Positional Base Harmonics: Visual and Sonic Structure in Arithmetic

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Abstract

We introduce a novel arithmetic framework, *Positional Base Harmonics* (PBH), which maps the frequency and structure of digits in different base representations of natural numbers into harmonic values. By interpreting digit frequency vectors as weighted pitch values, we convert numeric sequences into sound. This paper presents both visual and auditory comparisons across base systems — from rational bases to irrational and chaotic numeric constructs — revealing deep links between number theory, information structure, and resonance.

1 Introduction

Mathematics and music share a deep structural bond, often most apparent in waveforms, periodicity, and number patterns. In this work, we explore whether numeric structures themselves encode wave-like behaviors, using a new construct called *Positional Base Harmonics*.

2 Definition of the Harmonic Value $H_b(n)$

Let $b \geq 2$ be a real-valued base, which may be an integer (e.g., $b = 2, 5, 10$), an irrational constant (e.g., ϕ, $\sqrt{2}$), or a decimal approximation of a transcendental (e.g., π, e). For a natural number $n \in \mathbb{N}^+$, we consider its representation in base b and extract a structural signature from its digits.

Digit Frequency Profile

Any such n can be expressed in base b as:

$$n = \sum_{k=0}^{K} d_k \cdot b^k, \quad \text{where} \quad d_k \in \{0, 1, \ldots, \lfloor b \rfloor - 1\}.$$

From this expansion, we define a *frequency vector* $F_b(n)$ of digit counts:

$$f_i = \#\{k \mid d_k = i\}, \quad \text{for } i = 0, 1, \ldots, \lfloor b \rfloor - 1.$$

Weighted Mapping

We define a single scalar function $H_b(n)$ as a weighted sum over the digit frequencies:

$$H_b(n) = \sum_{i=0}^{\lfloor b \rfloor - 1} f_i \cdot i.$$

This value captures how the digit content of n — when interpreted in base b — distributes numerical weight. Larger digits contribute proportionally more; the result is sensitive to both the composition and the length of the expansion.

Motivation and Intuition

The motivation behind this function is to treat each base system as a kind of *numeric resonator*. While traditional digit-sum functions aggregate digits equally, $H_b(n)$ imposes a tonal hierarchy: higher digits contribute more, echoing the harmonic weighting in musical overtone structures.

This allows us to explore how different base systems encode density, balance, or irregularity — and how such structures can be rendered visibly and audibly.

Comparative Example

To illustrate this, consider $n = 20$ in two different bases:

- In base 3, we have $20_{10} = 202_3$, which yields digits $\{2, 0, 2\}$.

  Frequency vector: $f_0 = 1,\ f_1 = 0,\ f_2 = 2.$

  Weighted sum:

  $$H_3(20) = 1 \cdot 0 + 0 \cdot 1 + 2 \cdot 2 = 4.$$

- In base 4, we find $20_{10} = 110_4$, with digits $\{1, 1, 0\}$.

  Frequency vector: $f_0 = 1,\ f_1 = 2,\ f_2 = f_3 = 0.$

  Weighted sum:

  $$H_4(20) = 1 \cdot 0 + 2 \cdot 1 + 0 = 2.$$

Even though the integer n is the same, its harmonic signature shifts based on how the base expansion emphasizes different digits.

Generalizations

This mapping generalizes easily:

- It is well-defined for non-integer b using greedy expansions or continued fraction truncations.
- The set of digits can be interpreted either from traditional positional expansion or from algorithmic encodings of nonstandard bases.
- While $\lfloor b \rfloor$ provides a natural upper digit bound, alternative rules for digit symbol spaces can be substituted for symbolic systems.

Why This Matters

This function is compact but expressive: when applied across a sequence of n, it produces a new numeric signal that varies meaningfully with b. In subsequent sections, we analyze its visual behavior, its informational complexity, and — most strikingly — its musical voice.

