

About Atiyah and MacDonald's Book

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1 Introduction

Atiyah and MacDonald's book referred to in the title is of course the famous **Introduction to Commutative Algebra**.

Here are some links to texts related to this book:

- Errata <https://mathoverflow.net/q/42241/461>
- Allen Altman and Steven Kleiman, A term of Commutative Algebra, https://www.researchgate.net/publication/325591008_A_term_of_Commutative_Algebra
See also <https://mathoverflow.net/a/385313/461>
- Jeffrey Daniel Kasik Carlson, Exercises to Atiyah and MacDonald's Introduction to Commutative Algebra, <https://spaces.ac.cn/usr/uploads/2017/07/4208763092.pdf>
- Thomas Lam and Dustin Clausen
http://abel.math.harvard.edu/archive/221_spring_08/Math221.html
- Brent R. Doran
https://www2.math.ethz.ch/education/bachelor/lectures/hs2014/math/comm_alg.html
- Thomas J. Haines, Lectures on Commutative Algebra
<http://www.math.umd.edu/~tjh/CommAlg.pdf>
- Sarah Glaz <https://www2.math.uconn.edu/~glaz/math5020f14/>
- Athanasios Papaioannou, Solutions to Atiyah and MacDonald's Introduction to Commutative Algebra <https://tinyurl.com/r3y453b>
- Shengtian Yang, http://arxiv.codlab.net/book/note-am-ica/note-am-ica_0.1.2.pdf
- Dave Karpuk
https://mycourses.aalto.fi/pluginfile.php/426996/mod_resource/content/1/chap1solutions.pdf
- Joshua Ruiter <https://tinyurl.com/3hj9btp3>
- J. David Taylor
https://www.math.arizona.edu/~jtaylor/notes/atiyah_macdonald_solutions.pdf

2 About Chapter 1

2.1 Comments

2.1.1 Page 6

Proposition 1.9 will be used several times below. Here is the statement:

Proposition 1 (Proposition 1.9). $x \in \mathfrak{R} \iff 1 - xy$ is a unit in A for all $y \in A$.

[Recall that \mathfrak{R} is the Jacobson radical of A .]

Note 2. *The intersection and the product of the empty family of ideals is the unit ideal.*

It is written

In the ring \mathbb{Z} , \cap and $+$ are distributive over each other. This is not the case in general.

Here is an example: In the ring $K[x, y]$, where K is a field and x and y are indeterminates, we have

$$(x + y) \cap ((x) + (y)) \not\subset ((x + y) \cap (x)) + ((x + y) \cap (y)).$$

2.1.2 Page 7, Proposition 1.10

Chinese Remainder Theorem. *Let A be a commutative ring and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals such that $\mathfrak{a}_i + \mathfrak{a}_j = A$ for $i \neq j$. Then the natural morphism from A to the product of the A/\mathfrak{a}_i is surjective. Moreover the intersection of the \mathfrak{a}_i coincides with their product.*

Proof. We have

$$A = \mathfrak{a}_1 + \mathfrak{a}_2 \cdots \mathfrak{a}_n. \tag{1}$$

Indeed, this can be checked either by multiplying together the equalities $A = \mathfrak{a}_1 + \mathfrak{a}_i$ for $i = 2, \dots, n$, or by noting that a prime ideal containing a product of ideals contains one of the factors. Then (1) implies the existence of an a_1 in A such that $a_1 \equiv 1 \pmod{\mathfrak{a}_1}$ and $a_1 \equiv 0 \pmod{\mathfrak{a}_i}$ for all $i > 1$. Similarly we can find elements a_i in A such that $a_i \equiv \delta_{ij} \pmod{\mathfrak{a}_j}$ (Kronecker delta). This proves the first claim.

Let \mathfrak{a} be the intersection of the \mathfrak{a}_i . Multiplying (1) by \mathfrak{a} we get

$$\mathfrak{a} = \mathfrak{a}_1 \mathfrak{a} + \mathfrak{a} \mathfrak{a}_2 \cdots \mathfrak{a}_n \subset \mathfrak{a}_1 (\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n) \subset \mathfrak{a}.$$

This gives the second claim, directly for $n = 2$, by induction for $n > 2$. □

2.1.3 Page 8

• Proposition 1.11i, Prime Avoidance

The following is taken from Wikipedia:

https://en.wikipedia.org/wiki/Prime_avoidance_lemma

Let A be a commutative ring and S a multiplicatively closed additive subgroup of A . Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, $n \geq 1$ be ideals such that \mathfrak{a}_i is prime for $i \geq 3$. If S is not contained in any \mathfrak{a}_i , then S is not contained in their union.

Proof. We argue by induction on n . It suffices to find an element s that is in S and not in \mathfrak{a}_i for any i .

1. The case $n = 1$ is trivial.

2. Suppose $n \geq 2$. For each i choose s_i in $S \setminus \bigcup_{j \neq i} \mathfrak{a}_j$, this set being nonempty by inductive hypothesis. We can assume $s_i \in \mathfrak{a}_i$ for all i ; otherwise, some s_i avoids all the \mathfrak{a}_j 's and we are done.

Claim: the element $s := s_1 \cdots s_{n-1} + s_n$ is in S but not in \mathfrak{a}_i for any i .

2.1. If s is in \mathfrak{a}_i for some $i \leq n-1$, then s_n is in \mathfrak{a}_i , contradiction.

2.2. If s is in \mathfrak{a}_n , then $s_1 \cdots s_{n-1}$ is in \mathfrak{a}_n .

2.2.1. If n is 2, we get $s_1 \in \mathfrak{a}_2$, contradiction.

2.2.2. If $n > 2$ then, since \mathfrak{a}_n is prime, there is an i less than n such that s_i is in \mathfrak{a}_n , contradiction.

This proves the claim, and thus the statement. \square

Here is a version which is slightly weaker but sufficient for our purpose:

Proposition 3 (Proposition 1.11i p. 8 of the book). *Let A be a ring, let \mathfrak{a} be an ideal, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, $n \geq 1$ be prime ideals. If \mathfrak{a} is not contained in any \mathfrak{p}_i , then \mathfrak{a} is not contained in their union.*

Proof. We can assume that $n \geq 2$ and that there are elements a_1, \dots, a_n in \mathfrak{a} such that

$$a_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j.$$

Then $a := a_1 \cdots a_{n-1} + a_n$ is in \mathfrak{a} but not in \mathfrak{p}_i for any i . \square

• Proposition 1.11ii

Note that \mathfrak{p} is prime \iff

$$\mathfrak{p} \supset \mathfrak{a}\mathfrak{b} \implies \mathfrak{p} \supset \mathfrak{a} \text{ or } \mathfrak{p} \supset \mathfrak{b}.$$

Recall Proposition 1.11ii:

Proposition 4 (Proposition 1.11ii p. 8 of the book). *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals and let \mathfrak{p} be a prime ideal containing $\bigcap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{p} \supset \mathfrak{a}_i$ for some i . If $\mathfrak{p} = \bigcap \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i .*

• Quotient ideal

We have

$$\mathfrak{a} \subset \mathfrak{a}', \mathfrak{b}' \subset \mathfrak{b} \implies (\mathfrak{a} : \mathfrak{b}) \subset (\mathfrak{a}' : \mathfrak{b}'),$$

$$(1 : \mathfrak{b}) = (1) = (\mathfrak{a} : 0) \text{ and } (\mathfrak{a} : 1) = \mathfrak{a}.$$

2.1.4 Page 8, Exercise 1.12

Statement. (i) $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$

(ii) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subset \mathfrak{a}$

(iii) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$

(iv) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$

(v) $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap_i (\mathfrak{a} : \mathfrak{b}_i)$.

Solution. Left to the reader.

Comments.

(a) The inclusion $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$ in (i) is strict in general. To see this let A be nonzero and set $\mathfrak{a} = \mathfrak{b} = (0)$.

(b) The inclusion $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subset (\mathfrak{a} : \mathfrak{b})$ in (ii) is strict in general. To see this let A be nonzero and set $\mathfrak{a} = (1)$, $\mathfrak{b} = (0)$.

2.1.5 Page 9, Exercise 1.13

Statement. (i) $r(\mathfrak{a}) \supset \mathfrak{a}$

(ii) $r(r(\mathfrak{a})) = r(\mathfrak{a})$

(iii) $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$

(iv) $r(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$

(v) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$

(vi) if \mathfrak{p} is prime, $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.

Solution. Left to the reader.

2.1.6 Page 9, extended ideal

If $f : A \rightarrow B$ is a morphism of rings and \mathfrak{a} is an ideal of A , then we can define \mathfrak{a}^e by the formula

$$\mathfrak{a}^e := \sum_{\alpha \in \mathfrak{a}} (f(\alpha)).$$

2.1.7 Page 10, Exercise 1.18

Statement.

$$\begin{aligned}
 (\mathfrak{a}_1 + \mathfrak{a}_2)^e &= \mathfrak{a}_1^e + \mathfrak{a}_2^e, & (\mathfrak{b}_1 + \mathfrak{b}_2)^c &\supset \mathfrak{b}_1^c + \mathfrak{b}_2^c, \\
 (\mathfrak{a}_1 \cap \mathfrak{a}_2)^e &\subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e, & (\mathfrak{b}_1 \cap \mathfrak{b}_2)^c &= \mathfrak{b}_1^c \cap \mathfrak{b}_2^c, \\
 (\mathfrak{a}_1 \mathfrak{a}_2)^e &= \mathfrak{a}_1^e \mathfrak{a}_2^e, & (\mathfrak{b}_1 \mathfrak{b}_2)^c &\supset \mathfrak{b}_1^c \mathfrak{b}_2^c, \\
 (\mathfrak{a}_1 : \mathfrak{a}_2)^e &\subset (\mathfrak{a}_1^e : \mathfrak{a}_2^e), & (\mathfrak{b}_1 : \mathfrak{b}_2)^c &\subset (\mathfrak{b}_1^c : \mathfrak{b}_2^c), \\
 r(\mathfrak{a})^e &\subset r(\mathfrak{a}^e), & r(\mathfrak{b})^c &= r(\mathfrak{b}^c).
 \end{aligned}$$

The set of ideals E is closed under sum and product, and C is closed under the other three operations.

Hint. Let us prove that C is closed under quotients [the other claims being left to the reader]. For $\mathfrak{a}, \mathfrak{b} \in C$ we have

$$(\mathfrak{a} : \mathfrak{b}) \subset (\mathfrak{a} : \mathfrak{b})^{ec} \subset (\mathfrak{a}^e : \mathfrak{b}^e)^c \subset (\mathfrak{a}^{ec} : \mathfrak{b}^{ec}) = (\mathfrak{a} : \mathfrak{b}).$$

Indeed, the first inclusion follows from Proposition 1.17 p. 10, whereas the second and third inclusions follow from the first part of the exercise.

Comments.

(a) Let K be a field and let X and Y be indeterminates. If \mathfrak{a} is an ideal of $K[X, Y]$, we denote by x and y the images of X and Y in $K[X, Y]/\mathfrak{a}$.

- Let us show that the inclusion $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$ is strict in general. Set [with obvious notation]

$$A := K[X, Y]/(X^2, XY, Y^2), \quad B := K[X]/(X^2), \quad f : A \rightarrow B, \quad f(x) := x, \quad f(y) := 0,$$

$$\mathfrak{a}_1 := (x), \quad \mathfrak{a}_2 := (x - y).$$

- Let us show that the inclusion $\mathfrak{b}_1^c + \mathfrak{b}_2^c \subset (\mathfrak{b}_1 + \mathfrak{b}_2)^c$ is strict in general. Set [with obvious notation]

$$A := K[X]/(X^2), \quad B := K[X, Y]/(X^2, XY, Y^2),$$

$$f : A \rightarrow B, \quad f(x) := x, \quad \mathfrak{b}_1 := (y), \quad \mathfrak{b}_2 := (x - y).$$

- Let us show that the inclusion $\mathfrak{b}_1^c \mathfrak{b}_2^c \subset (\mathfrak{b}_1 \mathfrak{b}_2)^c$ is strict in general. Set $A := K[XY]$, $B := K[X, Y]$, and let $f : A \rightarrow B$ be the inclusion. Then we have $(X)^c(Y)^c = (X^2Y^2)$ and $(XY)^c = (XY)$.

- Let us show that the inclusion $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subset (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$ is strict in general. Set $A := K[X]$, $B := K[X, Y]/(XY)$, $f(X) := x$. Then we have

$$y \in ((0)^e : (X)^e) = ((0) : (X)^e), \quad y \notin (0) = (0)^e = ((0) : (X))^e.$$

- Let us show that the inclusion $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subset (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$ is strict in general. Set $A := K$, $B := K[X]$, $\mathfrak{b}_1 := (0)$, $\mathfrak{b}_2 := (X)$. Then we have $1 \in (\mathfrak{b}_1^c : \mathfrak{b}_2^c) \setminus (\mathfrak{b}_1 : \mathfrak{b}_2)^c$.

- Let us show that the inclusion $r(\mathfrak{a})^e \subset r(\mathfrak{a}^e)$ is strict in general. Set $A := K[X^2]$, $B := K[X]$, $\mathfrak{a} := (X^2)$. Then we have $X \in r(\mathfrak{a}^e) \setminus r(\mathfrak{a})^e$.

(b) For arbitrary sums and intersections we have

$$\left(\sum \mathfrak{a}_i\right)^e = \sum \mathfrak{a}_i^e, \quad \left(\sum \mathfrak{b}_i\right)^c \supset \sum \mathfrak{b}_i^c,$$

$$\left(\bigcap \mathfrak{a}_i\right)^e \subset \bigcap \mathfrak{a}_i^e, \quad \left(\bigcap \mathfrak{b}_i\right)^c = \bigcap \mathfrak{b}_i^c.$$

We prove the first of these four statements, leaving the others to the reader. We have

$$\left(\sum \mathfrak{a}_i\right)^e = \sum_{\alpha \in \sum \mathfrak{a}_i} (f(\alpha)), \quad \sum \mathfrak{a}_i^e = \sum_i \sum_{\alpha_i \in \mathfrak{a}_i} (f(\alpha_i)).$$

The inclusion \supset is clear. Let us prove the inclusion \subset . Let α be in $\sum \mathfrak{a}_i$. This means that α is of the form $\sum \alpha_i$ with $\alpha_i \in \mathfrak{a}_i$, $\alpha_i = 0$ for almost all i . This yields

$$f(\alpha) = \sum_i f(\alpha_i) \in \sum_i \sum_{\alpha_i \in \mathfrak{a}_i} (f(\alpha_i)) = \sum \mathfrak{a}_i^e.$$

2.2 Exercises

2.2.1 Page 10, Exercise 1.1

Statement. Let x be a nilpotent element of a ring A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.

