LAPLACIAN MINIMUM DOMINATING COLOR ENERGY OF A GRAPH

Abhilash H C[1],Shanmukha B[2] and Nithin J V[3]

[1] Research Scholar, PET Research Centre, Mandya-571401.
E-mail: abhilhc123@gmail.com.

[2] Department of Mathematics, P.E.S. College of Engineering, Mandya.
E-mail: shanmukhmaths@gmail.com

[3] Department of mathematics, PES college of science.Mandya.
E-mail:Nithinjvpatel@gmail.com

Abstract

In this paper, we introduced the concept of Laplacian minimum dominating color energy of a graph $LE_c^D(G)$ and compute the Laplacian minimum dominating color energy $LE_c^D(G)$ of few families of graphs. Further, the upper and lower bounds for $LE_c^D(G)$ are established.

Keywords : Minimum Dominating Set, Dominating Set, Color Energy, Minimum Dominating Color Energy, Laplacian Minimum Dominating Energy.
Mathematical Subject Classification : 05C50, 05C69.

1 Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges and let $A = (a_{i,j})$ be the adjacency matrix of the graph. The eigen values of graph G are the eigenvalues of its adjancey matrix $A(G)$, denoted by $\lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is equal to zero. For singular graphs, evidently, $detA = 0$. A graph is nonsingular if all its eigenvalues are different from zero. Then, $detA > 0$. A graph G is said to be k-regular graph if every vertex in G has degree k.
The concept of energy of a graph was introduced by I.Gutman [7]. Initially, the graph energy concept did not attract any noteworthy attention of mathematicians, but later they did realize its value and worldwide mathematical research of graph energy started. Nowadays, in connection with graph energy, energy like quantities were considered also for other matrices. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

I.Gutman and B.Zhou [8] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with n vertices and m edges. The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n. The elements of the Laplacian matrix are defined as

$$L_{ij} = \begin{cases} -1, & \text{if} \quad v_i \, and \, v_j \, are \, adjacent, \\ 0, & \text{if} \quad v_i \, and \, v_j \, are \, not \, adjacent, \\ d_i, & \text{if i=j.} \end{cases}$$

where d_i is the degree of the vertex v_i.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of Laplacian matrix G. Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|.$$

The basic properties Laplacian energy including various upper and lower bounds have been established in [13][14],[16],[17] and it has found that remarkable chemical application, high resolution satellite image classification and segmentation using Laplacian graph energy and finding semantic structures in image hierarchies using Laplacian graph energy.
Prof.Chandrashekara Adiga et al.[2] have defined color energy $E_c(G)$ of a graph G. P.Siva Kota Reddy et al.[9] have defined the Minimum Dominating Color Energy of a Graph. Motivated by these two papers, we introduced the concept of Laplacian minimum dominating color energy $E_c^D(G)$ of a graph G and computed Laplacian minimum dominating chromatic energies of some standard graphs are computed. Upper and lower bounds for $E_c^D(G)$ are also established.

2 Coloring and Color Energy

A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number and is denoted by $\chi(G)$ [11].

Consider the vertex colored graph. Then entries of the matrix $A_c(G)$ are as follows. If $c(v_i)$ is the color of v_i, then

$$a_{ij} = \begin{cases} 1, & \text{if } \; if\ v_i \ and\ v_j \ are\ adjacent\ with\ c(v_i) \neq c(v_j), \\ -1, & \text{if } \; if\ v_i \ and\ v_j \ are\ non-adjacent\ with\ c(v_i) = c(v_j), \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_c(G)$ is denoted by $f_n(G,\rho)$ and is defined by $f_n(G,\rho) = det(\rho I - A_c(G))$. The color eigenvalues of the graph G are the eigenvalues of $A_c(G)$. Since $A_c(G)$ ie real and symmetric, its eigen values are real numbers and are labelled in non-increasing order $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$. The color energy of G is defined as

$$E_c(G) = \sum_{i=1}^{n} |\rho_i|.$$

3 Minimum Dominating Color Energy of a Graph

Let G be a simple grpah of order n with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let D be the minimum dominating set of a graph G. The minimum dominating color matrix of G is the $n \times n$ matrix defined by $A_c^D(G) = (a_{ij})$. where