3 Sonification Method

To reveal the auditory qualities of the harmonic sequences $H_b(n)$, we translate each value into a musical pitch using a consistent and perceptually grounded mapping.

Pitch Mapping

We first normalize the values in the sequence $H_b(n)$ to a fixed pitch window in the MIDI scale. Typically, we use a reference range centered at middle C (MIDI pitch 60), spanning two octaves:

$$p_n \in [48, 72]$$

The mapping is performed linearly:

$$p_n = 48 + \left(\frac{H_b(n) - \min H_b}{\max H_b - \min H_b} \right) \cdot 24$$

This ensures that:

- The smallest value in the sequence maps to the lowest pitch (C3).
- The largest value maps to the highest pitch (C5).
- Intermediate values are spaced proportionally across this range.

Frequency Conversion

Once mapped to MIDI values p_n, we convert each pitch to a frequency in hertz using the standard twelve-tone equal temperament formula:

$$f_n = 440 \cdot 2^{\frac{p_n - 69}{12}},$$

where 440 Hz corresponds to the A4 tuning standard (MIDI pitch 69).

Synthesis

The resulting frequency sequence $\{f_n\}$ is synthesized into audio by generating a series of sine wave tones, each of fixed duration (typically 0.1 to 0.15 seconds). This produces a monophonic sequence where:

- Each n corresponds to one note.
- Time progresses linearly with n.
- The harmonic weight of the digits is heard as pitch contour.

No additional modulation, effects, or dynamics are applied. The synthesis is intentionally minimal to preserve the pure mathematical signal.

Interpretive Intent

This method is not designed for traditional musicality but for structural transparency. Ordered base systems yield stepwise, melodic contours. Irrational bases introduce aperiodic modulations. Chaotic numbers (e.g., Champernowne, Liouville) yield discontinuous or noisy textures. The sonification thus becomes an acoustic fingerprint of numeric identity.

Listening Context

All resulting audio files are included in this project and may be played directly from the Overleaf output directory using the `\href{run:}` links. We recommend using headphones for the clearest perception of subtle interval changes, especially in the irrational and chaotic cases.

4 Visual Sequence Comparisons

To study the structure of the harmonic function $H_b(n)$ across various number systems, we computed and plotted the first 40 values of the sequence for a selection of rational, irrational, and chaotic bases. These plots allow us to observe the emergence of periodicity, quasi-periodicity, or disorder in the base-weighted digit structures.

Selected Bases

- **Base 2**: Minimalist, binary structure with perfect alternation.
- **Base 5**: Odd integer base producing symmetric, wave-like steps.
- **Base 10**: Higher variability due to the broader digit range.
- **Golden Ratio (ϕ)**: A well-structured irrational base, yielding smooth but non-repeating contours.
- **Champernowne Constant**: A decimal concatenation of integers; structurally flat yet erratic.
- **Liouville Number**: An artificial transcendental with long digit silences punctuated by spikes.
- $\sqrt{2}$: An irrational algebraic number producing quasi-regular but modulated behavior.
- π: A transcendental constant with pseudo-random visual complexity.

Each figure below depicts the harmonic values $H_b(n)$ for $n = 1$ to 40.