Solution. If x is nilpotent, then $\sum_{n \geq 0} x^n$ is the inverse of $1 - x$. If u is a unit and x is nilpotent, then $u + x = u(1 + \frac{x}{u})$ is a unit.

2.2.2 Page 11, Exercise 1.2

Statement. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x , with coefficients in A . Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

(i) f is a unit in $A[x] \iff a_0$ is a unit in A and a_1, \dots, a_n are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f , prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]

(ii) f is nilpotent $\iff a_0, a_1, \dots, a_n$ are nilpotent.

(iii) f is a zero-divisor \iff there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ ($0 \leq r \leq n$).]

(iv) f is said to be primitive if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\iff f$ and g are primitive.

Solution. This will follow from Exercise 1.3.

2.2.3 Page 11, Exercise 1.3

Statement. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \dots, x_r]$ in several indeterminates.

Solution.

- (i) The condition “ a_0 is a unit in A and a_1, \dots, a_n are nilpotent” becomes “the constant term is a unit in A and the coefficients of the nonconstant monomials are nilpotent”.

If the constant term is a unit in A and the coefficients of the nonconstant monomials are nilpotent, then f is a unit by Exercise 1.1. Assume f is a unit in $A[x_1, \dots, x_r]$ and let \mathfrak{p} be a prime ideal of A . Then the image of f in $A/\mathfrak{p}[x_1, \dots, x_r]$ is again a unit, that is the constant term is a unit in A/\mathfrak{p} , and the coefficients of the nonconstant monomials are in \mathfrak{p} . Since \mathfrak{p} is arbitrary, this shows that these coefficients are nilpotent.

- (ii) The condition “ a_0, a_1, \dots, a_n are nilpotent” becomes “all the coefficients of f are nilpotent”.

If the coefficients of f are nilpotent, then f is nilpotent by Exercise 1.1. Assume f is nilpotent and let \mathfrak{p} be a prime ideal of A . Then the image of f in $A/\mathfrak{p}[x_1, \dots, x_r]$ is again nilpotent, that is zero, and the coefficients of f are in \mathfrak{p} . Since \mathfrak{p} is arbitrary, this shows that these coefficients are nilpotent.

- (iii) We will give two solutions to Part (iii), and these solutions will be spelled out in Section 2.2.4 below.

- (iv) Let $f, g \in A[x_1, \dots, x_r]$. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be respectively the ideals generated by the coefficients of f, g, fg . We must show $\mathfrak{c} = (1) \iff \mathfrak{a} = (1) = \mathfrak{b}$. Since we have $\mathfrak{c} \subset \mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, it suffices to prove $\mathfrak{a} = (1) = \mathfrak{b} \implies \mathfrak{c} = (1)$, or equivalently $\mathfrak{c} \neq (1) \implies (\mathfrak{a} \neq (1) \text{ or } \mathfrak{b} \neq (1))$. If $\mathfrak{c} \neq (1)$, then $\mathfrak{c} \subset \mathfrak{m}$ for some maximal ideal \mathfrak{m} , and the images \bar{f} and \bar{g} of f and g in $(A/\mathfrak{m})[x_1, \dots, x_r]$ satisfy $\bar{f}\bar{g} = 0$, so that we get $\bar{f} = 0$ or $\bar{g} = 0$, which implies $\mathfrak{a} \neq (1)$ or $\mathfrak{b} \neq (1)$.

2.2.4 Exercise 1.3iii

First solution. In the three statements below, A and B are rings, p, q, r, s are nonnegative integers satisfying $q > 0$ and $r < s$, and the x_i and y_j are indeterminates.

Recall the statement of Exercise 1.3iii:

(\star) If $f \in A[x_1, \dots, x_p]$ has a nonzero annihilator in $A[x_1, \dots, x_p]$, then f has a nonzero annihilator in A .

Lemma 5. *If $f \in B[y_1, \dots, y_q]$ has a nonzero annihilator in $B[y_1]$, then f has a nonzero annihilator in B .*

Lemma 6. *If $f \in A[x_1, \dots, x_s]$ has a nonzero annihilator in $A[x_1, \dots, x_{r+1}]$, then f has a nonzero annihilator in $A[x_1, \dots, x_r]$.*

Clearly Lemma 6 implies (\star). Moreover, Lemma 6 follows from Lemma 5 by setting $B = A[x_1, \dots, x_r]$, $q = s - r$, $y_i = x_{r+i}$. Thus it only remains to prove Lemma 5. To do so set

$$f = a_0 + \dots + a_n y_1^n,$$

where n is the y_1 -degree of f [we can assume $f \neq 0$] and $a_i \in B[y_2, \dots, y_q]$. Let

$$g = b_0 + \dots + b_m y_1^m \in B[y_1]$$

be a nonzero polynomial of least degree m such that

$$fg = 0. \quad (2)$$

It suffices to prove

$$m = 0. \quad (3)$$

It even suffices to prove

$$a_i g = 0 \quad \forall i. \quad (4)$$

Assume (4) is false, and let i be the largest index satisfying $a_i g \neq 0$. Then (2) implies $a_i b_m = 0$ [because $a_i b_m \neq 0$ would imply $\deg_{y_1}(fg) = i + m$, where \deg_{y_1} is the y_1 -degree, but we have $fg = 0$] and thus $\deg_{y_1}(a_i g) < m$. As $f \cdot (a_i g) = 0$, this entails $a_i g = 0$, contradiction. This proves (4) and thus (3), completing the proof of Lemma 5.

Second solution. Let $f \in A[x_1, \dots, x_n]$ be a nonzero zero-divisor. We must show that there is a nonzero b in A such that $bf = 0$. We equip the monomials with the graded lexicographic order, as defined in https://en.wikipedia.org/wiki/Monomial_order.

We denote the multi-degree of $g \in A[x_1, \dots, x_n]$ by $\deg g$. Put

$$f = a_0 + \dots + a_\alpha x^\alpha,$$

where α is the multi-degree of f . Let

$$g = b_0 + \dots + b_\beta x^\beta \in A[x_1, \dots, x_n]$$

be a nonzero polynomial of least multi-degree β such that

$$fg = 0. \quad (5)$$

It suffices to prove

$$\beta = 0. \quad (6)$$

It even suffices to prove

$$a_\gamma g = 0 \quad \forall \gamma. \quad (7)$$

Assume (7) is false, and let γ be the largest multi-index satisfying $a_\gamma g \neq 0$. Then (5) implies $a_\gamma b_\beta = 0$ [because $a_\gamma b_\beta \neq 0$ would imply $\deg(fg) = \gamma + \beta$, but we have $fg = 0$] and thus $\deg a_\gamma g < \beta$. As $f \cdot (a_\gamma g) = 0$, this entails $a_\gamma g = 0$, contradiction. This proves (7) and thus (6).

2.2.5 Page 11, Exercise 1.4

Statement. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution. Let f be in the Jacobson radical of $A[x]$. It suffices to show that f is nilpotent. By Proposition 1.9 p. 6 of the book [Proposition 1 p. 13], $1 - xf$ is a unit. Now use Exercise 1.2 p. 11 [Section 2.2.2 p. 17]: Part (i) implies that the coefficients of f are nilpotent, and Part (ii) implies that f itself is nilpotent.

2.2.6 Page 11, Exercise 1.5

Statement. Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in A . Show that

- (i) f is a unit in $A[[x]] \iff a_0$ is a unit in A .
- (ii) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true? (See Chapter 7, Exercise 2 [Section 8.2.2 p. 131].)
- (iii) f belongs to the Jacobson radical of $A[[x]] \iff a_0$ belongs to the Jacobson radical of A .
- (iv) The contraction of a maximal ideal \mathfrak{m} of $A[[x]]$ is a maximal ideal of A , and \mathfrak{m} is generated by \mathfrak{m}^c and x .
- (v) Every prime ideal of A is the contraction of a prime ideal of $A[[x]]$.

Solution.

- (i) The formal series $g = b_0 + b_1x + \cdots \in A[[x]]$ is an inverse of f if and only if we have $a_0b_0 = 1$ and $a_0b_n + \cdots + a_nb_0 = 0$ for $n \geq 1$, that is, if and only if b_0 is an inverse of a_0 and $b_n = -(a_1b_{n-1} + \cdots + a_nb_0)/a_0$ for $n \geq 1$. This proves the statement.
- (ii) Assume that f is nilpotent and let \mathfrak{p} be a prime ideal. Then the image of f in $A/\mathfrak{p}[[x]]$ is again nilpotent, that is zero. This implies that all the coefficients of f are in \mathfrak{p} . Since \mathfrak{p} is arbitrary, this entails that these coefficients are nilpotent.

Here is an example of a non-nilpotent formal power series all of whose coefficients are nilpotent. Put $B := \mathbb{Z}[y_2, y_3, \dots]$ where y_2, y_3, \dots are indeterminates. Let \mathfrak{a} be the ideal of B generated by the y_i^i for $i \geq 2$ and the $y_i y_j$ for $i \neq j$. Let a_i be the image of y_i in $A := B/\mathfrak{a}$. We clearly have $a_i^{i-1} \neq 0$, and $f := a_2x^2 + a_3x^3 + \cdots$ satisfies $f^n = \sum_{i \geq n} a_i^n x^{in} \neq 0$.

- (iii) [Other wording: $\mathfrak{R}(A[[x]]) = \mathfrak{R}(A) + (x)$.] Proof: Let $f \in A[[x]]$. Then we have: $f \in \mathfrak{R}(A[[x]]) \iff 1 - fg$ is a unit for all $g \in A[[x]] \iff 1 - a_0b$ is a unit in A for all $b \in A \iff a_0 \in \mathfrak{R}(A)$.
- (iv) Let \mathfrak{m} be a maximal ideal of $A[[x]]$. We must show that \mathfrak{m}^c is maximal and that $\mathfrak{m} = \mathfrak{m}^c + (x)$. We have

(a) $x \in \mathfrak{m}$. Proof: this follows from (iii).

(b) $A[[x]] = A + \mathfrak{m}$. Proof: $A[[x]] = A + (x) \stackrel{(a)}{\subset} A + \mathfrak{m}$.

(c) $A/\mathfrak{m}^c \simeq A[[x]]/\mathfrak{m}$. Proof: $A/\mathfrak{m}^c = A/(A \cap \mathfrak{m}) \simeq (A + \mathfrak{m})/\mathfrak{m} \stackrel{(b)}{=} A[[x]]/\mathfrak{m}$.

Then (c) implies that \mathfrak{m}^c is maximal.

(d) $\mathfrak{m} = \mathfrak{m}^c + (x)$. Proof: Let $f = a_0 + a_1x + \cdots \in A[[x]]$. We have $f = a_0 + xg$ for some $g \in A[[x]]$. Since $xg \in \mathfrak{m}$ by (a), we have $f \in \mathfrak{m} \iff a_0 \in \mathfrak{m} \iff a_0 \in \mathfrak{m}^c$.

- (v) In the next few lines, A^* shall designate the set of prime ideals of the ring A , and $f^* : B^* \rightarrow A^*$ shall denote the map induced by the ring morphism $f : A \rightarrow B$.

We must show that $A[[x]]^* \rightarrow A^*$ is surjective.

If $A \xrightarrow{f} B \xrightarrow{g} C$ are ring morphisms and if $(g \circ f)^* : C^* \rightarrow A^*$ is surjective, then so is $f^* : B^* \rightarrow A^*$. We solve Exercise 1.5v by applying this observation to the natural morphisms $A \rightarrow A[[x]] \rightarrow A$,

whose composition is the identity of A .

2.2.7 Page 11, Exercise 1.6

Statement. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent [that is, an element e such that $e^2 = e \neq 0$]. Prove that the nilradical and Jacobson radical of A are equal.

Hint. This is an immediate consequence of the following lemma:

Lemma 7. *Let $e = e^2$ be an idempotent in a ring A . If e is a unit, then $e = 1$. If $e \in \mathfrak{R}$, then $e = 0$.*

Proof. The first claim is clear. If $e \in \mathfrak{R}$, then the element $1 - e$, which is again idempotent, is a unit by Proposition 1.9 p. 6 of the book [Proposition 1 p. 13], and the first claim implies $1 - e = 1$, that is $e = 0$. [Other argument: If $e \in \mathfrak{R}$, since we have $(e)^2 = (e)$, Nakayama's Lemma, stated as Proposition 14 p. 34, implies $e = 0$.] \square

2.2.8 Page 11, Exercise 1.7

Statement. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution. We can assume that A is a domain, and it suffices to show that it is a field. We have, in above notation, $x(x^{n-1} - 1) = 0$. This implies that x is zero or a unit.

2.2.9 Page 11, Exercise 1.8

Statement. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

Hint. The intersection of a totally ordered set of prime ideals is a prime ideal.

Comment. Let X be the set of prime ideals of A . If Y is a subset of X , and if the intersection of any totally ordered subset of Y belongs to Y , then Y has a minimal element.

In particular, if $\mathfrak{p}_0 \in X$ and if \mathfrak{a} is an ideal of A , then the set $\{\mathfrak{p} \in X \mid \mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{p}_0\}$ has a minimal element.

2.2.10 Page 11, Exercise 1.9

Statement. Let \mathfrak{a} be an ideal $\neq (1)$ in a ring A . Show that $\mathfrak{a} = r(\mathfrak{a})$ if and only if \mathfrak{a} is an intersection of prime ideals.

Solution. Recall:

Proposition 8. [Proposition 1.14] *The radical of an ideal \mathfrak{a} is the intersection of the prime ideals which contain \mathfrak{a} .*

In particular $\mathfrak{a} = r(\mathfrak{a})$ implies that \mathfrak{a} is an intersection of prime ideals. Conversely, if \mathfrak{a} is an intersection of prime ideals, then it is the intersection of the prime ideals which contain \mathfrak{a} , intersection which coincides with $r(\mathfrak{a})$.