$$a_{ij} = \begin{cases} 1, & \text{if } \; if\ v_i \ and\ v_j \ are\ adjacent\ with\ c(v_i) \neq c(v_j)\ or\ if\ i = j\ and\, v_i \in D \\ -1, & \text{if } \; if\ v_i \ and\ v_j\ are\ non-adjacent\ with\ c(v_i) = c(v_j) \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_c^D(G)$ is denoted by $f_n(G,\lambda)$ and is defined by $f_n(G,\lambda) = det(\lambda I - A_c^D(G))$. The minimum dominating eigenvalues of the graph G are the eigenvalues of the matrix $A_c^D(G)$. We note that these eigenvalues are real numbers since $A_c^D(G)$ is real and symmetric. So, we

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

can label them in the non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The minimum dominating energy of G is defined to be the sum of the absolute eigenvalues of $A_c^D(G)$. In symbols, we write

$$E_c^D(G) = \sum_{i=1}^{n} |\lambda_i|.$$

If the color used is minimum then the energy is called minimum dominating chromatic energy and it is denoted by $E_\chi^D(G)$. For more details about Minimum Dominating Color Energy of a Graph refer [9].

4 The Laplacian Minimum Dominating Color Energy of a Graph

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then the Laplacian minimum dominating color matrix of G is denoted by $LE_c^D(G)$ and is defined as follows $LE_c^D(G) = D(G) - A_c^D(G)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values $LE_{pe}(G)$ arranged in non increasing order. These eigen values are called Laplacian minimum dominating color eigen values of G. The Laplacian minimum dominating energy of a graph G is defined as

$$LE_c^D(G) = \sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|,$$

where m is the number of edges of G and $\frac{2m}{n}$ is the average degree of G.
In this paper, we are interested in studying mathematical aspects of the Laplacian minimum dominating color energy of a graph. We are considering in this paper may have some applications in chemistry and computer science.

Example 4.1. *Let G be a graph on 6 vertices as shown in the Figure 1. The chromatic number of the graph G is 3, i.e., $\chi(G) = 3$. The possible minimum dominating sets are:*
(i)$D_1 = \{v_1, v_5\}$ ***(ii)***$D_2 = \{v_2, v_5\}$ ***(iii)***$D_3 = \{v_2, v_6\}$

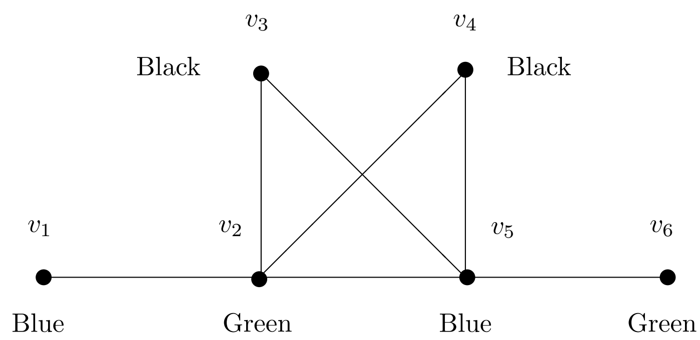

Figure 1: A Simple Graph with 6 Vertices

If the dominating set is $D_1 = \{v_1, v_5\}$, then

$$A_c^D D_1(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$LE_c^D(G) = D(G) - A_c^D D_1(G) = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 4 & -1 & -1 & -1 & 1 \\ 0 & -1 & 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 2 & -1 & 0 \\ 1 & -1 & -1 & -1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G, \lambda) = \lambda^6 - 12\lambda^5 + 45\lambda^4 - 53\lambda^3 + 9\lambda^2 + 17\lambda - 7 = 0$.
The Laplacian minimum dominating color eigen values are $\lambda_1 = -0.5508$, $\lambda_2 = 0.6377$, $\lambda_3 = 0.7823$, $\lambda_4 = 1$, $\lambda_5 = 4.6405$, $\lambda_6 = 5.4903$.
Average degree of the graph $= \frac{2m}{n} = \frac{2\times7}{6} = \frac{14}{6}$
Hence, Laplacian minimum dominating color energy, $LE_c^D(G) \approx 12.898$.
Note that the Laplacian minimum dominating color energy of the graph G depends on its minimum dominating set.

