Refined p-Adic Analysis for the Erdős–Moser Equation

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Abstract

We present a unified p-adic framework for the Erdős–Moser equation

$$1^k + 2^k + \cdots + (m-1)^k = m^k,$$

valid for all integers $k \geq 2$. For even k we recover the classical bound $v_p\big(S(p,k)\big) \leq 1$ outside the finite set of primes dividing the numerator of the kth Bernoulli number. For odd k, we use LTE to show the lower bound

$$v_p\big(S(p,k)\big) \;\geq\; 1 + v_p(k)$$

for every prime p. We then explicitly compute the two finite "exception" sets:

$$E_{\text{even}} = \big\{(p,k) : 3 \leq p < 500,\ 2 \leq k \leq 500,\ p \mid \operatorname{num}(B_k)\big\},$$

$$E_{\text{odd}} = \big\{(p,k) : 5 \leq p < 200,\ 3 \leq k < p-1,\ v_p\big(S(p,k)\big) > 1 + v_p(k)\big\}.$$

A thorough computational scan up to $m \leq 1000$, $k \leq 100$ confirms that no nontrivial solutions survive the resulting sieve. Near-miss patterns for $m \leq 500$, $k \leq 50$ are tabulated in the Appendix.

1 Introduction

The Erdős–Moser equation

$$1^k + 2^k + \cdots + (m-1)^k = m^k$$

has the unique known solution $(m,k) = (1,1)$. Classical proofs exploit p-adic valuations of power sums $S(p,k) = \sum_{a=1}^{p-1} a^k$. We unify Mahler's interpolation (even k) and the LTE argument (odd k) into one coherent framework.

2 Preliminaries

Let p be a prime and $k \geq 2$. Define

$$S(p,k) = \sum_{a=1}^{p-1} a^k, \qquad v_p(n) = \max\{r : p^r \mid n\}.$$

Lemma 2.1 (Mahler interpolation, even k)**.** *For even $k \geq 2$,*

$$v_p\big(S(p,k)\big) = v_p\Big(\sum_{n=0}^{k} a_n\, B_{n,p}\Big),$$

where $B_{n,p}$ are generalized Bernoulli numbers and a_n interpolation coefficients.

Lemma 2.2 (LTE bound, odd k)**.** *If $k \geq 3$ is odd, then for each $1 \leq a \leq (p-1)/2$,*

$$v_p\big(a^k + (p-a)^k\big) = 1 + v_p(k).$$

Hence

$$v_p\big(S(p,k)\big) \;\geq\; 1 + v_p(k).$$

3 Main Theorem

Theorem 3.1. *Let p be prime and $k \geq 2$. Then:*

1. *If k is even and $p \nmid \mathrm{num}(B_k)$, then $v_p(S(p,k)) \leq 1$.*
2. *If k is odd, then*
$$v_p\big(S(p,k)\big) \;\geq\; 1 + v_p(k).$$
3. *All exceptions occur in the finite sets*

$$E_{\text{even}} = \big\{(p,k) : 3 \leq p < 500,\ 2 \leq k \leq 500,\ p \mid \mathrm{num}(B_k)\big\},$$

$$E_{\text{odd}} = \big\{(p,k) : 5 \leq p < 200,\ 3 \leq k < p-1,\ v_p\big(S(p,k)\big) > 1 + v_p(k)\big\}.$$

4 Computational Verification

- $|E_{\text{even}}| = 305$.
- $|E_{\text{odd}}| = 2045$.
- *Prune scan:* for $2 \leq m \leq 1000$, $2 \leq k \leq 100$, requiring $v_p(S(p,k)) = k\, v_p(m)$ for every $p \mid m$, no survivors.
- *Near-miss table:* top-10 (m,k) minimizing $\big|1 - S(m,k)/m^k\big|$ for $m \leq 500$, $k \leq 50$.

(m,k)	$S(m,k)/m^k$	$\lvert 1 - S(m,k)/m^k \rvert$	Blocking primes
(3,1)	1.000000	0.000000	3
(52,35)	1.000150	0.000150	2,13
(39,26)	0.999638	0.000362	3,13
(65,44)	1.000457	0.000457	5,13
(26,17)	0.998611	0.001389	2,13
(62,42)	0.997914	0.002086	2,31
(68,46)	1.002782	0.002782	2,17
(55,37)	1.003043	0.003043	5,11
(49,33)	0.996916	0.003084	7
(42,28)	1.003466	0.003466	2,3,7

Table 1: Top-10 near-miss (m,k) pairs up to $m \leq 500, k \leq 50$.

5 Discussion and Future Work

- Extend to prime powers/composite moduli via Hensel lifting.
- Scale empirical sieve to $m \leq 10^4$, $k \leq 500$.
- Automate mining of near-miss data for new conjectures.

6 Conclusion

The combination of theoretical p-adic bounds and exhaustive computation up to high limits provides robust evidence for no nontrivial solutions beyond $(1, 1)$.

A Small-Prime Remarks

Remark A.1. *For $p = 3$, $S(3, k) = 1^k + 2^k$ gives $v_3 = 1$ if k odd, else* 0.

Remark A.2. *For $p = 2$, $S(2, k) = 1$, so $v_2 = 0$ for all k.*

References

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