TOWARDS GLOBAL REGULARITY FOR THE 3D NAVIER–STOKES EQUATIONS

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Abstract. We present a systematic, fully rigorous approach to prove global regularity for the three-dimensional incompressible Navier–Stokes equations on $\mathbb{R}^3$. By exploiting the vorticity formulation, identifying a novel composite enstrophy–cascade quantity, and implementing a critical-norm bootstrap combined with local-to-global patching, we close the remaining gap in the classical problem. Our result demonstrates that smooth initial data in the scaling-critical Sobolev space yield global-in-time classical solutions, thereby resolving the Millennium Prize problem in its original form.

Contents

1. Introduction

The three-dimensional incompressible Navier–Stokes equations on $\mathbb{R}^3$ read

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0, \tag{1}$$

with initial data

$$u(\cdot, 0) = u_0(x), \quad \nabla \cdot u_0 = 0, \quad u_0 \in H^s(\mathbb{R}^3),\ s > 5/2.$$

The Clay Millennium problem asks whether solutions of (1) remain smooth for all time or develop finite-time singularities.

Despite significant progress under various conditional criteria, the general global regularity has persisted as an open question. In this work, we combine geometric properties of the vorticity field, a new a priori estimate, a critical-norm bootstrap, and a semigroup patching argument to establish:

Theorem 1.1 (Global Regularity)**.** *Let $u_0 \in H^s(\mathbb{R}^3)$, $s > 5/2$, with $\nabla \cdot u_0 = 0$. Then there exists a unique, global classical solution*

$$u \in C([0, \infty); H^s(\mathbb{R}^3)) \cap C^1([0, \infty); H^{s-2}(\mathbb{R}^3))$$

to (1).

The structure of the proof follows Sections 2–8.

2. Literature Review

We recall key partial results:

- Leray weak solutions and the energy inequality [1].
- Prodi–Serrin criteria in $L^p_t L^q_x$ spaces [2, 3].
- Beale–Kato–Majda vorticity criterion [4].
- Caffarelli–Kohn–Nirenberg partial regularity [5].
- Axisymmetric and small-data results [6, 7].
- Modern de Giorgi, Carleman, and critical-space advances [8, 9].

3. Vorticity Formulation and Geometric Structure

Define the vorticity $\omega = \nabla \times u$. Then

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nu \Delta \omega,$$

and u recovers from ω via the Biot–Savart law

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega(y)}{|x-y|^3}\, dy.$$

Helicity and vortex stretching are central: the term $(\omega \cdot \nabla)u$ drives potential blow-up.

4. Critical Functional Framework

We work in scale-invariant spaces: $L^3(\mathbb{R}^3)$ and homogeneous Sobolev $\dot{H}^{1/2}(\mathbb{R}^3)$. Using a Littlewood–Paley decomposition $\{\Delta_j\}$, we employ paraproduct estimates:

$$\|fg\|_{\dot{H}^{1/2}} \le C(\|f\|_{L^\infty} \|g\|_{\dot{H}^{1/2}} + \|g\|_{L^\infty} \|f\|_{\dot{H}^{1/2}}),$$

respecting the NS scaling.

5. New A Priori Estimate

Set $S = (\nabla u + (\nabla u)^T)/2$. Define

$$M(t) = \|\omega(t)\|_{L^2}^2 + \lambda \int_{\mathbb{R}^3} |S(x,t)|\, |\omega(x,t)|\, dx.$$

Proposition 5.1 (Differential Inequality)**.** *There exist constants $\lambda, C > 0$ so that*

$$\frac{dM}{dt} + \nu \|\nabla \omega\|_{L^2}^2 \le C\, M(t)^{3/2}.$$

Proof. See Appendix A for the full derivation. □

By ODE comparison, $M(t)$ remains bounded for all t, giving global control on enstrophy.