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

5 Laplacian Minimum Dominating Color Energy of Some Standard Graphs

Theorem 5.1. *For $n > 2$, the Laplacian minimum dominating color energy of a complete graph K_n is $(n-2)+\sqrt{n^2-2n+5}$.*

Proof. Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Clearly, D=$\{v_1\}$ is a minimum dominating set of K_n. Then

$$A_c^D(K_n) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 1 & 1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}.$$

and

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & n-1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & n-1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & n-1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & n-1 \end{pmatrix}.$$

$$LE_c^D(K_n) - A_c^D(K_n) = \begin{pmatrix} n-2 & -1 & -1 & \ldots & -1 & -1 \\ -1 & n-1 & 1 & \ldots & -1 & -1 \\ -1 & -1 & n-1 & \ldots & -1 & -1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ -1 & -1 & -1 & \ldots & n-1 & -1 \\ -1 & -1 & -1 & \ldots & -1 & n-1 \end{pmatrix}.$$

The characteristic polynomial of $LE_c^D(K_n)$ is given by,
$(\lambda-n)^{n-2}(\lambda^2-(n-1)\lambda-1) = 0$.
The Laplacian minimum dominating color eigen values are:
$\lambda = n$ [(n-2) times], $\lambda = \frac{(n-1)\pm\sqrt{n^2-2n+5}}{2}$ [one time each].
Average degree of K_n= $\frac{2m}{n} = 2\frac{\frac{n(n-1)}{2}}{n} = n-1$.
Hence, the Laplacian minimum dominating color energy of K_n is

$$LE_c^D(K_n) = |n-(n-1)|(n-2) + \left|\frac{(n-2)+\sqrt{n^2-2n+5}}{2} - (n-1)\right| + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - (n-1)\right|$$

$$LE_c^D(K_n) = (n-2) + \left|\frac{-n+1+\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{-n+1-\sqrt{n^2-2n+5}}{2}\right|$$

Therefore, $LE_c^D(K_n) = (n-2)+\sqrt{n^2-2n+5}$. □

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

Definition 5.1. *The cocktail party graph, denoted by $K_{n\times 2}$, is graph having vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq n\}$. This graph is also called as complete n-partite graph*

Theorem 5.2. *For $n \geq 2$, the Laplacian minimum dominating color energy of a cocktail party graph $K_{n\times 2}$ is $(4n-5)+\sqrt{4n^2-8n+21}$.*

Proof. Let $K_{n\times 2}$ be a cocktail party graph with the vertex set V. The set $D = \{v_1, v_{2n}\}$ is the minimum dominating set of cocktail party graph. Then,

$$A_c^D(K_{n\times 2}) = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & -1 \\ 1 & 0 & 1 & ... & -1 & 1 \\ 1 & 1 & 0 & ... & 1 & 1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & -1 & 1 & ... & 0 & 1 \\ -1 & 1 & 1 & ... & 1 & 1 \end{pmatrix}.$$

and

$$D(K_{n\times 2}) = \begin{pmatrix} 2(n-1) & 0 & 0 & ... & 0 & 0 \\ 0 & 2(n-1) & 0 & ... & 0 & 0 \\ 0 & 0 & 2(n-1) & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 2(n-1) & 0 \\ 0 & 0 & 0 & ... & 0 & 2(n-1) \end{pmatrix}.$$

$$LE_c^D(K_{n\times 2}) = D(K_{n\times 2}) - A_c^D(K_{n\times 2}) = \begin{pmatrix} 2n-3 & -1 & -1 & ... & -1 & -1 \\ -1 & 2n-2 & 0 & ... & 0 & 0 \\ -1 & 0 & 2n-2 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -1 & 0 & 0 & ... & 2n-2 & 0 \\ -1 & 0 & 0 & ... & 0 & 2n-3 \end{pmatrix}.$$

The characteristic polynomial of $LE_c^D(K_{n\times 2})$ is given by,
$f_n(K_{n\times 2}, \lambda) = (\lambda-(2n-4))(\lambda-(2n-3))^{n-1}(\lambda-(2n+1))^{n-2}(\lambda^2-(2n+1)\lambda+(3n+5))$
The Laplacian minimum dominating color eigen values are:
$\lambda = (2n-4)$ [1 time], $\lambda = (2n-3)$ [(n-1) time], $\lambda = (2n+1)$ [(n-2) time], $\lambda = \frac{(2n+1)\pm\sqrt{4n^2-8n+21}}{2}$ [one time each].
Average degree of $K_{n\times 2} = \frac{2(2n)(n-1)}{2n} = 2(n-1)$.

