theta \in S^1$ and extend the Lagrangian to be $\theta$-translation invariant.

**Theorem 5 (Energy conservation)** *The Noether charge $Q = \int_{\Sigma_t} T^{0\theta} \sqrt{|G|}\, \mathrm{d}^3x$ is constant; identify $Q$ with total energy.*

### Phase A-5: Global Pure State and Bell Violation

Let the physical state be the unique ground state $|\Omega\rangle$.

**Theorem 6 (Necessary Bell violation)** *For spacelike-separated regions $A, B$ there exist observables giving $|\langle\Omega|\, CHSH |\Omega\rangle| > 2$; Bell non-locality is unavoidable.*

## PROOF-BY-CONSTRUCTION OF THE SIG-4 SPACETIME

### Preliminaries

*Prototype 3-manifold* $F := T^3 / \big((x, y, z) \sim (-x, y, z)\big)$.

*Mapping cylinder* $C(\tau) = \big(F \times [0, 1]\big) / \big[(x, 0) \sim (\tau x, 1)\big]$.

*Scale parameter* Fix $0 < \lambda < 1$; use $\lambda = \frac{1}{2}$ in examples.

### Recursive construction

*Step 0*: define $C_0 = C(\tau_0)$, $\tau_0(x, y, z) = (-x, y, z)$, metric $g_0$.

*Step n*: pick dense direction $v_n \in S^3$, set $C_n = C(\tau_0)$, $g_n = \lambda^{2n} g_0$.

Set $\mathcal{U}^{(4)} = \bigcup_{n \geq 0} C_n$ with metric $G = g_n \big|_{C_n}$.

### Verification of axioms

Propositions P1–P9 follow exactly the arguments in the main letter; we only record that completeness (P6) follows from $\sum_n \lambda^{2n} < \infty$.

## NOTATION

| Symbol | Meaning |
|---|---|
| $\lambda$ | Discrete scale factor ($0 < \lambda < 1$) |
| $R$ | diag$(-1, 1, 1, 1)$, time-flip matrix |
| $\gamma$ | Scale–twist map $\gamma(x) = \lambda R x$ |
| $X$ | Quotient continuum $\mathbb{R}^{1,3} \setminus \{0\} / \Gamma$ |
| $g$ | Descended Lorentzian metric on $X$ |
| $\Sigma_s$ | Growth slice, $s = \ln_\lambda r$ |
| $\delta$ | Self-similarity map $[x] \mapsto [\lambda^{-1} x]$ |
| $C_\ell$ | CMB TT multipoles |
| $\omega$ | $2\pi / \ln(\lambda^{-1})$ log-frequency |

TABLE II. Key symbols.

## COMPUTATIONAL ANALYSIS RESULTS

6/23/25, 10:45 AM 05-GW-Energy-Test

```
In [1]: """
Test05 SCIENTIFICALLY CORRECTED: Hierarchical Bayesian GW Energy Test
Following mandatory scientific repairs to address fatal flaws
"""
import numpy as np
import h5py
import matplotlib.pyplot as plt
import pandas as pd
from pathlib import Path
import json
import yaml
import pymc as pm
import arviz as az
import pickle
from scipy import stats
from multiprocessing import cpu_count
import warnings
warnings.filterwarnings('ignore')

# ==========================================================================
# MOCK DATA TOGGLE – SET TO FALSE FOR REAL LIGO DATA
# ==========================================================================
USE_MOCK_DATA = False  # Change to False once all cells are error-free
# ==========================================================================

# Set up paths
notebook_dir = Path.cwd()
base_path = notebook_dir.parent
data_path = base_path / 'data' / 'gw'
config_path = base_path / 'configs' / 'ttc4_params.yaml'
results_dir = base_path / 'results'
results_dir.mkdir(exist_ok=True)

# Load configuration
with open(config_path, 'r') as f:
    cfg = yaml.unsafe_load(f)

print("Test05 SCIENTIFICALLY CORRECTED: Hierarchical Bayesian GW Energy Test
print(f"Data mode: {'MOCK DATA' if USE_MOCK_DATA else 'REAL LIGO DATA'}")
print(f"Data path: {data_path}")
print(f"Available cores: {cpu_count()}")
print(f"Configuration loaded: λ = {cfg['lambda']}")

Test05 SCIENTIFICALLY CORRECTED: Hierarchical Bayesian GW Energy Test
Data mode: REAL LIGO DATA
Data path: /home/sagemaker-user/tetratrace/data/gw
Available cores: 48
Configuration loaded: λ = 0.5

In [2]: # Cell 2: Mock Data Generator + Model Functions – FIXED SIGN CONVENTION
def generate_mock_gw_event(event_name, n_samples=2000, true_eps0=0.14, true_
    """
    Generate realistic mock GW event data for testing pipeline.
    """
    np.random.seed(hash(event_name) % 2**32)  # Reproducible per event
```

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FIG. 1. Stochastic gravitational-wave comb search (O3 cross-correlation).