2.2.11 Page 11, Exercise 1.10

Statement. Let A be a ring, \mathfrak{N} its nilradical. Show that the following are equivalent:

- (i) A has exactly one prime ideal;
- (ii) every element of A is either a unit or nilpotent;
- (iii) A/\mathfrak{N} is a field.

Solution.

- (i) \implies (ii): The unique prime ideal of A is equal to \mathfrak{N} . Let a be in $A \setminus \mathfrak{N}$. If a was not a unit, it would be contained in a maximal ideal, which would coincide with \mathfrak{N} .
- (ii) \implies (iii): Let x be a nonzero element of A/\mathfrak{N} , and let $a \in A$ be a lift of x . Since a is not in \mathfrak{N} , it is a unit, and so is its image x .
- (iii) \implies (i): If \mathfrak{p} is a prime ideal, then $\mathfrak{N} \subset \mathfrak{p}$, and thus $\mathfrak{N} = \mathfrak{p}$ because \mathfrak{N} is maximal.

2.2.12 Page 11, Exercise 1.11

Statement. A ring A is Boolean if $x^2 = x$ for all $x \in A$. In a Boolean ring A , show that

- (i) $2x = 0$ for all $x \in A$;
- (ii) every prime ideal \mathfrak{p} is maximal, and A/\mathfrak{p} is a field with two elements;
- (iii) every finitely generated ideal in A is principal.

Solution.

- (i) We have $x + 1 = (x + 1)^2 = x^2 + 2x + 1 = 3x + 1$.
- (ii) We can assume A is a domain and it suffices that A/\mathfrak{p} is a field with two elements. Since the equation $x^2 = x$ has exactly two solutions, this is clear.
- (iii) Let A be boolean, and let us show that any finitely generated ideal \mathfrak{a} is principal. We can assume $\mathfrak{a} = (x, y)$. Then $z := x + y - xy \in \mathfrak{a}$ satisfies $xz = x, yz = y$, and we get $\mathfrak{a} = (z)$.

2.2.13 Page 11, Exercise 1.12

Statement. A local ring (A, \mathfrak{m}) contains no idempotent $\neq 0, 1$.

Solution. Let $e \in A$ be idempotent. If e is a unit, Lemma 7 p. 21 yields $e = 1$. If e is not a unit, then $e \in \mathfrak{m} = \mathfrak{N}$, and the same lemma implies $e = 0$.

2.2.14 Page 11, Exercise 1.13

Statement. [Construction of an algebraic closure of a field (E. Artin).] Let K be a field and let Σ be the set of all irreducible monic polynomials f in one indeterminate with coefficients in K . Let A be the polynomial ring over K generated by indeterminates x_f , one for each $f \in \Sigma$. Let \mathfrak{a} be the ideal of A generated by the polynomials $f(x_f)$ for all $f \in \Sigma$. Show that $\mathfrak{a} \neq (1)$.

Let \mathfrak{m} be a maximal ideal of A containing \mathfrak{a} , and let $K_1 = A/\mathfrak{m}$. Then K_1 is an extension field of K in which each $f \in \Sigma$ has a root. Repeat the construction with K_1 in place of K , obtaining a field K_2 , and so on. Let $L = \bigcup_{n=1}^{\infty} K_n$. Then L is a field in which each $f \in \Sigma$ splits completely into linear factors. Let \overline{K} be the set of all elements of L which are algebraic over K . Then \overline{K} is an algebraic closure of K .

Hints. To show $\mathfrak{a} \neq (1)$, let f_1, \dots, f_n be in Σ , set $x_i := x_{f_i}$ for $i = 1, \dots, n$. Assuming $\mathfrak{a} = (1)$ by contradiction, we can choose f_1, \dots, f_n as above in such a way that there are $g_i(x_1, \dots, x_n)$ in $K[x_1, \dots, x_n]$ such that

$$\sum_{i=1}^n g_i(x_1, \dots, x_n) f_i(x_i) = 1.$$

Letting L be an extension of K in which each f_i has a root α_i , we get the equality $0 = 1$ in L by evaluating the above display at $(\alpha_1, \dots, \alpha_n)$.

To show that each $f \in \Sigma$ has a root in K_1 , note the equality $f(\overline{x_f}) = 0$, where $\overline{x_f} \in K_1$ is the image of x_f .

To show that each $f \in \Sigma$ splits completely into linear factors in L , let g be an irreducible monic divisor of f in $L[x]$. It suffices to prove that g has a root in L . Note that g is an irreducible monic polynomial in $K_n[x]$ for some n , and thus g has a root in $K_{n+1} \subset L$.

Let f be in Σ . To show that \overline{K} is an algebraic closure of K , it suffices to check that f has a root in \overline{K} . We saw that f has a root in L , and this root is certainly in \overline{K} .

2.2.15 Page 12, Exercise 1.14

Statement. In a ring A , let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Hints. More precisely, for each $\mathfrak{a} \in \Sigma$ there is a maximal element $\mathfrak{b} \in \Sigma$ such that $\mathfrak{a} \subset \mathfrak{b}$. To show that Σ has maximal elements, note that the union of a totally ordered subset of Σ is in Σ . The proof that every maximal element of Σ is a prime ideal is similar to that of Proposition 1.8 p. 5 of the book. Here are some details. Let \mathfrak{p} be a maximal element of Σ , and let a and b be in $A \setminus \mathfrak{p}$. Since the ideals $\mathfrak{p} + (a)$ and $\mathfrak{p} + (b)$ contain elements which are not zero-divisors, so does their product $\mathfrak{p} + (ab)$. This implies that ab is not in \mathfrak{p} .

2.2.16 Page 12, Exercise 1.15

Statement. [The prime spectrum of a ring.] Let A be a ring and let X be the set of all prime ideals of A . For each subset E of A , let $V(E)$ denote the set of all prime ideals of A which contain E . Prove that

- (i) if \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$;
- (ii) $V(0) = X$, $V(1) = \emptyset$;
- (iii) if $(E_i)_{i \in I}$ is any family of subsets of A , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i);$$

- (iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A .

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the **Zariski topology**. The topological space X is called the **prime spectrum** of A , and is written $\text{Spec}(A)$.

Hints.

- (i) If \mathfrak{p} contains E , then \mathfrak{p} contains \mathfrak{a} , and $\mathfrak{p} = r(\mathfrak{p})$ contains $r(\mathfrak{a})$.

Say that a **radical ideal** is an ideal equal to its radical. The following proposition will be useful:

Proposition 9. (a) *The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ is a bijection between the radical ideals of A and the closed subsets of $\text{Spec}(A)$. Its inverse is given by $Y \mapsto \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$ where*

$$\bigcap Y := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

- (b) *The above bijections are decreasing.*
- (c) *If \mathfrak{a} and \mathfrak{b} are ideals of A , then $V(\mathfrak{a}) \subset V(\mathfrak{b}) \iff \mathfrak{b} \subset r(\mathfrak{a})$.*
- (d) *The set $V(\bigcap Y)$ is the closure of Y for all $Y \subset X := \text{Spec}(A)$.*

Proof. (a) As just observed, each $V(E)$ is equal to $V(\mathfrak{a})$ where \mathfrak{a} is the radical ideal generated by E . We have $\bigcap V(\mathfrak{a}) = \mathfrak{a}$ if \mathfrak{a} is a radical ideal by Proposition 8 p. 21 above [proposition which says that the radical of an ideal is the intersection of the prime ideals which contain that ideal]. Let us show that $V(\mathfrak{a}) \neq V(\mathfrak{b})$ if \mathfrak{a} and \mathfrak{b} are distinct radical ideals. We can assume there is an a in $\mathfrak{a} \setminus \mathfrak{b}$. Using Proposition 8 again we see that there is a prime ideal \mathfrak{p} which contains the ideal \mathfrak{b} but not the element a , and we get $\mathfrak{p} \in V(\mathfrak{b}) \setminus V(\mathfrak{a})$.

(b) Clear.

(c) Clear.

(d) Note that the inclusion $Y \subset V(\bigcap Y)$ is obvious. Let \mathfrak{a} be an ideal of A satisfying $Y \subset V(\mathfrak{a})$. It suffices to show $V(\bigcap Y) \subset V(\mathfrak{a})$. But this is clear because we have $\mathfrak{a} \subset \mathfrak{p}$ for all $\mathfrak{p} \in Y$, and thus $\mathfrak{a} \subset \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} = \bigcap Y$. \square

- (ii) Clear.
- (iii) Clear.
- (iv) If \mathfrak{p} contains $\mathfrak{a}\mathfrak{b}$, then \mathfrak{p} contains \mathfrak{a} or \mathfrak{b} . In both cases \mathfrak{p} contains $\mathfrak{a} \cap \mathfrak{b}$.

2.2.17 Page 12, Exercise 1.16

Statement. Draw pictures of $\text{Spec}(\mathbb{Z})$, $\text{Spec}(\mathbb{R})$, $\text{Spec}(\mathbb{C}[x])$, $\text{Spec}(R[x])$, $\text{Spec}(\mathbb{Z}[x])$.

Solution. Omitted.

2.2.18 Page 12, Exercise 1.17

Statement. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

- (i) $X_f \cap X_g = X_{fg}$;
- (ii) $X_f = \emptyset \iff f$ is nilpotent;
- (iii) $X_f = X \iff f$ is a unit;
- (iv) $X_f = X_g \iff r((f)) = r((g))$;
- (v) X is quasi-compact [that is, every open covering of X has a finite sub-covering].
- (vi) More generally, each X_f is quasi-compact.
- (vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f .

The sets X_f are called **basic open sets** of $X = \text{Spec}(A)$.

Hints.

To show that the X_f form a basis of open sets for the Zariski topology, let \mathfrak{p} be in X and \mathfrak{a} an ideal such that $\mathfrak{p} \not\supset \mathfrak{a}$, and note that there is an f in $\mathfrak{a} \setminus \mathfrak{p}$. This implies $\mathfrak{p} \in X_f \subset X \setminus V(\mathfrak{a})$.

- (i) Clear.
- (ii) It suffices to show $V(f) = X \iff f$ is nilpotent, which follows from Proposition 8 p. 21 [proposition which says that the radical of an ideal is the intersection of the prime ideals which contain that ideal].
- (iii) Clear.
- (iv) It suffices to show $V(f) = V(g) \iff r(f) = r(g)$. But this follows from Proposition 9.a p. 24.
- (v) Will follow from (vi).
- (vi) We must show that X_f is quasi-compact, that is, assuming $X_f \subset \bigcup_{i \in I} X_{g_i}$, we must show that there is a finite subset F of I such that

$$X_f \subset \bigcup_{i \in F} X_{g_i}. \quad (8)$$

Our assumption means $\bigcap_{i \in F} V(g_i) \subset V(f)$. Writing \mathfrak{a} for the ideal generated by the g_i , the above display is equivalent to $V(\mathfrak{a}) \subset V(f)$, that is to $f^n \in \mathfrak{a}$ for some n [see Proposition 9.c p. 24]. But

this holds if and only if f^n belongs to the ideal generated by $\{g_i \mid i \in F\}$ for some finite subset F of I , and F clearly satisfies (8).

(vii) If U is a quasi-compact open subset of X , then U is covered by basic open subsets, and thus by finitely many such subsets. Conversely a finite union of quasi-compact open subsets is quasi-compact and open.

2.2.19 Page 13, Exercise 1.18

Statement. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \text{Spec}(A)$. When thinking of x as a prime ideal of A , we denote it by \mathfrak{p}_x (logically, of course, it is the same thing). Show that

- (i) the set $\{x\}$ is closed (we say that x is a “closed point”) in $\text{Spec}(A) \iff \mathfrak{p}_x$ is maximal;
- (ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
- (iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subset \mathfrak{p}_y$;
- (iv) Prove that $\text{Spec}(A)$ is a T0-space [this means that if x, y are distinct points of $\text{Spec}(A)$, then either there is a neighborhood of x which does not contain y , or else there is a neighborhood of y which does not contain x].

Solution.

- (i) Will follow from (ii).
- (ii) Follows from Proposition 9.d p. 24 above.
- (iii) Follows from (ii).
- (iv) Let $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(A)$ be distinct. It suffices to show that there is an ideal \mathfrak{a} such that either $\mathfrak{p} \supset \mathfrak{a}$ and $\mathfrak{q} \not\supset \mathfrak{a}$ or $\mathfrak{q} \supset \mathfrak{a}$ and $\mathfrak{p} \not\supset \mathfrak{a}$. We can assume $\mathfrak{q} \not\subset \mathfrak{p}$, and it suffices to set $\mathfrak{a} := \mathfrak{p}$.

2.2.20 Page 13, Exercise 1.19

Statement. A topological space X is said to be **irreducible** if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Observations.

- The statement implies that the closed irreducible subsets of $\text{Spec}(A)$ are the closures of the singletons, i.e. the $V(\mathfrak{p})$.
- $X \neq \emptyset$ is irreducible if and only if $X = C \cup D$ with C and D closed implies $C = X$ or $D = X$.

Assume $X \neq \emptyset$. Let us show that X is irreducible if and only if every non-empty open set is dense in X , or, equivalently, let us prove that X is reducible if and only if some non-empty open set is not dense in X .

If X is reducible, there are disjoint non-empty open subsets U and V . Then V is contained in the complement of the closure of U , and U is not dense.

If some non-empty open subset U is not dense in X , the complement of the closure of U is non-empty open subset disjoint from U .

Hint. The following conditions are equivalent:

- (a) X is reducible,
- (b) there are ideals \mathfrak{a} and \mathfrak{b} such that $\mathfrak{a} \not\subset \mathfrak{N}$, $\mathfrak{b} \not\subset \mathfrak{N}$, $\mathfrak{ab} \subset \mathfrak{N}$,
- (c) \mathfrak{N} is not prime.

2.2.21 Page 13, Exercise 1.20

Statement. Let X be a topological space.