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

Hence, the Laplacian minimum dominating color energy is

$$\begin{aligned} LE_c^D(K_{n\times 2}) &= \Big|(2n-4)-2(n-1)\Big| + \Big|(2n-3)-2(n-1)\Big|(n-1) + \Big|(2n+1)-2(n-1)\Big|(n-2) \\ &+ \Big|\frac{(2n+1)+\sqrt{4n^2-8n+21}}{2} - 2(n-1)\Big| + \Big|\frac{(2n+1)+\sqrt{4n^2-8n+21}}{2} - 2(n-1)\Big| \\ &= 2+\mid -1 \mid (n-1) + 3(n-2) + \Big|\frac{(-2n+5)+\sqrt{4n^2-8n+21}}{2}\Big| \\ &+ \Big|\frac{(-2n+5)-\sqrt{4n^2-8n+21}}{2}\Big|. \\ &= (4n-5)+\sqrt{4n^2-8n+21} \end{aligned}$$

Therefore, $LE_c^D(K_{n\times 2}) = (4n-5)+\sqrt{4n^2-8n+21}$. □

Theorem 5.3. *For $n \geq 4$, the Laplacian minimum dominating color energy of a star graph is $\frac{2(n-1)(n-2)}{n} + \sqrt{4n-3}$.*

Proof. Let $K_{1,n-1}$ be a star graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ having the vertex v_1 at the center. The set $D = \{v_1\}$ is the minimum dominating set of star graph. Then,

$$A_c^D(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & -1 & \ldots & -1 & -1 \\ 1 & -1 & 0 & \ldots & -1 & -1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & -1 & -1 & \ldots & 0 & -1 \\ 1 & -1 & -1 & \ldots & -1 & 0 \end{pmatrix}.$$

and

$$D(K_{1,n-1}) = \begin{pmatrix} n-1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

$$LE_c^D(K_{1,n-1}) = D(K_{1,n-1}) - A_c^D(K_{1,n-1}) = \begin{pmatrix} n-2 & -1 & -1 & \ldots & -1 & -1 \\ -1 & 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & 1 & \ldots & 1 & 1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ -1 & 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}.$$

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

The characteristic polynomial of $LE_c^D(K_{1,n-1})$ is given by,
$f_n(K_{1,n-1},\lambda) = (\lambda)^{n-2}(\lambda^2 - (2n-3)\lambda + (n^2-4n+3))$
The Laplacian minimum dominating color eigen values are:
$\lambda = 0$ [(n-2)times], $\lambda = \frac{(2n-3)\pm\sqrt{4n-3}}{2}$ [one time each].
Average degree of $K_{1,n-1} = \frac{2(n-1)}{n}$.
Hence, the Laplacian minimum dominating color energy is

$$\begin{aligned} LE_c^D(K_{1,n-1}) =& \left|0 - \frac{2(n-1)}{n}\right|(n-2) + \left|\frac{(2n-3)+\sqrt{4n-3}}{2} - \frac{2(n-1)}{n}\right| \\ &+ \left|\frac{(2n-3)-\sqrt{4n-3}}{2} - \frac{2(n-1)}{n}\right| \\ =& \left|\frac{2(n-1)(n-2)}{n}\right| + \left|\frac{(2n^2-7n+4)+\sqrt{4n-3}}{2}\right| + \left|\frac{(2n^2-7n+4)-\sqrt{4n-3}}{2}\right|. \\ =& \frac{2(n-1)(n-2)}{n} + \sqrt{4n-3}. \end{aligned}$$

Therefore, $LE_c^D(K_{1,n-1}) = \frac{2(n-1)(n-2)}{n} + \sqrt{4n-3}$. □

Definition 5.2. *The friendship graph, denoted by $F_3^{(n)}$, is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.*

Theorem 5.4. *For $n \geq 2$, the Laplacian minimum dominating color energy of a friendship graph $F_3^{(n)}$ is at most $\frac{6n^2-3n}{(2n+1)} + \sqrt{n^2+6n+1}$.*