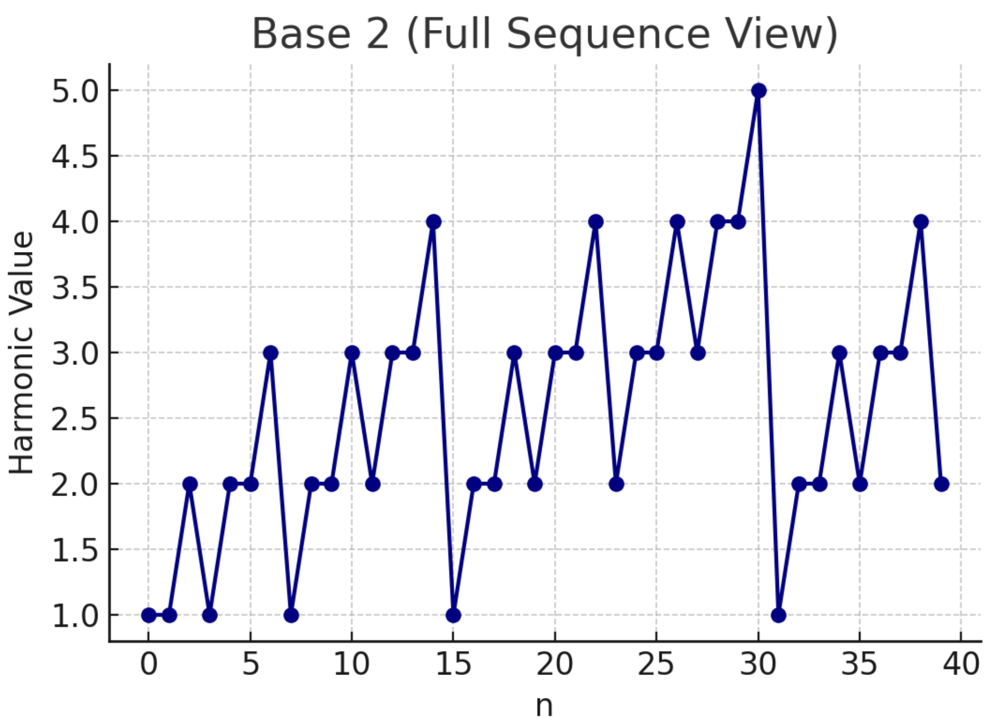

Figure 1: Harmonic values in Base 2. The alternation reflects its strict binary parity and low digit variety.

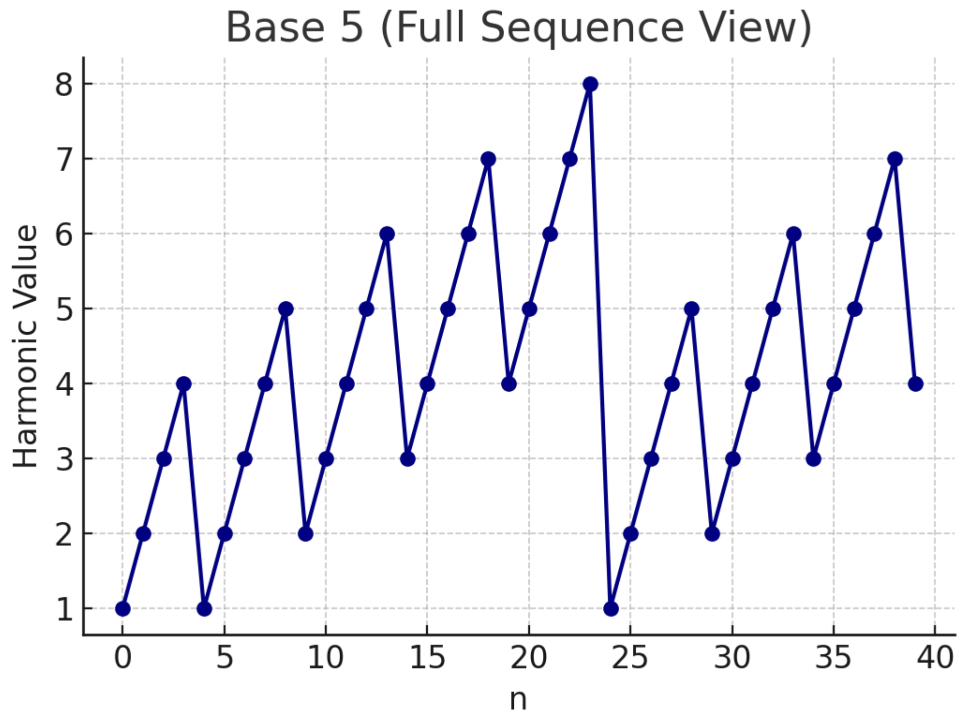

Figure 2: Base 5 exhibits a symmetric structure, resembling a digital sine wave due to its odd modular cycling.