- (i) If Y is an irreducible (Exercise 19) subspace of X , then the closure \overline{Y} of Y in X is irreducible.
- (ii) Every irreducible subspace of X is contained in a maximal irreducible subspace.
- (iii) The maximal irreducible subspaces of X are closed and cover X . They are called the **irreducible components** of X . What are the irreducible components of a Hausdorff space?
- (iv) If A is a ring and $X = \text{Spec}(A)$, then the irreducible components of X are the closed sets $V(\mathfrak{p})$, where \mathfrak{p} is a minimal prime ideal of A (Exercise 8 [Exercise 2.2.9 p. 21]).

Solution. (i) This results immediately from the following observations:

- Y is irreducible if and only if for all open subsets U, V of X we have: U and V meet Y implies that $U \cap V$ meets Y .
- An open subset of X which meets \overline{Y} meets Y . [Proof: If U does not meet Y , then $Y \subset X \setminus U$, and thus $\overline{Y} \subset X \setminus U$.]

Parts (ii) and (iii) are left to the reader.

- (iv) This follows from the first observation made in Section 2.2.20 p. 26 above.

2.2.22 Page 13, Exercise 1.21

Statement. Let $\phi : A \rightarrow B$ be a ring homomorphism. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A , i.e., a point of X . Hence ϕ induces a mapping $\phi^* : Y \rightarrow X$. Show that

- (i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$ and hence that ϕ^* is continuous.
- (ii) If \mathfrak{a} is an ideal of A , then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.
- (iii) If \mathfrak{b} is an ideal of B , then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^e)$.
- (iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\text{Ker}(\phi))$ of X . (In particular, $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{N})$ (where \mathfrak{N} is the nilradical of A) are naturally homeomorphic.)
- (v) If ϕ is injective, then $\phi^*(Y)$ is dense in X . More precisely, $\phi^*(Y)$ is dense in $X \iff \text{Ker}(\phi) \subset \mathfrak{N}$.
- (vi) Let $\psi : B \rightarrow C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
- (vii) Let A be an integral domain with just one non-zero prime ideal \mathfrak{p} and let K be the field of

fractions of A . Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \rightarrow B$ by $\phi(x) = (\bar{x}, x)$, where \bar{x} is the image of x in A/\mathfrak{p} . Show that ϕ^* is bijective but not a homeomorphism.

Hints. Parts (iv), (vi) and (vii) are left to the reader. The solutions to (i), (iii) and (v) are given below.

- (i) $\mathfrak{q} \in \phi^{*-1}(X_f) \iff \phi^*(\mathfrak{q}) \in X_f \iff f \notin \phi^*(\mathfrak{q}) \iff \phi(f) \notin \mathfrak{q} \iff \mathfrak{q} \in Y_{\phi(f)}$.
- (ii) $\mathfrak{q} \in \phi^{*-1}(V(\mathfrak{a})) \iff \phi^*(\mathfrak{q}) \in V(\mathfrak{a}) \iff \phi^*(\mathfrak{q}) \supset \mathfrak{a} \iff \mathfrak{q} \supset \mathfrak{a}^e \iff \mathfrak{q} \in V(\mathfrak{a}^e)$.
- (iii) We have

$$\overline{\phi^*(V(\mathfrak{b}))} = V\left(\bigcap_{\mathfrak{p} \in \phi^*(V(\mathfrak{b}))} \mathfrak{p}\right) = V\left(\bigcap_{\mathfrak{q} \supset \mathfrak{b}} \mathfrak{q}^c\right) = V\left(\left(\bigcap_{\mathfrak{q} \supset \mathfrak{b}} \mathfrak{q}\right)^c\right) = V(r(\mathfrak{b})^c) = V(r(\mathfrak{b}^c)) = V(\mathfrak{b}^c).$$

- (v) We must show $\overline{\phi^*(Y)} = X \iff \text{Ker } \phi \subset \mathfrak{N}(A)$. More generally we have $\overline{\phi^*(Y)} = V(\text{Ker } \phi)$ by Part (iii) of the same exercise.

2.2.23 Page 13, Exercise 1.22

Statement. Let $A = \prod_{i=1}^n A_i$ be the direct product of rings A_i . Show that $\text{Spec}(A)$ is the disjoint union of open (and closed) subspaces X_i , where X_i is canonically homeomorphic with $\text{Spec}(A_i)$.

Conversely, let A be any ring. Show that the following statements are equivalent:

- (i) $X = \text{Spec}(A)$ is disconnected.
- (ii) $A \simeq A_1 \times A_2$ where neither of the rings A_1, A_2 is the zero ring.
- (iii) A contains an idempotent $\neq 0, 1$.

In particular, the spectrum of a local ring is always connected (Exercise 12 [Section 2.2.13 p. 22]).

Solution. We can assume $n \geq 2$ and $A_i \neq 0$ for all i . A prime ideal of A is of the form $\prod_{i=1}^n \mathfrak{a}_i$ where one of the \mathfrak{a}_i , say \mathfrak{a}_j is a prime ideal of A_j , and each of the other \mathfrak{a}_i is equal to A_i .

(i) \implies (iii): We assume that $\text{Spec}(A)$ is disconnected and we must show that A has a nontrivial idempotent. We have $X = V(\mathfrak{a}) \sqcup V(\mathfrak{b})$ with $\mathfrak{a}, \mathfrak{b} \neq (1)$. This implies $\mathfrak{a} + \mathfrak{b} = (1)$ and $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{N}$. Hence there are $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$ and $ab \in \mathfrak{N}$. We have

$$(a^n) + (b^n) = (1), \quad a^n b^n = 0, \quad (a^n) \neq (1) \neq (b^n)$$

for some n , hence $e + f = 1$, $ef = 0$ and $e \neq 1 \neq f$ for some $e \in (a^n)$ and some $f \in (b^n)$. This gives $e - e^2 = ef = 0$, and thus $e^2 = e$. Similarly $f^2 = f$.

(iii) \implies (ii): If e is an idempotent, then each of the ideals (e) and $(1 - e)$ has a structure of ring which makes the natural isomorphism of A -modules $A \simeq (e) \times (1 - e)$ an isomorphism of rings.

(ii) \implies (i): This follows from the first part of the exercise.

2.2.24 Page 14, Exercise 1.23

Statement. Let A be a Boolean ring (Exercise 11 [Exercise 2.2.12 p. 22]), and let $X = \text{Spec}(A)$.

- (i) For each $f \in A$, the set X_f (Exercise 17 [Exercise 2.2.18 p. 25]) is both open and closed in X .
- (ii) Let $f_1, \dots, f_n \in A$. Show that $X_{f_1} \cup \dots \cup X_{f_n} = X_f$ for some $f \in A$.
- (iii) The sets X_f are the only subsets of X which are both open and closed. [Let $Y \subset X$ be both open and closed. Since Y is open, it is a union of basic open sets X_f . Since Y is closed and X is quasi-compact (Exercise 17 [Exercise 2.2.18 p. 25]), Y is quasi-compact. Hence Y is a finite union of basic open sets; now use (ii) above.]
- (iv) X is a compact Hausdorff space.

Solution.

- (i) In fact we have $X = X_f \sqcup X_{1-f}$ for all f in A .
- (ii) See the proof of Statement (iii) in Section 2.2.12 p. 22.
- (iii) The hint is a complete solution.
- (iv) If $f \in \mathfrak{p} \setminus \mathfrak{q}$ then $\mathfrak{p} \in X_{1-f}$ and $\mathfrak{q} \in X_f$.

2.2.25 Page 14, Exercise 1.24

Statement. Let L be a lattice, in which the sup and inf of two elements a, b are denoted by $a \vee b$ and $a \wedge b$ respectively. L is a **Boolean lattice** (or **Boolean algebra**) if

- (i) L has a least element and a greatest element (denoted by $0, 1$ respectively).
- (ii) Each of \vee and \wedge is distributive over the other.
- (iii) Each $a \in L$ has a unique “complement” $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. (For example, the set of all subsets of a set, ordered by inclusion, is a Boolean lattice.)

Let L be a Boolean lattice. Define addition and multiplication in L by the rules $a + b = (a \wedge b') \vee (a' \wedge b)$, $ab = a \wedge b$. Verify that in this way L becomes a Boolean ring, say $A(L)$.

Conversely, starting from a Boolean ring A , define an ordering on A as follows: $a \leq b$ means that $a = ab$. Show that, with respect to this ordering, A is a Boolean lattice. [The sup and inf are given by $a \vee b = a + b + ab$ and $a \wedge b = ab$, and the complement by $a' = 1 - a$.] In this way we obtain a one-to-one correspondence between (isomorphism classes of) Boolean rings and (isomorphism classes of) Boolean lattices.

Solution. Left to the reader.

2.2.26 Page 14, Exercise 1.25

Statement. From the last two exercises deduce Stone’s theorem, that every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.

Hint. Let a and b be elements of a boolean ring A . We claim

$$(a) = (b) \iff a = b. \quad (9)$$

In words: In a boolean ring, any principal ideal has a unique generator. Let us first show

$$a \in (b) \iff ab = a. \quad (10)$$

Assuming $a \in (b)$, that is $a = bc$ for some c , we get $ab = b^2c = bc = a$. The other implication is trivial. This proves (10). To prove the non-obvious implication in (9), assume $(a) = (b)$, that is $a \in (b)$ and $b \in (a)$. By (10) this gives $a = ab = b$. \square

Since ideals in boolean rings are radical, (9) and Proposition 9 p. 24 imply $X_f = X_g \iff f = g$.

Note also the following: Let A be a boolean ring. Then the map $a \mapsto (a)$ is a bijection from A to the set P of principal ideals of A . Moreover we have $(a) + (b) = (a + b + ab)$ and $(a) \cap (b) = (ab)$, showing that P is the lattice attached to A and also a sublattice of the lattice of ideals of A .

2.2.27 Page 14, Exercise 1.26

Statement. Let A be a ring. The subspace of $\text{Spec}(A)$ consisting of the **maximal** ideals of A , with the induced topology, is called the **maximal spectrum** of A and is denoted by $\text{Max}(A)$. For arbitrary commutative rings it does not have the nice functorial properties of $\text{Spec}(A)$ (see Exercise 21 [Section 2.2.22 p. 27]), because the inverse image of a maximal ideal under a ring homomorphism need not be maximal.

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous functions on X (add and multiply functions by adding and multiplying their values). For each $x \in X$, let \mathfrak{m}_x be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal \mathfrak{m}_x is maximal, because it is the kernel of the (surjective) homomorphism $C(X) \rightarrow \mathbb{R}$ which takes f to $f(x)$. If \tilde{X} denotes $\text{Max}(C(X))$, we have therefore defined a mapping $\mu : X \rightarrow \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto \tilde{X} .

(i) Let \mathfrak{m} be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in \mathfrak{m} : that is,

$$V = \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say U_{x_1}, \dots, U_{x_n} , cover X . Let

$$f = f_{x_1}^2 + \dots + f_{x_n}^2.$$

Then f does not vanish at any point of X , hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty.

Let x be a point of V . Then $\mathfrak{m} \subset \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because \mathfrak{m} is maximal. Hence μ is surjective.

(ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X . Hence $x \neq y \implies \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.

(iii) Let $f \in C(X)$; let

$$U_f = \{x \in X \mid f(x) \neq 0\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} \mid f \notin \mathfrak{m}\}.$$

Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. \tilde{U}_f) form a basis of the topology of X (resp. \tilde{X}) and therefore μ is a homeomorphism.

Thus X can be reconstructed from the ring of functions $C(X)$.

Solution. Omitted. [The hint in the book is fairly complete.]

2.2.28 Page 14, Exercise 1.27

Statement. Let k be an algebraically closed field and let

$$f_\alpha(t_1, \dots, t_n) = 0$$

be a set of polynomial equations in n variables with coefficients in k . The set X of all points $x = (x_1, \dots, x_n) \in k^n$ which satisfy these equations is an **affine algebraic variety**.

Consider the set of all polynomials $g \in k[t_1, \dots, t_n]$ with the property that $g(x) = 0$ for all $x \in X$. This set is an ideal $I(X)$ in the polynomial ring, and is called the **ideal of the variety** X . The quotient ring

$$P(X) = k[t_1, \dots, t_n]/I(X)$$

is the ring of polynomial functions on X , because two polynomials g, h define the same polynomial function on X if and only if $g - h$ vanishes at every point of X , that is, if and only if $g - h \in I(X)$.

Let ξ_i be the image of t_i in $P(X)$. The ξ_i ($1 \leq i \leq n$) are the **coordinate functions** on X : if $x \in X$, then $\xi_i(x)$ is the i th coordinate of x . $P(X)$ is generated as a k -algebra by the coordinate functions, and is called the **coordinate ring** (or affine algebra) of X .

As in Exercise 26, for each $x \in X$ let \mathfrak{m}_x be the ideal of all $f \in P(X)$ such that $f(x) = 0$; it is a maximal ideal of $P(X)$. Hence, if $\tilde{X} = \text{Max}(P(X))$, we have defined a mapping $\mu : X \rightarrow \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

It is easy to show that μ is injective: if $x \neq y$, we must have $x_i \neq y_i$ for some i ($1 \leq i \leq n$), and hence $\xi_i - x_i$ is in \mathfrak{m}_x , but not in \mathfrak{m}_y , so that $\mathfrak{m}_x \neq \mathfrak{m}_y$. What is less obvious (but still true) is that μ is **surjective**. This is one form of Hilbert's Nullstellensatz (see Chapter 7).

Solution. Omitted. [This is more a comment than an exercise.]

2.2.29 Page 16, Exercise 1.28

Statement. Let f_1, \dots, f_m be elements of $k[t_1, \dots, t_n]$. They determine a **polynomial mapping** $\phi : k^n \rightarrow k^m$: if $x \in k^n$, the coordinates of $\phi(x)$ are $f_1(x), \dots, f_m(x)$.

Let X, Y be affine algebraic varieties in k^n, k^m respectively. A mapping $\phi : X \rightarrow Y$ is said to be **regular** if ϕ is the restriction to X of a polynomial mapping from k^n to k^m .

If η is a polynomial function on Y , then $\eta \circ \phi$ is a polynomial function on X . Hence ϕ induces a k -algebra homomorphism $P(Y) \rightarrow P(X)$, namely $\eta \mapsto \eta \circ \phi$. Show that in this way we obtain a

one-to-one correspondence between the regular mappings $X \rightarrow Y$ and the k -algebra homomorphisms $P(Y) \rightarrow P(X)$.