Proof. Let $F_3^{(n)}$ be a friendship graph with the vertex set $V(F_3^{(n)}) = \{v_0, v_1, v_2, ..., v_n\}$. The minimum dominating set is $D = \{v_0\}$. Then

$$A_c^D(F_3^{(n)}) = \begin{pmatrix} 0 & 1 & -1 & ... & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 1 \\ -1 & 1 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & -1 \\ 1 & 1 & 0 & ... & -1 & 0 \end{pmatrix}.$$

and

$$D(F_3^{(n)}) = \begin{pmatrix} 2 & 0 & 0 & ... & 0 & 0 \\ 0 & 2n & 0 & ... & 0 & 0 \\ 0 & 0 & 2 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 2 & 0 \\ 0 & 0 & 0 & ... & 0 & 2 \end{pmatrix}.$$

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

$$LE_c^D(F_3^{(n)}) = D(F_3^{(n)}) - A_c^D(F_3^{(n)}) = \begin{pmatrix} 2 & -1 & 1 & ... & 0 & -1 \\ -1 & 2n-1 & -1 & ... & 0 & 0 \\ 1 & -1 & 2 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & -1 & 0 & ... & 2 & 1 \\ -1 & 0 & 0 & ... & 1 & 2 \end{pmatrix}.$$

The characteristic polynomial of $LE_c^D(F_3^{(n)})$ is given by

$$f_n(F_3^{(n)}, \lambda) = (\lambda - 0)^{(n-1)}(\lambda - 2)^{(n-1)}(\lambda - (n+2))(\lambda^2 - (3n-1)\lambda + (2n^2 - 3n))$$

The Laplacian minimum dominating color eigen values are:
$\lambda = 0$, [(n-1)times], $\lambda = 2$, [(n-1)times], $\lambda = (n+2)$, [1 time],$\lambda = \frac{(3n)\pm\sqrt{n^2+6n+1}}{2}$ [1 time].
Average degree of $LE_c^D(F_3^{(n)}) = \frac{2(3n)}{2n+1} = \frac{6n}{2n+1}$.
Hence, the Laplacian minimum dominating color energy is
$LE_c^D F_3^{(n)} = \left|0 - \frac{6n}{2n+1}\right|(n-1) + \left|2 - \frac{6n}{2n+1}\right|(n-1) + \left|(n+2) - \frac{6n}{2n+1}\right| + \left|\frac{(3n-1)+\sqrt{n^2+6n+1}}{2} - \frac{6n}{2n+1}\right| + \left|\frac{(3n-1)-\sqrt{n^2+6n+1}}{2} - \frac{6n}{2n+1}\right|$

$LE_c^D(F_3^{(n)}) \leq \frac{6n^2-3n}{(2n+1)} + \sqrt{n^2+6n+1}$.

Therefore, Laplacian minimum dominating color energy of a friendship graph $F_3^{(n)}$ is at most $\frac{6n^2-3n}{(2n+1)} + \sqrt{n^2+6n+1}$. □

6 Properties of Laplacian Minimum Dominating Color Eigen Values of a Graph

Theorem 6.1. *If D is a minimum dominating set of a graph G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of $LE_c^D(G)$ then*

(i) $\sum_{i=1}^{n} \lambda_i = 2m- \mid D \mid$.

(ii) $\sum_{i=1}^{n} \lambda_i^2 = 2(m + m_c) + \sum_{i=1}^{n}(d_i - c_i)^2$ *where* $c_i = \begin{cases} 1, & if v_i \in D, \\ 0, & if v_i \notin D. \end{cases}$

and m_c is the number of pairs of non-adjacent vertices receiving the same color in G.

Proof. **(i)** By definition, the sum of the principal diagonal elements of $LE_c^D(G)$ is equal to $\sum_{i=1}^{n} d_i- \mid D \mid= 2m- \mid D \mid$.
Also the sum of eigen values of $LE_c^D(G)$ is trace of $LE_c^D(G)$.
(ii) The sum of squares of eigen values of $LE_c^D(G)$ is the trace of $LE_c^D(G)^2$