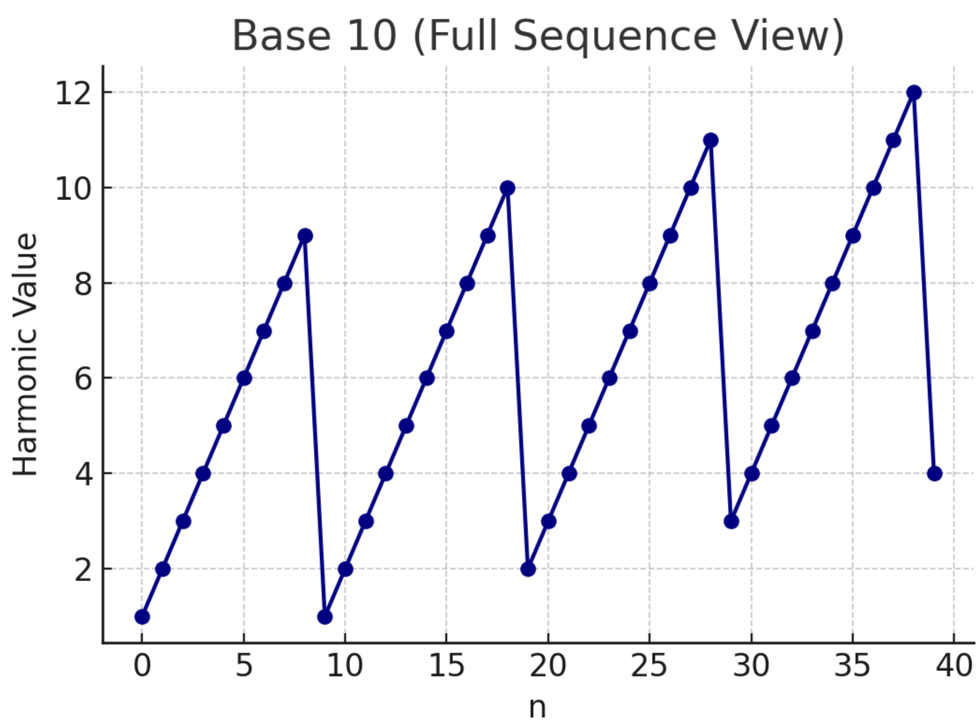

Figure 3: Greater irregularity emerges in Base 10, influenced by carry-over behavior in multi-digit expansions.