Solution. Assume that k is infinite [not necessarily algebraically closed] and set $t := (t_1, \dots, t_n)$, $u := (u_1, \dots, u_m)$, where the t_i and u_j are indeterminates.

To a polynomial map $\phi : k^n \rightarrow k^m$ we attach a k -algebra morphism $\psi : k[u] \rightarrow k[t]$ by setting $\psi(g) := g \circ \phi$, where $g \in k[u]$ is viewed as a polynomial map $g : k^m \rightarrow k$.

Conversely, to a k -algebra morphism $\psi : k[u] \rightarrow k[t]$ we attach a polynomial map $\phi : k^n \rightarrow k^m$ by setting $\phi_i(x) := \psi(u_i)(x)$.

We claim that $\phi \mapsto \psi$ and $\psi \mapsto \phi$ are inverse bijections.

Let $\phi : k^n \rightarrow k^m$ be a polynomial map, let $\psi : k[u] \rightarrow k[t]$ be the corresponding k -algebra morphism, and let $\phi' : k^n \rightarrow k^m$ be the polynomial map attached to ψ . For $x \in k^n$ we have

$$\phi'_i(x) = \psi(u_i)(x) = u_i(\phi(x)) = \phi_i(x).$$

Conversely, let $\psi : k[u] \rightarrow k[t]$ be a k -algebra morphism, let $\phi : k^n \rightarrow k^m$ be the corresponding polynomial map, and let $\psi' : k[u] \rightarrow k[t]$ be the k -algebra morphism attached to ϕ . For $g \in k[u]$ and $x \in k^n$ we have

$$\psi'(g)(x) = g(\phi(x)) = g(\phi_1(x), \dots, \phi_m(x)) = g(\psi(u_1)(x), \dots, \psi(u_m)(x)).$$

As the k -algebra morphisms $\alpha, \beta : k[u] \rightrightarrows k$ defined by

$$\alpha(g) := g(\psi(u_1)(x), \dots, \psi(u_m)(x)) \quad \text{and} \quad \beta(g) := \psi(g)(x)$$

coincide on the u_i , they are equal, so that we get

$$g(\psi(u_1)(x), \dots, \psi(u_m)(x)) = \psi(g)(x), \tag{11}$$

and thus $\psi'(g)(x) = \psi(g)(x)$. This shows that $\phi \mapsto \psi$ and $\psi \mapsto \phi$ are inverse bijections, proving the claim. To complete the solution to the exercise, it suffices to show [using obvious notation]:

(a) If $\phi : k^n \rightarrow k^m$ maps X into Y , then $\psi : k[u] \rightarrow k[t]$ maps $I(Y)$ into $I(X)$.

(b) If $\psi : k[u] \rightarrow k[t]$ maps $I(Y)$ into $I(X)$, then $\phi : k^n \rightarrow k^m$ maps X into Y .

Proof of (a): For $g \in I(Y)$ and $x \in X$ we have $\psi(g)(x) = g(\phi(x)) = 0$.

Proof of (b): For $x \in X$ and $g \in I(Y)$ we have

$$g(\phi(x)) = g(\phi_1(x), \dots, \phi_m(x)) = g(\psi(u_1)(x), \dots, \psi(u_m)(x)) = \psi(g)(x) = 0,$$

the penultimate equality being justified by (11).

3 About Chapter 2

3.1 Comments

3.1.1 Page 21, Proposition 2.4

Proposition 2.4 reads:

Let M be a finitely generated A -module, let \mathfrak{a} be an ideal of A , and let ϕ be an A -module endomorphism of M such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form

$$\phi^n + a_1 \phi^{n-1} + \cdots + a_n = 0$$

where the a_i are in \mathfrak{a} .

Strictly speaking, this makes no sense because ϕ and the a_i belong to different rings. We suggest the following restatement:

Let M be a finitely generated A -module, let \mathfrak{a} be an ideal of A , let ϕ be an A -module endomorphism of M such that $\phi(M) \subseteq \mathfrak{a}M$, and let $\psi : A \rightarrow \text{End}_A(M)$ be the natural morphism. Then ϕ satisfies an equation of the form

$$\phi^n + \psi(a_1) \phi^{n-1} + \cdots + \psi(a_n) = 0$$

where the a_i are in \mathfrak{a} .

[We have used the symbol \subseteq above to make the quote accurate, but in general we denote inclusions by \subset .]

Another fix would be to equip $\text{End}_A(M)$ with its natural A -module structure and change the display to

$$\phi^n + a_1 \phi^{n-1} + \cdots + a_n \phi^0 = 0.$$

If $\phi = \psi(y)$ for some y in A we get

$$\psi(y^n + a_1 y^{n-1} + \cdots + a_n) = 0. \tag{12}$$

This yields the following:

If $yM \subset \mathfrak{a}M$ for some y in A , then there is an x in A such that $xM = 0$ and $x \equiv y^n \pmod{\mathfrak{a}}$ for some n .

Corollary 2.5 reads:

Corollary 10 (Corollary 2.5 p. 21). *Let M be a finitely generated A -module and let \mathfrak{a} be an ideal of A such that $\mathfrak{a}M = M$. Then there exists $x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$.*

The proof reads: Take $\phi = \text{identity}$, $x = 1 + a_1 + \cdots + a_n$ in (2.4).

I suggest the following restatement of the proof:

Since $\phi = \text{identity}$, we can take $y = 1$ in (12). This gives $\psi(1 + a_1 + \cdots + a_n) = 0$, and we can set $x := 1 + a_1 + \cdots + a_n$.

Note that Corollary 2.5 can also be stated as follows:

Corollary 11. *Let M be a finitely generated A -module and let \mathfrak{a} be an ideal of A such that $\mathfrak{a}M = M$. Then there exists $\alpha \in \mathfrak{a}$ such that $\alpha m = m$ for all $m \in M$.*

In other words, we go from $\mathfrak{a}M = M$ to $\alpha m = m$. Here is a particular case [take $\mathfrak{a} := (a)$]:

Corollary 12. *Let M be a finitely generated A -module and let $a \in A$ satisfy $aM = M$. Then there is a $b \in A$ such that $abm = m = bam$ for all $m \in M$. In particular, if the endomorphism $m \mapsto am$ of M is surjective, then it is bijective.*

Here is a particular case of the particular case:

Corollary 13. *Let M be a finitely generated A -module and ϕ a surjective endomorphism of M . Then ϕ is bijective.*

Proof. Let x be an indeterminate, view M as an $A[x]$ -module on which x acts by ϕ , and apply Corollary 12 to the ring $A[x]$ and the element x . \square

Let us also state Nakayama's Lemma [Proposition 2.6 p. 21 of the book]:

Proposition 14 (Nakayama's Lemma). *Let M be a finitely generated A -module and \mathfrak{a} an ideal of A contained in the Jacobson radical of A . Then $\mathfrak{a}M = M$ implies $M = 0$.*

Let us rewrite the proof of Proposition 2.4 to turn it into a proof of the corrected statement:

Let x_1, \dots, x_n be a set of generators of M . Then each $\phi(x_i) \in \mathfrak{a}M$, so that we have say $\phi(x_i) = \sum_{j=1}^n a_{ij}x_j$ ($1 \leq i \leq n; a_{ij} \in \mathfrak{a}$), i.e., $\sum_{j=1}^n (\delta_{ij}\phi - \psi(a_{ij}))x_j = 0$, where δ_{ij} is the Kronecker delta. Set $b_{ij} := \delta_{ij}\phi - \psi(a_{ij})$, and let us regard the matrix (b_{ij}) as a matrix with entries in the subring B of $\text{End}_A(M)$ generated by $\psi(A)$ and ϕ , subring which is clearly commutative. Letting (c_{ij}) be the adjoint of (b_{ij}) , we get

$$0 = \sum_j c_{ij} \sum_k b_{jk} x_k = \sum_{j,k} c_{ij} b_{jk} x_k = \sum_k \left(\sum_j c_{ij} b_{jk} \right) x_k = \sum_k \delta_{ik} \det(b_{j\ell}) x_k = \det(b_{j\ell}) x_i.$$

It follows that $\det(b_{j\ell})$ annihilates each x_i , hence is the zero endomorphism of M . Expanding out the determinant, we have an equation of the required form.

[The underlying reasoning is that we consider the natural morphism from the ring of n by n matrices with entries in B to the endomorphism ring of the A -module M^n .]

3.1.2 Page 23, proof of Proposition 2.9 (i)

Let us prove: \bar{v} injective $\implies v$ surjective. If $\phi : M'' \rightarrow M''/\text{Im}(v)$ is the canonical projection, we get

$$0 = \phi \circ v = \bar{v}(\phi) \implies \phi = 0 \implies v \text{ surjective.}$$

3.1.3 Page 24, vanishing tensors

This is taken from Lemma 10, Chapter 1, Section 2, Subsection 11, page 41 in Nicolas Bourbaki, **Algèbre commutative: Chapitres 1 à 4**, Masson, Paris 1985:

Let A be a (non necessarily commutative) associative ring with 1, let M be a right A -module, let N be a left A -module, let $(y_i)_{i \in I} \subset N$ be a generating family, let $(x_i)_{i \in I} \subset M$ be a finitely supported family, and assume that we have $\sum_{i \in I} x_i \otimes y_i = 0$ in $M \otimes_A N$. Then there is a finite set J and there are finitely supported families $(x'_j)_{j \in J} \subset M$ and $(a_{ij})_{i \in I, j \in J} \subset A$ such that $\sum_{j \in J} x'_j a_{ij} = x_i$ for all i and $\sum_{i \in I} a_{ij} y_i = 0$ for all j .

Proof. Let F be the free left A -module whose basis is the family of symbols $(e_i)_{i \in I}$, and consider the exact sequence $0 \rightarrow R \xrightarrow{\iota} F \xrightarrow{\varphi} N \rightarrow 0$, where φ is defined by $\varphi(e_i) = y_i$. It induces the exact sequence $M \otimes_A R \xrightarrow{\iota'} M \otimes_A F \xrightarrow{\varphi'} M \otimes_A N \rightarrow 0$, and we get successively

$$\varphi' \left(\sum_{i \in I} x_i \otimes e_i \right) = \sum_{i \in I} x_i \otimes y_i = 0,$$

$$\sum_{i \in I} x_i \otimes e_i = \iota' \left(\sum_{j \in J} x'_j \otimes r_j \right) = \sum_{j \in J} x'_j \otimes \iota(r_j)$$

where J is a finite set, where x'_j is in M and where r_j is in R , $\iota(r_j) = \sum_{i \in I} a_{ij} e_i$ for some finitely supported family $(a_{ij})_{i \in I, j \in J} \subset A$,

$$\sum_{i \in I} x_i \otimes e_i = \sum_{j \in J} x'_j \otimes \iota(r_j) = \sum_{j \in J} x'_j \otimes \sum_{i \in I} a_{ij} e_i = \sum_{i \in I} \sum_{j \in J} x'_j \otimes a_{ij} e_i = \sum_{i \in I} \left(\sum_{j \in J} x'_j a_{ij} \right) \otimes e_i,$$

$$x_i = \sum_{j \in J} x'_j a_{ij} \quad \text{and} \quad 0 = \varphi(\iota(r_j)) = \sum_{i \in I} a_{ij} y_i.$$

□

In the same spirit, we have:

Let A be a (non necessarily commutative) associative ring with 1; let I be a set; let J be a finite set; let $A^{\oplus I} \xrightarrow{\phi} A^{\oplus J} \xrightarrow{\psi} N \rightarrow 0$ be an exact sequence of left A -modules; let ϕ be given by $\phi(e_i) = \sum_j a_{ij} f_j$, where (e_i) and (f_j) are the obvious canonical bases; for j in J set $v_j := \psi(f_j) \in N$; let M be a right A -module; let u in $M^{\oplus J}$ satisfy $\sum u_j \otimes v_j = 0$ in $M \otimes_A N$. Then there is a w in $M^{\oplus I}$ such that $\sum_i w_i a_{ij} = u_j$ for all j .

Proof. Applying $M \otimes_A -$ to the above exact sequence we get the exact sequence

$$M^{\oplus I} \xrightarrow{\phi'} M^{\oplus J} \xrightarrow{\psi'} M \otimes_A N \rightarrow 0$$

and $\psi'(u) = 0$. Thus there is a w in $M^{\oplus I}$ such that $\phi'(w) = u$, and it's easy to see that this w does the job. □

3.1.4 Page 27, contracted ideals

If $A \rightarrow B$ is a morphism, then an ideal \mathfrak{a} of A is contracted if and only if the natural map $A/\mathfrak{a} \rightarrow B \otimes_A A/\mathfrak{a}$ is injective. In particular, if a proper ideal \mathfrak{a} is contracted, we have $B \otimes_A A/\mathfrak{a} \neq 0$. Indeed, $B \otimes_A A/\mathfrak{a} \simeq B/\mathfrak{a}^e$ and $\text{Ker}(A/\mathfrak{a} \rightarrow B \otimes_A A/\mathfrak{a}) \simeq \mathfrak{a}^{ec}/\mathfrak{a}$.

3.1.5 Page 29, Proposition 2.19

I do not understand the proof that (ii) implies (i). Here is another argument.

We start by proving (ii) \iff (iii) as in the book. Then we prove (iii) \implies (i) as follows:

Let $P \xrightarrow{f} Q \xrightarrow{g} R$ be exact, let $Q \xrightarrow{g'} g(Q)$ and $g(Q) \xrightarrow{i} R$ be the obvious maps, and let T be the functor $N \otimes_A -$. We must show that

$$T(P) \xrightarrow{T(f)} T(Q) \xrightarrow{T(g)} T(R)$$

is exact. The sequences

$$\begin{aligned} P \xrightarrow{f} Q \xrightarrow{g'} g(Q) \rightarrow 0, \quad 0 \rightarrow g(Q) \xrightarrow{i'} R, \\ T(P) \xrightarrow{T(f)} T(Q) \xrightarrow{T(g')} T(g(Q)) \rightarrow 0, \quad 0 \rightarrow T(g(Q)) \xrightarrow{T(i')} T(R), \end{aligned}$$

being exact, we get $\text{Ker}(T(g)) = \text{Ker}(T(i) \circ T(g')) = \text{Im}(T(f))$.