Therefore $\sum_{i=1}^{n}\lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ji} = \sum_{i=1}^{n}(l_{ij})^2 + \sum_{j=1}^{n}(l_{ji})^2$

$$= 2\sum_{i<j}(l_{ij})^2 + \sum_{i=1}^{n}(l_{ji})^2$$

$$= 2(m+m_c) + \sum_{i=1}^{n}(d_i - c_i)^2 \ \textit{where}\ c_i = \begin{cases} 1, & \text{if } v_i \in D \\ 0, & \text{if } v_i \notin D \end{cases}$$

$$= 2N,\ \textit{where}\ \ N = (m+m_c) + \frac{1}{2}\sum_{i=1}^{n}(d_i - c_i)^2$$

□

7 Upper and Lower Bounds

Theorem 7.1. *If G be a graph with n vertices, m edges and D is a minimum dominating set of a graph G. Then $LE_c^D(G) \leq \sqrt{2Nn} + 2m$.*

Proof. Let G be a graph with n vertices and m edges and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of G. By using Cauchy's - Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Put $a_i = 1$, $b_i = \lambda_i$ then,

$$\left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right)$$

$$\left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \leq n\ 2N$$

$$\therefore\ \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right) \leq \sqrt{2Nn}$$

By Triangle inequality $\left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \left|\frac{2m}{n}\right|\ \ \forall i = 1, 2, \ldots, n$

$$\textit{i.e.},\ \left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \frac{2m}{n}\ \ \forall\, i$$

Zhuzao/Foundry[ISSN:1001-4977] VOLUME 28 ISSUE 3

$$\begin{aligned}\Big(\sum_{i=1}^{n}\Big|\lambda_i - \frac{2m}{n}\Big|\Big) &\leq \Big(\sum_{i=1}^{n}\lambda_i\Big) + \Big(\sum_{i=1}^{n}\frac{2m}{n}\Big)\\ &\leq \sqrt{2Nn} + 2m\\ \therefore LE_c^D(G) &\leq \sqrt{2Nn} + 2m\end{aligned}$$

□

Theorem 7.2. *Let G be a graph with n vertices and m edges and D be a minimum dominating set of G. Then $LE_c^D \leq \sqrt{2Nn + 4m(\mid D \mid - m)}$*

Proof. By using Cauchy's - Schwarz inequality

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big)$$

Put $a_i = 1$, $b_i = \Big|\lambda_i - \frac{2m}{n}\Big|$ then,

$$\Big(\sum_{i=1}^{n}\Big|\lambda_i - \frac{2m}{n}\Big|\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n}\Big|\lambda_i - \frac{2m}{n}\Big|^2\Big).$$

$$\begin{aligned} i.e., [LE_c^D(G)]^2 &= n\Big[\sum_{i=1}^{n}\lambda_i^2 + \sum_{i=1}^{n}\frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n}\lambda_i\Big]\\ &= n\Big[2N + \frac{4m^2}{n^2}.n - \frac{4m}{n}(2m - \mid D \mid)\Big]\\ &= n\Big[2N + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m \mid D \mid}{n}\Big]\\ &= 2Nn + 4m(\mid D \mid - m)\\ \therefore\ LE_c^D(G) &\leq \sqrt{2Nn + 4m(\mid D \mid - m)}\end{aligned}$$

□

Theorem 7.3. *Let G be a graph with n vertices and m edges and D is a minimum dominating set of G. If $D = \mid detLE_c^D(G) \mid$ then*
$LE_c^D(G) \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m.$

Proof. Consider

$$\begin{aligned}\Big[\sum_{i=1}^{n}\mid \lambda_i \mid\Big]^2 &= \Big(\sum_{i=1}^{n}\mid \lambda_i \mid\Big).\Big(\sum_{j=1}^{n}\mid \lambda_j \mid\Big)\\ &= \sum_{i=1}^{n}\mid \lambda_i \mid^2 + \sum_{i\neq j}\mid \lambda_i \mid\mid \lambda_j \mid\end{aligned}$$

$$\therefore \sum_{i \neq j} |\lambda_i||\lambda_j| = \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 - \sum_{i=1}^{n} |\lambda_i|^2 \tag{7.1}$$

Applying Arithmtic and Geometric means for $n(n-1)$ terms, we have

$$\frac{\sum_{i \neq j} |\lambda_i||\lambda_j|}{n(n-1)} \geq \Big[\prod_{i \neq j} |\lambda_i||\lambda_j|\Big]^{\frac{1}{n(n-1)}}$$

$$i.e., \sum_{i \neq j} |\lambda_i||\lambda_j| \geq n(n-1)\Big[\prod_{i \neq j} |\lambda_i||\lambda_j|\Big]^{\frac{1}{n(n-1)}}$$