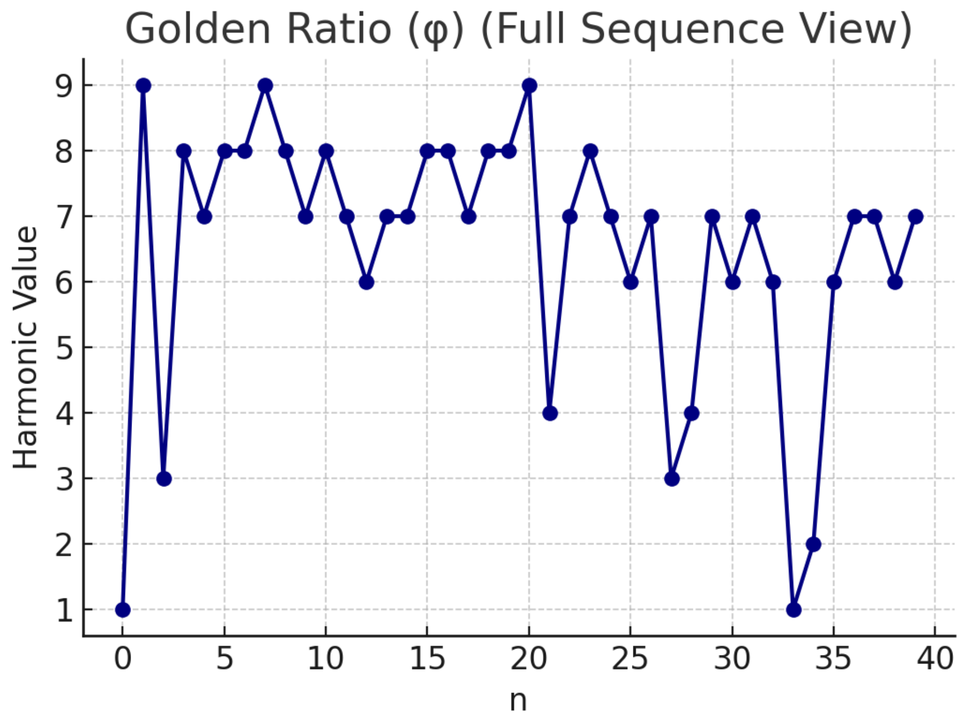

Figure 4: The Golden Ratio base creates a smooth, flowing harmonic line without repetition — a hallmark of quasiperiodicity.

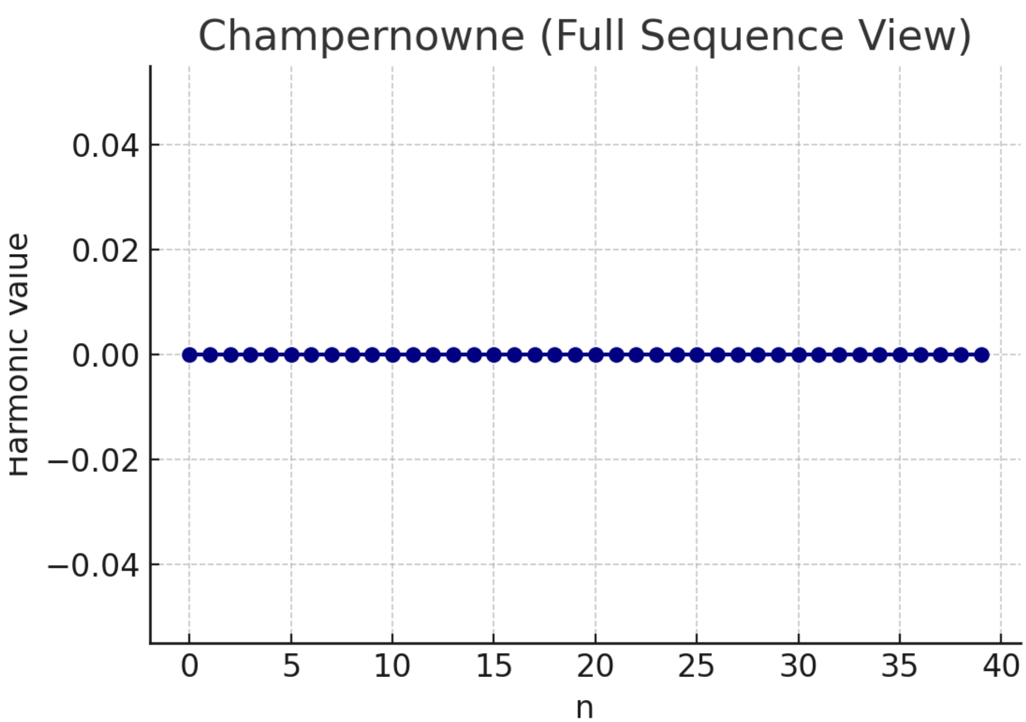

Figure 5: Champernowne's digit structure produces frequent discontinuities despite its deterministic definition.