3.1.6 Page 30, Tensor product of algebras

The claim on p. 31 that “the mapping $a \mapsto f(a) \otimes g(a)$ is a ring homomorphism $A \rightarrow D$ ” is incorrect. The expression $f(a) \otimes g(a)$ should be replaced with $f(a) \otimes 1$ [which is equal to $1 \otimes g(a)$].

Note that the tensor product $B \otimes_A C$ of the A -algebras B and C “is” their coproduct. The tensor product of an arbitrary family $(B_\lambda)_{\lambda \in \Lambda}$ of A -algebras defined in Exercise 23 p. 34 [Section 3.2.24 p. 47] is also the coproduct of the family: see this Mathematics Stackexchange post of Eric Wofsey: <https://math.stackexchange.com/a/2441323/660>.

3.2 Exercises

3.2.1 Page 31, Exercise 2.1

Statement. Show that $(\mathbb{Z}/m\mathbb{Z}) \times_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Generalization: compute $A/\mathfrak{a} \otimes_A A/\mathfrak{b}$ [obvious notation].

Solution. Using Exercise 2.2 in Section 3.2.2 below, we get

$$\frac{A}{\mathfrak{a}} \otimes_A \frac{A}{\mathfrak{b}} \simeq \frac{A/\mathfrak{b}}{\mathfrak{a}(A/\mathfrak{b})} \simeq \frac{A/\mathfrak{b}}{\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})} \simeq \frac{A/\mathfrak{b}}{(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}} \simeq \frac{A}{\mathfrak{a} + \mathfrak{b}}.$$

More precisely this means that the solid diagram

$$\begin{array}{ccc} A \otimes A & \longrightarrow & A \\ \downarrow & & \downarrow \\ \frac{A}{\mathfrak{a}} \otimes \frac{A}{\mathfrak{b}} & \dashrightarrow & \frac{A}{\mathfrak{a}+\mathfrak{b}} \end{array}$$

where the solid arrows are the obvious ones, can be completed as indicated in a unique way, and that the induced map is bijective.

3.2.2 Page 31, Exercise 2.2

Statement. Let A be a ring, \mathfrak{a} an ideal, M an A -module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

[Tensor the exact sequence $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0$ with M .]

Solution. Here is a more precise statement: there is a unique A -linear map $f : (A/\mathfrak{a}) \otimes_A M \rightarrow M/\mathfrak{a}M$ such that $f(\bar{a}) \otimes x = \overline{ax}$ for all $a \in A, x \in M$, where $\bar{a} \in A/\mathfrak{a}$ is the image of $a \in A$ and $\overline{ax} \in M/\mathfrak{a}M$ is the image of $ax \in M$. Moreover f is bijective.

Let us sketch the proof. Consider the sequence

$$\mathfrak{a} \otimes_A M \xrightarrow{g} M \xrightarrow{h} (A/\mathfrak{a}) \otimes_A M \rightarrow 0,$$

where g and h are defined by $g(a \otimes x) = ax$ and $h(x) = 1 \otimes x$. This sequence is exact, and the image of g is $\mathfrak{a}M \subset M$.

3.2.3 Page 31, Exercise 2.3

Statement. Let A be a local ring, M and N finitely generated A -modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

[Let \mathfrak{m} be the maximal ideal, $k = A/\mathfrak{m}$ the residue field. Let $M_k = k \otimes_A M \simeq M/\mathfrak{m}M$ by Exercise 2. By Nakayama's lemma (Proposition 14 p. 34), $M_k = 0 \implies M = 0$. But $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0 \implies M_k = 0$ or $N_k = 0$, since M_k, N_k are vector spaces over a field.]

Solution. Omitted. [The hint is fairly complete.]

3.2.4 Page 31, Exercise 2.4

Statement. Let M_i ($i \in I$) be any family of A -modules, and let M be their direct sum. Prove that M is flat \iff each M_i is flat.

Solution. This will follow from the solution to Exercise 2.20 given below in Section 3.2.21 p. 45.

3.2.5 Page 32, Exercise 2.5

Statement. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A . Prove that $A[x]$ is a flat A -algebra. [Use Exercise 4.]

Solution. As an A -module $A[x]$ is a direct sum of copies of A . Since A is A -flat, this implies that $A[x]$ is also A -flat.

3.2.6 Page 32, Exercise 2.6

Statement. For any A -module, let $M[x]$ denote the set of all polynomials in x with coefficients in M , that is to say expressions of the form

$$m_0 + m_1x + \cdots + m_rx^r \quad (m_i \in M).$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$ -module.

Show that $M[x] \simeq A[x] \otimes_A M$.

Solution. Left to the reader.

3.2.7 Page 32, Exercise 2.7

Statement. Let \mathfrak{p} be a prime ideal in A . Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If \mathfrak{m} is a maximal ideal in A , is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Solution. By the previous two exercises we have an isomorphism of A -modules $A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})[x]$, and it is easy to check that it is in fact an isomorphism of A -algebras [and in particular an isomorphism of rings]. If A is a field, then (0) is a maximal ideal of A but $(0)[x] = (0)$ is not a maximal ideal of $A[x]$.

3.2.8 Page 32, Exercise 2.8

Statement. (i) If M and N are flat A -modules, then so is $M \otimes_A N$.

(ii) If B is a flat A -algebra and N is a flat B -module, then N is flat as an A -module.

Hints. (i) We have $(M \otimes_A N) \otimes_A P \simeq M \otimes_A (N \otimes_A P)$.

(ii) We have $N \otimes_A P \simeq N \otimes_B (B \otimes_A P)$.

3.2.9 Page 32, Exercise 2.9

Statement. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules. If M' and M'' are finitely generated, then so is M .

Solution. Let f be the map from M' to M occurring in the above exact sequence; let x'_1, \dots, x'_m be generators of M' ; let x''_1, \dots, x''_n be generators of M'' ; and let $x_j \in M$ be a pre-image of x''_j ($1 \leq j \leq n$). It is easily seen that the $f(x'_i)$ together with the x_j generate M .

3.2.10 Page 32, Exercise 2.10

Statement. Let A be a ring, \mathfrak{a} an ideal contained in the Jacobson radical of A ; let M be an A -module and N a finitely generated A -module, and let $u : M \rightarrow N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \rightarrow N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution. We have $N = \mathfrak{a}N + u(M)$, and Corollary 2.7 p. 22 of the book implies $N = u(M)$. Recall that Corollary 2.7 is a corollary to Proposition 2.6, which is Nakayama's lemma. Here is the statement of Corollary 2.7:

Corollary 15 (Corollary 2.7 p. 22 of the book). *Let M be a finitely generated A -module, N a submodule of M , $\mathfrak{a} \subset \mathfrak{R}$ an ideal. Then $M = \mathfrak{a}M + N \implies M = N$.*

3.2.11 Page 32, Exercise 2.11

Statement. Let A be a ring $\neq 0$. Show that $A^m \simeq A^n \implies m = n$.

[Let \mathfrak{m} be a maximal ideal of A and let $\phi : A^m \rightarrow A^n$ be an isomorphism. Then $1 \otimes \phi : (A/\mathfrak{m}) \otimes_A^m \rightarrow (A/\mathfrak{m}) \otimes_A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $k = A/\mathfrak{m}$. Hence $m = n$.] (Cf. Chapter 3, Exercise 15 [Section 4.2.15 p. 69].)

If $\phi : A^m \rightarrow A^n$ is surjective, then $m \geq n$.

If $\phi : A^m \rightarrow A^n$ is injective, is it always the case that $m \leq n$?

Hint. We only solve the last part of Exercise 2.11. We claim that if $f : A^m \rightarrow A^n$ is an A -linear injection, then $m \leq n$. The claim is an immediate consequence of the following lemma.

Lemma 16. *If A is a ring, M_1 and M_2 are finitely generated A -modules, M_2 is faithful and $f : M_1 \oplus M_2 \rightarrow M_1$ is a morphism, then f is not injective.*

Proof. Suppose for the sake of contradiction that f is injective. Setting $M := M_1 \oplus M_2$, define $g : M \rightarrow M$ by $g(x + y) := f(x + y)$ for $x \in M_1, y \in M_2$ [we regard M_1 and M_2 as submodules of M]. Note that g is injective. By Proposition 2.4 p. 21 of the book [see Section 3.1.1 p. 33 above] we have

$$g^n + a_1 g^{n-1} + \cdots + a_{n-1} g^1 + a_n g^0 = 0 \quad (13)$$

for some $a_1, \dots, a_n \in A$. We can assume that n is minimal [note that we have $n \geq 2$]. Applying (13) to $y \in M_2$ yields

$$\left(g^n(y) + a_1 g^{n-1}(y) + \cdots + a_{n-1} g(y) \right) + (a_n y) = 0.$$

Since the first parenthesis is in M_1 , both parenthesis vanish. Since y is arbitrary and M_2 is faithful, this implies $a_n = 0$, and (13) becomes

$$g \left(g^{n-1} + a_1 g^{n-2} + \cdots + a_{n-1} g^0 \right) = 0.$$

The parenthesis being zero by injectivity of g , we get a contradiction with the minimality of n . \square

3.2.12 Page 32, Exercise 2.12

Statement. Let M be a finitely generated A -module and $\phi : M \rightarrow A^n$ a surjective homomorphism. Show that $\text{Ker}(\phi)$ is finitely generated.

[Let e_1, \dots, e_n be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$ ($1 \leq i \leq n$). Show that M is the direct sum of $\text{Ker}(\phi)$ and the submodule generated by u_1, \dots, u_n .]

Solution. We follow the hint. Let N be submodule of M generated by u_1, \dots, u_n . Let x be in M . We have $\phi(x) = \sum a_i e_i$ for some $a_i \in A$. Set $y := \sum a_i u_i \in N$. We get $x = (x - y) + y$ with $\phi(x - y) = 0$. This shows $M = \text{Ker}(\phi) + N$. Let x be in $\text{Ker}(\phi) \cap N$. We have $x = \sum a_i u_i$ for some $a_i \in A$, hence $0 = \phi(x) = \sum a_i e_i$, hence $x = 0$. This shows $M = \text{Ker}(\phi) \oplus N$.

3.2.13 Page 32, Exercise 2.13

Statement. Let $f : A \rightarrow B$ be a ring homomorphism, and let N be a B -module. Regarding N as an A -module by restriction of scalars, form the B -module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \rightarrow N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B .

[Define $p : N_B \rightarrow N$ by $p(b \otimes y) = by$, and show that $N_B = \text{Im}(g) \oplus \text{Ker}(p)$.]

Hint. Note that $p \circ g = \text{id}_N$.

3.2.14 A digression about limits and colimits

The purpose of this section is to give a naive motivation for the expression of limits in terms of products and equalizers, and, dually, for the expression of colimits in terms of coproducts and coequalizers [when such exist]. These expressions are used in some of the subsequent sections.

We shall describe only limits of sets, but we shall try to do it in a way that suggests an obvious generalization for limits in categories with products and equalizers, and for colimits in categories with coproducts and coequalizers.

Let Φ be a set such that each $f \in \Phi$ is a map $f : X_f \rightarrow Y_f$ from a set X_f to a set Y_f . Put

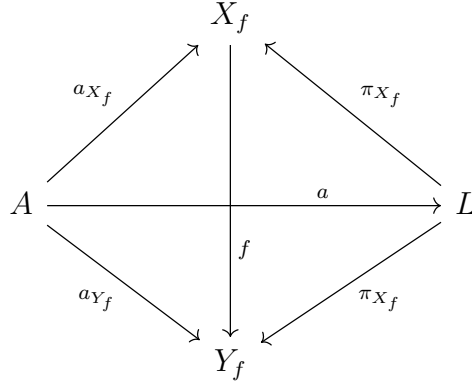
$$\Omega := \{X_f \mid f \in \Phi\} \cup \{Y_f \mid f \in \Phi\}.$$

Let L be a set equipped with a map $\pi_Z : L \rightarrow Z$ for each $Z \in \Omega$ such that $\pi_{Y_f} \circ f = \pi_{X_f}$:

$$\begin{array}{ccc} X_f & \xrightarrow{f} & Y_f \\ & \searrow \pi_{X_f} & \downarrow \pi_{Y_f} \\ & & L \end{array}$$

for all $f \in \Phi$. Such a pair $(L, (\pi_Z)_{Z \in \Omega})$ is a **limit** of the set Φ if for each family $(A, (a_Z)_{Z \in \Omega})$ such that for each a_Z is a map $a_Z : A \rightarrow Z$ satisfying $a_{Y_f} \circ f = a_{X_f}$ for all $f \in \Phi$ there is a unique map

$a : A \rightarrow L$ making the diagram



commutes.

It is clear that if such a limit exists it is unique up to unique isomorphism [we leave the precise definition of the phrase “unique up to unique isomorphism” to the reader]. At first we assume the such a limit exists. This will yields a recipe to construct this limit, and will be easy [end left again to the reader] to verify that this recipe works.

Note that the π_Z define a map $\pi : L \rightarrow \prod_{Z \in \Omega} Z$.

The trick is to consider the case when A is a singleton. We then see that L can be identified with the set of all $a = (a_Z)_{Z \in \Omega} \in \prod_{Z \in \Omega} Z$ such that $f(a_{X_f}) = a_{Y_f}$ for all $f \in \Phi$.