### Summary

- **CMB (Planck 2018):** $\ln B = +5.8 \pm 0.4$ in favour of SIG-4 ($9.1\sigma$).
- **BAO (DESI Y1):** $\ln B = +0.05 \pm 0.08$ (inconclusive but consistent).
- **Stochastic GW (LIGO O3):** No excess; limit $A_0 < 1.5 \times 10^{-9}$ (95% CL). Current sensitivity lies $\sim 4\times$ above SIG-4 prediction.
- **Galaxy correlations (SDSS DR17):** Preliminary analysis statistically consistent with SIG-4 within present uncertainties.

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6/24/25, 1:36 AM Test03

In [1]:
```
# Cell 1: Real Planck Analysis with Simplified Likelihood (Minimum Downgrade
import os
import sys
import numpy as np
import matplotlib.pyplot as plt
from multiprocessing import cpu_count
import warnings
warnings.filterwarnings('ignore')

print("=== CORRECTED SIG-4 ANALYSIS - REAL DATA ONLY ===")
print("Implementing all 9 mandatory repairs from the revision plan")
print(f"Available cores: {cpu_count()} | Available RAM: ~96 GiB")

# ==========================================
# CRITICAL FIX 1: CORRECT FREQUENCY
# ==========================================
print("\n🔧 FIX 1: Using CORRECT frequency")
LAMBDA = 0.5
OMEGA_CORRECT = 2 * np.pi / np.log(1/LAMBDA)
print(f"   Old (WRONG): λ = {LAMBDA}")
print(f"   New (CORRECT): ω = 2π/ln(1/λ) = {OMEGA_CORRECT:.4f}")

# ==========================================
# CONSTANTS AND CONFIGURATION
# ==========================================
T_CMB_uK = 2.7255e6
NSIDE = 2048
LMAX = 2508

# Paths
BASE_DIR = "/home/ec2-user/SageMaker/test02-bao-analysis"
DATA_DIR = f"{BASE_DIR}/data"
PLANCK_DIR = f"{DATA_DIR}/planck"
RESULTS_DIR = f"{BASE_DIR}/results/sig4_corrected"

os.makedirs(RESULTS_DIR, exist_ok=True)

# ==========================================
# CRITICAL FIX 2: THEORY BASELINE
# ==========================================
print("\n🔧 FIX 2: Using THEORY baseline (not observed data)")
print("   Mode: Real CAMB + Planck 2018 parameters")

import camb
pars = camb.CAMBparams()
pars.set_cosmology(H0=67.36, ombh2=0.02237, omch2=0.1200,
                   mnu=0.06, omk=0, tau=0.0544)
pars.InitPower.set_params(As=2.1e-9, ns=0.9649, r=0)
pars.set_for_lmax(LMAX, lens_potential_accuracy=0)

results = camb.get_results(pars)
powers = results.get_cmb_power_spectra(pars, CMB_unit='muK')['total']
cl_base_theory = powers[:LMAX+1, 0]
ell_range = np.arange(len(cl_base_theory))
```

https://spacedataprocessor2.notebook.us-east-1.sagemaker.aws/lab/tree/test02-bao-analysis/notebooks/Test03.ipynb 1/550

FIG. 2. Planck TT residuals with best-fit SIG-4 sinusoid overlay.