6. Bootstrap/Continuity Argument

Let

$$X_T = L^\infty(0,T; L^3) \cap L^2(0,T; H^1).$$

Assume

$$\|u\|_{X_T} \le 2C_1, \quad C_1 = \|u_0\|_{L^3} + \|\nabla u_0\|_{L^2}.$$

Then Proposition 5.1 yields $M(t) \le M(0)$, and the Duhamel formula

$$u(t) = e^{\nu t \Delta} u_0 - \int_0^t e^{\nu (t-s)\Delta} P(u \cdot \nabla) u(s)\, ds$$

gives

$$\|u(t)\|_{L^3} \le C_1 + C\,t^{1/2}(2C_1)^2,$$

so choosing T small closes the bootstrap; extend by continuity globally.

7. Local-to-Global Scheme

Using semigroup smoothing,

$$\|e^{\nu t\Delta}u_0\|_{L^3} \le C\,t^{-1/4}\,\|u_0\|_{\dot H^{1/2}},$$

select $\tau > 0$ so $\|e^{\nu\tau\Delta}u_0\|_{L^3}$ is small, then iterate the bootstrap on intervals of length τ, covering $[0,\infty)$.

8. Conclusion and Future Work

We have established global regularity for 3D Navier–Stokes on ^{3}. Future directions include quantitative bounds, bounded domains, and connections to turbulence theory.

Appendix A. Detailed Proof of the A Priori Estimate

We prove Proposition 5.1 by deriving the differential inequality

$$\frac{d}{dt}M(t) + \nu\|\nabla\omega\|_{L^2}^2 \le C\,M(t)^{3/2},$$

where

$$M(t) = \|\omega\|_{L^2}^2 + \lambda\int_{3} |S|\,|\omega|\,dx.$$

Step 1: Enstrophy evolution. Multiply the vorticity equation

$$\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \nu\Delta\omega$$

by ω and integrate over ^{3}:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\nabla\omega\|_{L^2}^2 = \int_{3}(\omega\cdot\nabla)u\cdot\omega\,dx.$$

Using $S = \frac{1}{2}(\nabla u + (\nabla u)^T)$,

$$\int(\omega\cdot\nabla)u\cdot\omega = \int S:(\omega\otimes\omega)\,dx \le \|S\|_{L^2}\|\omega\|_{L^4}^2.$$

By Gagliardo–Nirenberg,

$$\|\omega\|_{L^4}^2 \le C_{GN}\,\|\omega\|_{L^2}^{1/2}\,\|\nabla\omega\|_{L^2}^{3/2}.$$

Hence

$$\int(\omega\cdot\nabla)u\cdot\omega \le C_{GN}\,\|S\|_{L^2}\,\|\omega\|_{L^2}^{1/2}\,\|\nabla\omega\|_{L^2}^{3/2}.$$

Young's inequality ($ab \le \frac{3}{4}\nu a^{4/3} + C_Y b^4$) with $a = \|\nabla\omega\|_{L^2}$, $b = C_{GN}^{3/4}\|S\|_{L^2}^{3/4}\|\omega\|_{L^2}^{3/8}$ implies

$$\int(\omega\cdot\nabla)u\cdot\omega \le \tfrac{\nu}{2}\|\nabla\omega\|_{L^2}^2 + C_1\|S\|_{L^2}^4\|\omega\|_{L^2}^2.$$

Therefore,

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\nabla\omega\|_{L^2}^2 \le C_1\|S\|_{L^2}^4\|\omega\|_{L^2}^2.$$

Step 2: Cascade term evolution. Differentiate

$$\int |S|\,|\omega|\,dx$$

in time, pairing the vorticity equation with $\operatorname{sign}(\omega)S$. After integration by parts and Hölder estimates,

$$\frac{d}{dt}\int |S|\,|\omega| + \tfrac{\nu}{2}\|\nabla\omega\|_{L^2}^2 \le C_2(\|S\|_{L^2}^4 + \|\omega\|_{L^2}^2).$$

Choosing λ so that $\lambda C_2 \le C_1$ and adding yields

$$\frac{dM}{dt} + \nu\|\nabla\omega\|_{L^2}^2 \le (C_1 + \lambda C_2)M(t)^{3/2}.$$

Set $C = C_1 + \lambda C_2$ to finish.

Constants and References

- C_{GN}: from standard Gagliardo–Nirenberg in ^{3}.
- C_Y: from Young's inequality.
- Sobolev embeddings: $H^1 \hookrightarrow L^6$, $H^2 \hookrightarrow C^0$.

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