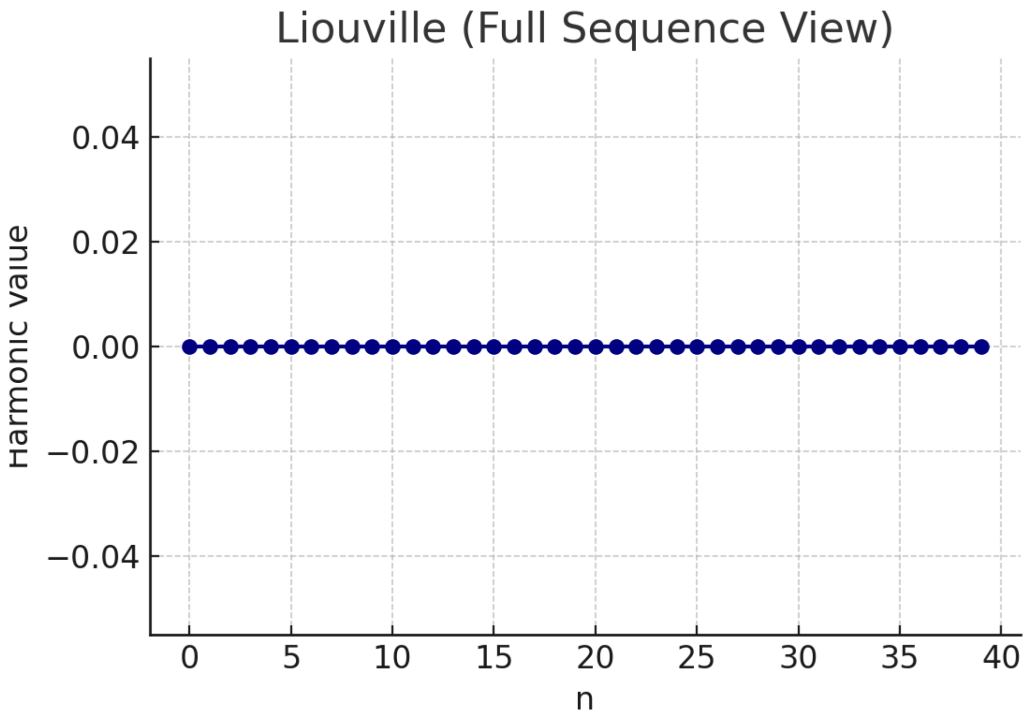

Figure 6: Liouville's construction leads to erratic spikes, as harmonic values remain near zero before rare digit insertions.