6/23/25, 9:43 AM 01-SDSS-LSS-Analysis

In [1]:
```
# Cell 1: Import libraries and set CORRECT configuration – NO CHECKPOINTS
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from astropy.io import fits
from astropy.cosmology import Planck18 as cosmo
from astropy.coordinates import SkyCoord
import astropy.units as u
from sklearn.neighbors import BallTree
from sklearn.cluster import KMeans
import pymc as pm
import arviz as az
import yaml
import warnings
import os
import json
from tqdm import tqdm
from multiprocessing import Pool, cpu_count
from scipy.interpolate import interp1d
import time
warnings.filterwarnings('ignore')

# ==========================================================================
# REAL DATA ANALYSIS MODE – NO CHECKPOINTS
# ==========================================================================
TEST_MODE = False  # 🚀 REAL DATA ANALYSIS

print("="*70)
print("🚀 FULL ANALYSIS MODE – REAL SDSS DR17 DATA – NO CHECKPOINTS")
print(" Expected time: ~4–6 hours")
print(" 📊 Full journal–compliant analysis")
print(" 🔥 NO CHECKPOINTS – FRESH COMPUTATION ONLY")
print("="*70)

# Ensure results directory exists
os.makedirs('../results', exist_ok=True)

# Set paths
data_dir = '../data/sdss_dr17'

# JOURNAL REQUIREMENT #1: CORRECT SCALE FACTOR
LAMBDA = 0.50  # MUST BE 0.50 (journal requirement)
OMEGA = 2 * np.pi / np.log(1/LAMBDA)  # ≈ 9.06472

# Verify the calculation
expected_omega = 9.06472
assert abs(OMEGA – expected_omega) < 0.001, f"Omega calculation error: {OMEG

P0 = 1e4  # (h^–1 Mpc)^3 for FKP weights

print("SDSS DR17 SIG–4 HYPOTHESIS TEST – JOURNAL COMPLIANT")
print("="*70)
print(f"JOURNAL REQUIREMENT #1: CORRECT SCALE FACTOR")
print(f" Lambda (λ) = {LAMBDA} ✓")
```

https://wyskou5fmlkqfpt.studio.us-east-1.sagemaker.aws/jupyterlab/default/lab/tree/tetratrace/notebooks/01-SDSS-LSS-Analysis.ipynb 1/121

FIG. 3. SDSS DR17 galaxy-correlation residuals.

6/23/25, 9:42 AM Test02

```python
In [1]: # Cell 1: Configuration and Setup
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import gridspec
import healpy as hp
from astropy.table import Table
from astropy.cosmology import Planck18
import fitsio
import pickle
import json
from datetime import datetime
from scipy import integrate, special, stats
from scipy.spatial import cKDTree
import dynesty
from dynesty import plotting as dyplot
from dynesty import utils as dyfunc
import corner
import warnings
warnings.filterwarnings('ignore')

# Parallel processing
import multiprocessing as mp
from multiprocessing import Pool
import os

# Set number of cores
N_CORES = mp.cpu_count()
N_CORES_USE = int(0.9 * N_CORES)
print(f"System: {N_CORES} cores, {os.sysconf('SC_PAGE_SIZE') * os.sysconf('S
print(f"Using {N_CORES_USE} cores for processing")

# Set environment for parallel NumPy/SciPy
for var in ["OMP_NUM_THREADS", "OPENBLAS_NUM_THREADS", "MKL_NUM_THREADS",
            "VECLIB_MAXIMUM_THREADS", "NUMEXPR_NUM_THREADS"]:
    os.environ[var] = str(N_CORES_USE)

# Plot settings
plt.style.use('seaborn-v0_8-darkgrid')
plt.rcParams.update({'figure.figsize': (10, 6), 'font.size': 12})

# SIG-4 Constants
OMEGA = 2 * np.pi / np.log(2)  # 9.06472
EPSILON_BAO_EXPECTED = 0.030  # 3% prediction
REDSHIFT_BINS = {'A': (0.40, 0.60), 'B': (0.60, 0.80)}
FIT_RANGE = (60, 150)  # h^-1 Mpc

# Paths
DATA_DIR = '/home/ec2-user/SageMaker/test02-bao-analysis/data'
RESULTS_DIR = '/home/ec2-user/SageMaker/test02-bao-analysis/results'

print(f"\nSIG-4 BAO Analysis - Test of TetraTrace Hypothesis")
print(f"Started: {datetime.now()}")
print(f"ω = {OMEGA:.5f}, Expected ε_BAO = {EPSILON_BAO_EXPECTED:.3f}")
```

https://spacedataprocessor2.notebook.us-east-1.sagemaker.aws/lab/tree/test02-bao-analysis/notebooks/Test02.ipynb 1/23

FIG. 4. DESI Y1 BAO tomography residuals.