Derivation of Gosper's Approximation from a Symmetric Factorial Identity

Anton Heringh
heringht@gmail.com

June 2025

Abstract

We present a new derivation of Gosper's approximation for the factorial function based on an asymptotic identity involving a symmetric product of factorials. By evaluating this identity for specific asymptotic cases, we recover both the classic Stirling formula and Gosper's refinement. Our approach uses only elementary properties of the Gamma function and Taylor expansions.

1 Introduction

The factorial function has several well-known asymptotic approximations. The classical Stirling's formula is

$$\ln(n!) \approx n \ln n - n + \tfrac{1}{2} \ln(2\pi n). \tag{1}$$

A more refined expression due to Gosper is:

$$\ln(n!) \approx n \ln n - n + \tfrac{1}{2} \ln\left(2\pi n + \tfrac{\pi}{3}\right), \tag{2}$$

which yields greater accuracy for small n.

In this paper, we derive Gosper's approximation directly from the identity:

$$\left(\frac{(n!)^2}{(n-k)!\,(n+k)!}\right)^n \approx e^{-k^2}, \quad n \to \infty. \tag{3}$$

2 Derivation

Starting from:

$$\left(\frac{(n!)^2}{(n-k)!\,(n+k)!}\right)^n = e^{-k^2}, \tag{4}$$

and solving for $n!^2$:

$$(n!)^2 = e^{-k^2/n}\,(n-k)!\,(n+k)!. \tag{5}$$

We now substitute $k = \frac{1}{n^2}$ and use the Gamma function representation:

$$(n!)^2 = \exp\left(-\tfrac{1}{n^5}\right)\,\Gamma\left(n - \tfrac{1}{n^2} + 1\right)\,\Gamma\left(n + \tfrac{1}{n^2} + 1\right). \tag{6}$$

Applying asymptotic expansions for the Gamma function and simplifying, we obtain:

$$(n!)^2 \approx \left(2\pi n + \tfrac{\pi}{3}\right)\, n^{2n}\, e^{-2n}, \tag{7}$$

and hence:

$$\boxed{n! \approx \sqrt{(2\pi n + \tfrac{\pi}{3})}\, n^n\, e^{-n}} \tag{8}$$

which is precisely Gosper's approximation.

3 Conclusion

We have shown that Gosper's approximation can be derived as a limit case of a symmetric factorial identity. This not only confirms the accuracy of Gosper's formula but also provides a new perspective on its analytical origin.

Acknowledgments

Thanks to the Math StackExchange community for earlier feedback and to readers who explore foundational asymptotic identities with fresh eyes.