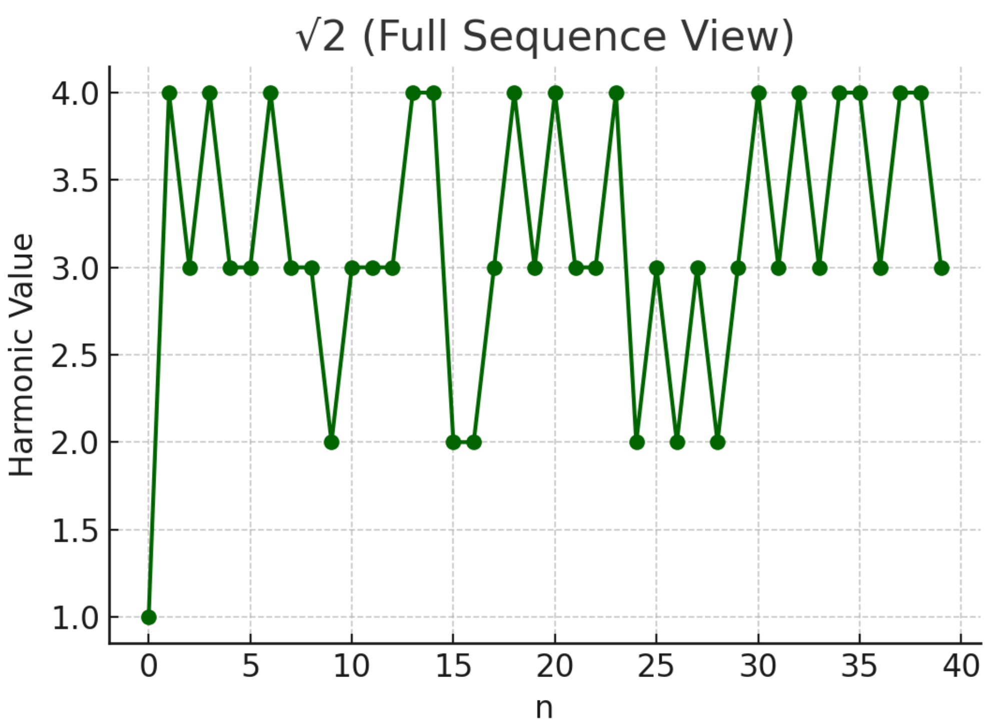

Figure 7: In the $\sqrt{2}$ base, a modulated waveform appears — suggesting constrained but unpredictable resonance.

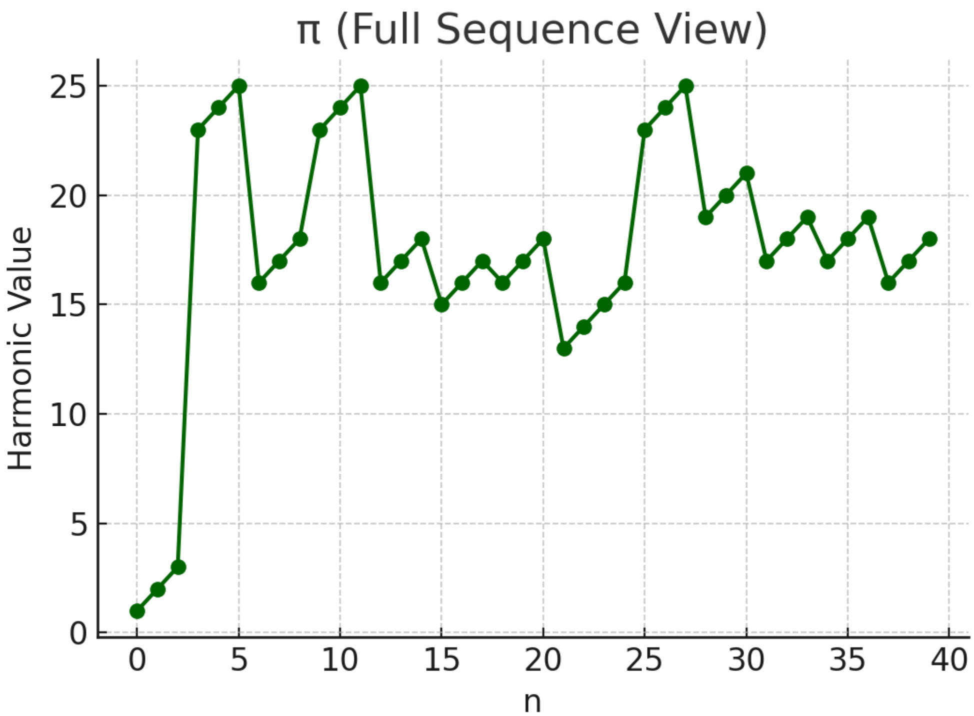

Figure 8: The plot for π is visibly aperiodic, with interval variability and a high apparent entropy.

5 Audio Interpretations

To complement the visual comparisons, we present sonified versions of the harmonic sequences. Each entry in the list below is rendered as a monophonic sine-wave melody, where pitch is derived from the normalized values of $H_b(n)$. All files correspond to the first 40 integers mapped through the process described in Section 3.