Denoting by $p_T : \prod_{Z \in \Omega} Z \rightarrow T$ (for $T \in \Omega$) the canonical projection, we can rewrite the condition

$$f(a_{X_f}) = a_{Y_f} \text{ for all } f \in \Phi$$

as

$$(f \circ p_{X_f})(a) = p_{Y_f}(a) \text{ for all } f \in \Phi.$$

We thus have two natural maps from $\prod_{Z \in \Omega} Z$ to Y_f parametrized by $f \in \Phi$, namely $f \circ p_{X_f}$ and p_{Y_f} . To organize them into two maps from $\prod_{Z \in \Omega} Z$ to $\prod_{f \in \Phi} Y_f$ as follows. Let $q_g : \prod_{f \in \Phi} Y_f \rightarrow Y_g$ be the canonical projection, and define the maps u and v from $\prod_{Z \in \Omega} Z$ to $\prod_{f \in \Phi} Y_f$ by the commutative diagrams

$$\begin{array}{ccc}
 \prod_{Z \in \Omega} Z & \xrightarrow{u} & \prod_{f \in \Phi} Y_f \\
 p_{X_g} \downarrow & & \downarrow q_g \\
 X_g & \xrightarrow{f} & Y_g
 \end{array}
 \qquad
 \begin{array}{ccc}
 \prod_{Z \in \Omega} Z & \xrightarrow{v} & \prod_{f \in \Phi} Y_f \\
 & \searrow p_{Y_g} & \downarrow q_g \\
 & & Y_g
 \end{array}$$

Then we see L is the equalizer of u and v .

We can summarize this by the single diagram

$$\begin{array}{ccccc}
 & X_g & \xrightarrow{f} & Y_g & \\
 & \uparrow p_{X_g} & & \uparrow q_g & \\
 L & \xrightarrow{\pi} & \prod_{Z \in \Omega} Z & \xrightleftharpoons[u]{u} & \prod_{f \in \Phi} Y_f \\
 & & \searrow p_{Y_g} & & \downarrow q_g \\
 & & & & Y_g
 \end{array}$$

This diagram makes sense whenever the appropriate products and equalizers exist.

We can also reverse the arrows and replace products with coproducts and equalizers with coequalizers to define colimits.

3.2.15 Page 32, Exercise 2.14

Statement. A partially ordered set I is said to be a **directed** set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A -modules indexed by I . For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \rightarrow M_j$ be an A -homomorphism, and suppose that the following axioms are satisfied:

- (1) μ_{ii} is the identity mapping of M_i for all $i \in I$;
- (2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a **direct system** $M = (M_i, \mu_{ij})$ over the directed set I .

We shall construct an A -module M called the **direct limit** of the direct system M . Let C be the direct sum of the M_i and identify each module M_i with its canonical image in C . Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \rightarrow M$ be the projection and let μ_i be the restriction of μ to M_i .

The module M , or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \rightarrow M$, is called the **direct limit** of the direct system M , and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.

Solution. Here is a slightly more general construction.

Assume we have an inductive system (M_i) of A -modules indexed by a category I , that is, for each object i of I we have an A -module M_i , and for each morphism $f : d(f) \rightarrow c(f)$ in I we have an A -linear map $M_f : M_{c(f)} \rightarrow M_{d(f)}$.

Consider the commutative diagram

$$\begin{array}{ccccc}
 M_{d(f)} & & & M_i & \\
 \alpha_f \downarrow & \searrow \beta_{d(f)} & & \downarrow \beta_i & \\
 \bigoplus_g M_{d(g)} & \xrightarrow[u]{u} & \bigoplus_j M_j & \xrightarrow{\pi} & C \\
 \alpha_f \uparrow & & \uparrow \beta_{c(f)} & & \\
 M_{d(f)} & \xrightarrow{M_f} & M_{c(f)}, & &
 \end{array}$$

where the α_f and the β_i are the coprojections, and the middle row is exact [i.e. the last arrow is a coequalizer].

We claim that C is a **colimit** of our system [see Section 3.2.14 p. 40].

Let $h : \bigoplus M_j \rightarrow N$ be A -linear. We have

$$h \circ u = h \circ v \iff h \circ u \circ \alpha_f = h \circ v \circ \alpha_f \ \forall f \iff h \circ \beta_{d(f)} = h \circ \beta_{c(f)} \circ M_f \ \forall f.$$

This shows that C is indeed a colimit of our system.

The following definition is taken from the Stacks Project
<https://stacks.math.columbia.edu/tag/002V>:

Definition 17. We say that a category I is **filtered** if the following conditions hold:

1. the category I has at least one object,
2. for every pair of objects x, y of I there exists an object z and morphisms $x \rightarrow z$, $y \rightarrow z$, and
3. for every pair of objects x, y of I and every pair of morphisms $a, b : x \rightarrow y$ of I there exists a morphism $c : y \rightarrow z$ of I such that $c \circ a = c \circ b$ as morphisms in \mathcal{C} .

Assume now that the category I is filtered, and form the commutative diagram

$$\begin{array}{ccccc}
 M_{d(f)} & & & M_i & \\
 \alpha'_f \downarrow & \searrow \beta'_{d(f)} & & \downarrow \beta'_i & \searrow \gamma_i \\
 \bigsqcup_g M_{d(g)} & \xrightarrow[u]{u} & \bigsqcup_i M_i & \xrightarrow{\pi'} & C' \\
 \alpha'_f \uparrow & & \uparrow \beta'_{c(f)} & & \\
 M_{d(f)} & \xrightarrow{M_f} & M_{c(f)}, & &
 \end{array}$$

which is the “set theoretical analog” of the previous one, that is, \bigsqcup denotes disjoint union, and the middle row is exact in the category of sets [i.e. the last arrow is a set theoretical coequalizer]. Then C' is the set theoretical colimit of our inductive system [see Section 3.2.14 p. 40].

We claim that the natural set theoretical map $C' \rightarrow C$ is bijective.

To prove this we define a structure of A -module on C' . To define the addition it suffices to define $\gamma_i(x_i) + \gamma_j(x_j)$ for $x_i \in M_i, x_j \in M_j$. To do this we choose morphisms $f : i \rightarrow k, g : j \rightarrow k$, we check that the element $\gamma_k(M_f(x_i) + M_g(x_j)) \in C'$ does not depend on the choice of k, f and g , and we set

$$\gamma_i(x_i) + \gamma_j(x_j) := \gamma_k(M_f(x_i) + M_g(x_j)) \in C'.$$

Then we define the map $A \times C' \rightarrow C'$ [details left to the reader], we check that we have indeed defined a structure of A -module on C' , we use it to define a morphism $C \rightarrow C'$, and we check that this morphism is inverse to the morphism $C' \rightarrow C$ previously defined. Again, the details are left to the reader.

3.2.16 Page 33, Exercise 2.15

Statement. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j .

Solution. This follows from the comments made in the previous section [Section 3.2.15 p. 42].

3.2.17 Page 33, Exercise 2.16

Statement. Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A -module and for each $i \in I$ let $\alpha_i : M_i \rightarrow N$ be an A -module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \rightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Solution. This follows from the comments made in Section 3.2.15 p. 42.

3.2.18 Page 33, Exercise 2.17

Statement. Let $(M_i)_{i \in I}$ be a family of submodules of an A -module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subset M_k$. Define $i \leq j$ to mean $M_i \subset M_j$ and let $\mu_{ij} : M_i \rightarrow M_j$ be the embedding of M_i in M_j . Show that

$$\varinjlim M_i = \sum M_i = \bigcup M_i.$$

In particular, any A -module is the direct limit of its finitely generated submodules.

Solution. This follows from the comments made in Section 3.2.15 p. 42.

3.2.19 Page 33, Exercise 2.18

Statement. Let $\mathbf{M} = (M_i, \mu_{ij}), \mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A -modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \rightarrow M, \nu_i : N_i \rightarrow N$ the associated homomorphisms.

A **homomorphism** $\Phi : \mathbf{M} \rightarrow \mathbf{N}$ is by definition a family of A -module homomorphisms $\phi_i : M_i \rightarrow N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that Φ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \rightarrow N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Solution. Left to the reader.

3.2.20 Page 33, Exercise 2.19

Statement. A sequence of direct systems and homomorphisms

$$\mathbf{M} \rightarrow \mathbf{N} \rightarrow \mathbf{P}$$

is **exact** if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \rightarrow N \rightarrow P$ of direct limits is then exact. [Use Exercise 15 (Section 3.2.16 p. 44).]

Hint. Using obvious notation, writing $\Phi : \mathbf{M} \rightarrow \mathbf{N}$, $\Psi : \mathbf{N} \rightarrow \mathbf{P}$, $\phi : M \rightarrow N$ and $\psi : N \rightarrow P$ for the morphisms in the statement and denoting by $\mu_{ji}, \nu_{ji}, \pi_{ji}$ what the book designates by $\mu_{ij}, \nu_{ij}, \pi_{ij}$, let $y \in \text{Ker } \psi$. Then y is of the form $\nu_i(y_i)$ for some $i \in I$ and $y_i \in N_i$ such that $\pi_i(\psi_i(y_i)) = 0$. Thus there is a $j \geq i$ with $\pi_{ji}(\psi_i(y_i)) = 0$, that is $\psi_j(\nu_{ji}(y_i)) = 0$. Hence there is an $x_j \in M_j$ with $\nu_{ji}(y_i) = \phi_j(x_j)$, and it is straightforward to check that we have $y = \phi(\mu_j(x_j))$.

Note that colimits are **not** exact in general. For instance cokernels are **not** exact, as shown by the following example. Consider the exact sequence

$$0 \rightarrow 4\mathbb{Z} \rightarrow \mathbb{Z} \text{ and } 0 \rightarrow 2\mathbb{Z} \rightarrow \mathbb{Z},$$

where the morphisms are the inclusions. A morphism from the first exact sequence to the second one is given by the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & 4\mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 2\mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

[the morphisms are again the inclusions]. The sequence of cokernels is clearly not exact.

3.2.21 Page 33, Exercise 2.20

Statement. Keeping the same notation as in Exercise 14, let N be any A -module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \varinjlim (M_i \otimes N)$ be its direct limit.

For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \rightarrow M \otimes N$, hence by Exercise 16 [Section 3.2.17 p. 44] a homomorphism $\psi : P \rightarrow M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim (M_i \otimes N) \simeq (\varinjlim M_i) \otimes N.$$

[For each $i \in I$, let $g_i : M_i \times N \rightarrow M_i \otimes N$ be the canonical bilinear mapping. Passing to the limit we obtain a mapping $g : M \times N \rightarrow P$. Show that g is A -bilinear and hence define a homomorphism $\phi : M \otimes N \rightarrow P$. Verify that $\phi \circ \psi$ and $\psi \circ \phi$ are identity mappings.]

Solution. In view of Section 3.2.15 p. 42, to prove that

$$\text{the tensor product commutes with colimits,} \quad (14)$$

it suffices to check that it commutes with direct sums.

To do so, let M be an A -module, let (N_i) be a family of A -module, let $\alpha_i : N_i \rightarrow \bigoplus_i N_i$ and $\beta_i : M \otimes N_i \rightarrow M \otimes (\bigoplus_i N_i)$ be the canonical coprojections [in this section the tensor products are taken over A], and define the morphism $f : M \otimes (\bigoplus_i N_i) \rightarrow \bigoplus_i (M \otimes N_i)$ by the commutative diagram

$$\begin{array}{ccc} \bigoplus_i (M \otimes N_i) & \xrightarrow{f} & M \otimes (\bigoplus_i N_i) \\ \beta_i \uparrow & & \uparrow 1 \otimes \alpha_i \\ M \otimes N_i & \xlongequal{\quad} & M \otimes N_i. \end{array}$$

It suffices to show that f is bijective. We will define a morphism

$$g : M \otimes (\bigoplus_i N_i) \rightarrow \bigoplus_i (M \otimes N_i)$$

and show that it is inverse to f . Let x be in M . We define

$$g_x : \bigoplus_i N_i \rightarrow \bigoplus_i (M \otimes N_i)$$

by the commutative diagram

$$\begin{array}{ccc} \bigoplus_i N_i & \xrightarrow{g_x} & \bigoplus_i (M \otimes N_i) \\ \alpha_i \uparrow & & \uparrow \beta_i \\ N_i & \xrightarrow{g_{x,i}} & M \otimes N_i, \end{array}$$

where $g_{x,i}$ sends y to $x \otimes y$. Since $g_x(y)$ depends linearly on x , the family $(g_x)_{x \in M}$ induces the desired morphism g , and we leave it to the reader to check that g is inverse to f .

3.2.22 Page 34, Exercise 2.21

Statement. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I , and for each pair $i \leq j$ in I let $\alpha_{ij} : A_i \rightarrow A_j$ be a ring homomorphism, satisfying conditions (1) and (2) of Exercise 14. Regarding each A_i as a \mathbb{Z} -module we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \rightarrow A$ are ring homomorphisms. The ring A is the **direct limit** of the system (A_i, α_{ij}) .

If $A = 0$ prove that $A_i = 0$ for some $i \in I$. [Remember that all rings have identity elements!]

Solution. Follows from the hint and Exercise 2.15 [Section 3.2.16 p. 44].

3.2.23 Page 34, Exercise 2.22

Statement. Let $(A_i)_{i \in I}$ be a direct system of rings and let \mathfrak{N}_i be the nilradical of A_i . Show that $\varinjlim \mathfrak{N}_i$ is the nilradical of $\varinjlim A_i$.

If each A_i is an integral domain, then $\varinjlim A_i$ is an integral domain.

Solution. Follows from Exercise 2.15 [Section 3.2.16 p. 44].

3.2.24 Page 34, Exercise 2.23

Statement. Let $(B_\lambda)_{\lambda \in \Lambda}$ be a family of A -algebras. For each finite subset J of Λ let B_J denote the tensor product (over A) of the B_λ for $\lambda \in J$. If J' is another finite subset of Λ and $J \subset J'$, there is a canonical A -algebra homomorphism $B_J \rightarrow B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ . The ring B has a natural A -algebra structure for which the homomorphisms $B_J \rightarrow B$ are A -algebra homomorphisms. The A -algebra B is the **tensor product** of the family $(B_\lambda)_{\lambda \in \Lambda}$.

Solution. Follows from Exercise 2.21 [Section 3.2.22 p. 46].

Note. As mentioned in Section 3.1.6 p. 36 above, the A -algebra B defined in the statement of the exercise is the **coproduct** of the family $(B_\lambda)_{\lambda \in \Lambda}$. We repeat the reference to this Mathematics Stackexchange post of Eric Wofsey: <https://math.stackexchange.com/a/2441323/660>.