Using (7.1) we get,

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 - \sum_{i=1}^{n} |\lambda_i|^2 \geq n(n-1)\Big[\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 - 2N \geq n(n-1)\Big[\prod_{i=1}^{n} |\lambda_i|\Big]^{\frac{2}{n}}$$

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 \geq 2N + n(n-1)\Big[\prod_{i=1}^{n} |\lambda_i|\Big]^{\frac{2}{n}}$$

$$\therefore \sum_{i=1}^{n} |\lambda_i| \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} \tag{7.2}$$

We know that

$$|\lambda_i| - \Big|\frac{2m}{n}\Big| \leq \Big|\lambda_i - \frac{2m}{n}\Big| \ \forall i$$

$$\sum_{i=1}^{n} |\lambda_i| - \sum_{i=1}^{n} \frac{2m}{n} \leq \sum_{i=1}^{n} \Big|\lambda_i - \frac{2m}{n}\Big|$$

$$i.e., \sum_{i=1}^{n} \Big|\lambda_i\Big| - 2m \leq LE_c^D(G)$$

$$i.e., LE_c^D(G) \geq \sum_{i=1}^{n} \Big|\lambda_i\Big| - 2m$$

$$\geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m \quad (from 7.2)$$

$$\therefore LE_c^D(G) \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m$$

□

References

[1] C.Adiga, A.Bayad, I.Gutman, S.A.Srinivas, *The minimum covering energy of a graph*, Kragujevac J. Sci. 34 (2012) 39-56.

[2] C. Adiga, E. Sampathkumar and M. A. Sriraj, *Color energy of a graph*, Proc.Jangjeon Math.Soc., 16(3)(2013), 335-351.

[3] C. Adiga and C. S. Shivakumaraswamy, *Bounds on the largest minimum degree eigenvalues of graphs*, Int.Math.Forum., 5(37)(2010), 1823-1831.

[4] Akbar Jahanbani, *Some new lower bounds for energy of graphs*, Applied Mathematics and Computation. 296(2017) 233-238.

[5] R.B. Bapat, S.Pati, *Energy of a graph is never an odd integer*. Bull. Kerala Math. Assoc. 1 (2011) 129 - 132.

[6] C. A. Coulson, *On the calculation of the energy in unsaturated hydrocarbons molecules*. Proc.Cambridge.Phil.Soc, 36(1940), 201-203.

[7] I.Gutman, *The energy of a graph*. Ber. Math-Statist. Sekt. Forschungsz.Graz 103 (1978) 1-22.

[8] I.Gutman, B.Zhou *Laplacian energy of a graph*. Lin. Algebra Appl. 414, 29-37(2006).

[9] P.Siva Kota Reddy, K.N.Prakasha and Gavirangaiah K, *Minimum Dominating Color Energy of a Graph*, International J.Math. Combin. Vol.3(2017), 22-31.

[10] M.R.Rajesh Kanna, B.N. Dharmendra, G.Sridhara, *Laplacian Minimum Dominating Energy of a Graph*, IJPAM Vol. 89 No.4 (2013), 565-518.

[11] E. Sampathkumar and M. A. Sriraj, *Vertex labeled/colored graphs, matrices and signed graphs*, J. of Combinatorics, Information and System Sciences, to appear.

[12] H.Wang, H. Hua, *Note on Laplacian energy of graphs*. MATCH Commun. Math. Comput. Chem., 59(2008), 373-380. [28]

[13] B. Zhou, *New upper bounds for Laplacian energy*. MATCH Commun. Math. Comput.Chem., 62(2009), 553-560.

[14] B. Zhou, I. Gutman, *Nordhaus-Gaddum-type relations for the energy and Laplacian energy of graphs*. Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.) 134(2007), 1-11.

[15] B. Zhou, *On the sum of powers of the Laplacian eigenvalues of graphs*. Lin. Algebra Appl., 429(2008), 2239-2246.

[16] B. Zhou, I. Gutman, T. Aleksic, *A note on Laplacian energy of graphs*. MATCH Commun. Math. Comput. Chem., 60(2008), 441-446.

[17] B. Zhou, I. Gutman, *On Laplacian energy of graphs.* MATCH Commun. Math. Comput.