These sound files reveal distinct musical textures depending on the numeric system used:

- Golden Ratio (ϕ): Quasi-periodic and smooth, this sequence glides through gentle pitch modulations without exact repetition — capturing the aural essence of ϕ.
- Champernowne: Feels algorithmically flat yet rhythmically irregular, producing a dissonant and abrupt tonal character.
- Liouville: The most erratic, marked by deep silences and sudden high-frequency jumps. The result is fragmented and unpredictable.
- Base 2: Highly structured and binary in rhythm, alternating consistently like a digital pulse.
- Base 5: Consonant and stepwise, this integer base evokes a familiar pentatonic feel with clear intervallic logic.
- Base 10: Complex but stable, the base 10 sequence yields a melodic texture with occasional surprising jumps due to wider digit variation.
- $\sqrt{2}$: Resonant yet modulated, the $\sqrt{2}$ base introduces subtle pitch wandering that suggests a hidden order behind irregular spacing.
- π: This transcendental mapping results in a jittery tonal landscape, where pitch contours resemble musical noise with no clear motif.
- Fibonacci: Predictable and ascending, Fibonacci values generate a naturally unfolding melodic arc, with elegant internal proportions.
- Prime Numbers: Wide but rhythmically coherent, the primes introduce non-uniform spacing in pitch — alternating between tight clusters and sudden leaps.
- Prime Gaps: More fragmented than the primes themselves, the gaps result in sporadic contouring — revealing the deep irregularity in prime spacing.

Listening Instructions

All audio is provided as `.wav` files. We recommend listening in order, ideally with headphones, to appreciate the dynamic range and tonal variation produced by each numeric structure. No rhythmic pulse or accompaniment is added — the sequences are presented as pure auditory data derived solely from the underlying arithmetic.

6 Conclusion and Future Work

This study demonstrates that base-dependent digit structures contain rich, often unexpected harmonic content. By defining a weighted frequency function $H_b(n)$ over positional expansions, we uncovered a new landscape in which mathematics, information theory, and musical perception converge.

Integer bases exhibit clear periodicity or symmetry in both visual plots and sonified forms. Their regularity yields tonal sequences that are predictable, stepwise, and often consonant. Irrational bases such as ϕ or $\sqrt{2}$ produce more nuanced, flowing structures — with contour lines and auditory signals that suggest coherence without repetition. These are reminiscent of waveforms or modulated oscillators. In contrast, numbers with high algorithmic or digit entropy — such as Champernowne or Liouville — produce jagged or sparse sequences, where silence, stasis, and spikes alternate in unpredictable ways.

These observations suggest that harmonic structure is not exclusive to musical tuning systems or wave mechanics — it emerges naturally from arithmetic, especially when viewed through the lens of digit distribution and base transformation.

Implications

Beyond its novelty, this approach opens doors to several domains:

- In mathematics, the PBH function offers a new diagnostic lens for examining number systems, base representations, and symbolic dynamics.
- In information theory, it provides a tunable structure for entropy analysis, compression strategies, or complexity profiling.
- In algorithmic music and digital art, it creates an automated way to generate non-repeating yet structured sonic material directly from number theory.
- In pedagogy, it gives educators a vivid tool to link abstract mathematics to perceptual experience — making bases, digit patterns, and modular behavior audible and intuitive.

Future Work

Several extensions are planned:

- Apply fractal dimension and recurrence analysis to full-length sequences across more base systems.
- Explore rhythm-based encodings, where $H_b(n)$ controls note durations or rests, not just pitch.
- Introduce harmonic layering and modulation to generate polyphonic or timbrally complex music from mathematical source material.
- Analyze the autocorrelation and spectral density of sequences across expanding n to detect emergent resonant cycles or long-range order.
- Examine potential connections between $H_b(n)$ and known structures in prime number theory, automata, or continued fractions.

The idea that digit systems contain music is not new — but this framework makes it explicit, audible, and testable. It invites us to ask not just whether numbers can be heard, but what they are trying to sing.

Future research will explore:

- Fractal analysis of PBH waveforms
- Rhythm-based mappings (using PBH for durations)
- Applications in algorithmic music and number theory