Here is an important consequence of the above note:

Theorem 18. *If A is a ring, then the category of A -algebras has all limits and all colimits.*

Proof (sketch). In view of the argument described in Section 3.2.14 p. 40 and Section 3.2.15 p. 42, it suffices to show that the following particular limits and colimits exist in the category of A -algebras: products, equalizers, coproducts and coequalizers. Products and equalizers can be computed in the category of sets. As just indicated, arbitrary coproducts are given by tensor products. The coequalizer of a pair of parallel morphisms $f, g : B \rightrightarrows C$ is furnished by the canonical projection $C \rightarrow C/\mathfrak{c}$ where \mathfrak{c} is the ideal generated by the elements of the form $f(b) - g(b)$. The details are left to the reader. \square

Comment. Exercises 2.21 and 2.22 of the book imply that $B = 0$ if and only if $B_{\lambda_1} \otimes_A \cdots \otimes_A B_{\lambda_n} = 0$ for some family $(\lambda_1, \dots, \lambda_n)$ of distinct elements of Λ .

3.2.25 Page 34, Exercise 2.24

Statement. If M is an A -module, the following are equivalent:

- (i) M is flat;
- (ii) $\mathrm{Tor}_n^A(M, N) = 0$ for all $n > 0$ and all A -modules N ;
- (iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A -modules N .

[To show that (i) \implies (ii), take a free resolution of N and tensor it with M . Since M is flat, the resulting sequence is exact and therefore its homology groups, which are the $\text{Tor}_n^A(M, N)$, are zero for $n > 0$. To show that (iii) \implies (i), let $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ be an exact sequence. Then, from the Tor exact sequence,

$$\text{Tor}_1^A(M, N'') \rightarrow M \otimes N' \rightarrow M \otimes N \rightarrow M \otimes N'' \rightarrow 0$$

is exact. Since $\text{Tor}_1^A(M, N'') = 0$ it follows that M is flat.]

Solution. Omitted.

3.2.26 Page 34, Exercise 2.25

Recall the statement of Exercise 2.25:

Proposition 19 (Exercise 2.25). *Let $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ be an exact sequence, with N'' flat. Then N' is flat if and only if N is flat.*

Here is a [very long!] solution to Exercise 2.25 which does **not** use the Tor functor. Of course, the solution using the Tor functor [and assuming the Tor functor and some of its basic properties are known] is much simpler.

The proof below follows closely the proof of Proposition 1.2.5.5 in Bourbaki's **Algèbre commutative**.

In this section, the ground ring is denoted by R , not by A .

Part 1: The Snake Lemma. Let

$$\begin{array}{ccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' \end{array} \quad (15)$$

be a commutative diagram of R -modules with exact rows.

Lemma 20. *If γ is injective, we have $\text{Im}(\beta) \cap \text{Im}(u') = \text{Im}(u' \circ \alpha) = \text{Im}(\beta \circ u)$.*

Proof. We clearly have $\text{Im}(u' \circ \alpha) = \text{Im}(\beta \circ u) \subset \text{Im}(\beta) \cap \text{Im}(u')$. Conversely, let $b' \in \text{Im}(\beta) \cap \text{Im}(u')$. There is a $b \in B$ such that $b' = \beta(b)$. As $v' \circ u' = 0$, we have $0 = v'(b') = v'(\beta(b)) = \gamma(v(b))$, whence $v(b) = 0$ since γ is injective. The first row of (15) being exact, there is an $a \in A$ such that $b = u(a)$, whence $b' = \beta(u(a))$. \square

Lemma 21. *If α is surjective, we have $\text{Ker}(\beta) + \text{Im}(u) = \text{Ker}(v' \circ \beta) = \text{Ker}(\gamma \circ v)$.*

Proof. As $v \circ u = 0$ and $v' \circ u' = 0$, it is clear that $\text{Ker}(\beta) + \text{Im}(u) \subset \text{Ker}(v' \circ \beta) = \text{Ker}(\gamma \circ v)$. Conversely, let $b \in \text{Ker}(v' \circ \beta)$. Then $\beta(b) \in \text{Ker}(v')$, and there is a $a' \in A'$ such that $u'(a') = \beta(b)$ since the bottom row of (15) is exact. As α is surjective, there is an $a \in A$ such that $\alpha(a) = a'$, whence $\beta(b) = u'(\alpha(a)) = \beta(u(a))$; this implies that $b - u(a)$ is in $\text{Ker}(\beta)$. \square

We extend the commutative diagram (15) as follows:

$$\begin{array}{ccccc}
\text{Ker}(\alpha) & \xrightarrow{u_1} & \text{Ker}(\beta) & \xrightarrow{v_1} & \text{Ker}(\gamma) \\
\downarrow & & \downarrow j & & \downarrow k \\
A & \xrightarrow{u} & B & \xrightarrow{v} & C \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' \\
\downarrow p & & \downarrow q & & \downarrow \\
\text{Coker}(\alpha) & \xrightarrow{u_2} & \text{Coker}(\beta) & \xrightarrow{v_2} & \text{Coker}(\gamma),
\end{array} \tag{16}$$

the new maps being the natural ones.

Recall that we are working under the assumption that the second and third rows of (16) are exact.

Lemma 22. *The sequence $\text{Ker}(\alpha) \xrightarrow{u_1} \text{Ker}(\beta) \xrightarrow{v_1} \text{Ker}(\gamma)$ is a complex. Moreover, if u' is injective, this complex is exact.*

Proof. The first claim is clear. We have $\text{Ker}(v_1) = \text{Ker}(\beta) \cap \text{Ker}(v) = \text{Ker}(\beta) \cap \text{Im}(u) = \text{Im}(j) \cap \text{Im}(u)$. Assume that u' is injective. As Lemma 20 implies $\text{Im}(j) \cap \text{Im}(u) = \text{Im}(j \circ u_1) = \text{Im}(u_1)$, we get $\text{Ker}(v_1) = \text{Im}(u_1)$. \square

Lemma 23. *The sequence $\text{Coker}(\alpha) \xrightarrow{u_2} \text{Coker}(\beta) \xrightarrow{v_2} \text{Coker}(\gamma)$ is a complex. Moreover, if v is surjective, this complex is exact.*

Proof. As u_2 and v_2 are obtained from u and v by taking quotients, it is clear that $v_1 \circ u_2 = 0$. Suppose v is surjective; q and p being surjective, we get, in view of the assumptions and Lemma 21,

$$\begin{aligned}
\text{Ker}(v_2) &= q(\text{Ker}(v_2 \circ q)) = q(\text{Ker}(v') + \text{Im}(\beta)) = q(\text{Ker}(v')) \\
&= q(\text{Im}(u')) = \text{Im}(q \circ u') = \text{Im}(u_2 \circ p) = \text{Im}(u_2).
\end{aligned}$$

(Lemma 21 is used to prove the second equality.) \square

Theorem 24 (Snake Lemma). *Assume that u' is injective and that v is surjective. Then the correspondence [see Bourbaki's **Théorie des ensembles**, Section II.3]*

$$\delta := p \circ u'^{-1} \circ \beta \circ v^{-1} \circ k$$

is an R -linear map. On other words, there is a unique R -linear map $\delta : \text{Ker}(\gamma) \rightarrow \text{Coker}(\alpha)$ having the following property: if $c \in \text{Ker}(\gamma)$, $b \in B$ and $a' \in A'$ satisfy $v(b) = k(c)$ and $u'(a') = \beta(b)$, then we have $\delta(c) = p(a')$. Moreover the sequence

$$\begin{array}{ccccc}
\text{Ker}(\alpha) & \xrightarrow{u_1} & \text{Ker}(\beta) & \xrightarrow{v_1} & \text{Ker}(\gamma) \\
& & & \searrow \delta & \\
\text{Coker}(\alpha) & \xrightarrow{u_2} & \text{Coker}(\beta) & \xrightarrow{v_2} & \text{Coker}(\gamma).
\end{array}$$

is exact.

Part 2.

Lemma 25. *If M is a flat R -module, if $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M \rightarrow 0$ is an exact sequence of R -modules, and if N is an R -module, then the sequence*

$$0 \rightarrow M_1 \otimes_R N \rightarrow M_2 \otimes_R N \rightarrow M \otimes_R N \rightarrow 0$$

is exact.

Proof. Let $0 \rightarrow P \rightarrow F \rightarrow N \rightarrow 0$ be an exact sequence of R -modules such that F is free, and form the diagram

$$\begin{array}{ccccccc} M_1 \otimes P & \longrightarrow & M_2 \otimes P & \longrightarrow & M \otimes P & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow g & & \\ 0 \longrightarrow & M_1 \otimes F & \longrightarrow & M_2 \otimes F & \longrightarrow & M \otimes F & \\ \downarrow & & \downarrow & & & & \\ M_1 \otimes N & \longrightarrow & M_2 \otimes N & & & & \\ \downarrow & & \downarrow & & & & \\ 0 & & 0, & & & & \end{array}$$

where the tensor products are taken over R , and where the maps are the natural ones. This diagram is clearly commutative and exact. By the Snake Lemma [Theorem 24] there is an exact sequence $\text{Ker}(f) \rightarrow \text{Ker}(g) \rightarrow M_1 \otimes N \rightarrow M_2 \otimes N$. As M is flat, g is injective. Thus $M_1 \otimes N \rightarrow M_2 \otimes N$ is also injective. \square

Clearly the following lemma implies Exercise 2.25 [stated as Proposition 19 p. 48].

Lemma 26. *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules with M'' flat. Then M' is flat if and only if M is flat.*

Proof. Let $N' \rightarrow N$ be a monomorphism of R -modules and form the commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 \longrightarrow & M' \otimes N' & \xrightarrow{f} & M \otimes N' & \longrightarrow & M'' \otimes N' & \longrightarrow 0 \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 0 \longrightarrow & M' \otimes N & \xrightarrow{g} & M \otimes N & \longrightarrow & M'' \otimes N & \longrightarrow 0, \end{array}$$

the maps being the natural ones. The diagram is exact by flatness of M'' and Lemma 25.

Assume that M is flat. Then β is injective, and so is $\beta \circ f = g \circ \alpha$. This shows that α is injective, and thus that M' is flat.

Assume that M' is flat. Then α is injective, and Lemma 22 implies that β is injective, and thus that M is flat.

The solution to Exercise 2.25 p. 34 of the book is now complete. \square

The goal is to show:

Theorem 27. *In the above setting, M is flat if and only if for all finitely generated ideal \mathfrak{a} the morphism $M \otimes \mathfrak{a} \rightarrow M$ sending $x \otimes a$ to xa is injective.*

Here is the sketch of a solution [following Bourbaki's **Algèbre commutative**].

Given A -modules M and N we say that M is **N -flat** if for all submodule N' of N the natural morphism $M \otimes N' \rightarrow M \otimes N$ is injective. [Here and in the sequel “ \otimes ” means “ \otimes_A ”.]

(a) If $M \otimes N' \rightarrow M \otimes N$ is injective for all **finitely generated** submodule N' of N , then M is N -flat.

Proof. Let N'' be an arbitrary submodule of N ; let x_1, \dots, x_n be in M ; let y_1, \dots, y_n be in N'' ; define $t'' \in M \otimes N''$ by $t'' = \sum x_i \otimes y_i$; define $t \in M \otimes N$ by $t = \sum x_i \otimes y_i$; and assume $t = 0$. It suffices to show $t'' = 0$. Let N' be the submodule of N'' generated by the y_i . By assumption the tensor $t' \in M \otimes N'$ defined by $t' = \sum x_i \otimes y_i$ vanishes. As the natural map $M \otimes N' \rightarrow M \otimes N''$ sends t' to t'' , we have indeed $t'' = 0$. \square

(b) If M is N -flat and if P is a submodule or a quotient of N , then M is P -flat.

Proof. The case of the submodules is left to the reader. Let $0 \rightarrow R \xrightarrow{i} N \xrightarrow{p} Q \rightarrow 0$ be exact, and let us show that M is Q -flat. Let Q' be a submodule of Q , and set $N' := p^{-1}(Q')$. We get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{i'} & N' & \xrightarrow{p'} & Q' & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow n & & \downarrow q & & \\ 0 & \longrightarrow & R & \xrightarrow{i} & N & \xrightarrow{p} & Q & \longrightarrow & 0, \end{array}$$

where r is the identity of R , and the maps i', p', n and q are the obvious ones. Letting $S \mapsto \bar{S}$ be the functor $M \otimes -$ we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} \bar{R} & \xrightarrow{\bar{i}'} & \bar{N}' & \xrightarrow{\bar{p}'} & \bar{Q}' & \longrightarrow & 0 \\ & & \downarrow \bar{n} & & \downarrow \bar{q} & & \\ \bar{R} & \xrightarrow{\bar{i}} & \bar{N} & \xrightarrow{\bar{p}} & \bar{Q} & \longrightarrow & 0, \end{array}$$

where \bar{r} is the identity of \bar{R} . As \bar{n} is injective, so is \bar{q} . \square

(c) If $N = \bigoplus_{i \in I} N_i$ is the direct sum of a family of submodules, and if M is N_i -flat for each i , then M is N -flat.

Proof. Let $(N_i)_{i \in I}$ be our family.

- First assume $I = \{1, 2\}$. Consider the **split** exact sequence $0 \rightarrow N_1 \xrightarrow{i} N_1 \oplus N_2 \xrightarrow{p} N_2 \rightarrow 0$, let N' be a submodule of N , set $N'_1 := N' \cap N_1$, let N'_2 be the image of N' in N_2 , and let

$$\begin{array}{ccccccc} 0 & \longrightarrow & N'_1 & \xrightarrow{i'} & N' & \xrightarrow{p'} & N'_2 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow f_2 \\ 0 & \longrightarrow & N_1 & \xrightarrow{i} & N & \xrightarrow{p} & N_2 \longrightarrow 0 \end{array}$$

be the obvious commutative diagram with exact rows. Writing again $S \mapsto \bar{S}$ for the functor $M \otimes -$, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccc} \bar{N}'_1 & \xrightarrow{\bar{i}'} & \bar{N}' & \xrightarrow{\bar{p}'} & \bar{N}'_2 \\ \downarrow \bar{f}_1 & & \downarrow \bar{f} & & \downarrow \bar{f}_2 \\ \bar{N}_1 & \xrightarrow{\bar{i}} & \bar{N} & \xrightarrow{\bar{p}} & \bar{N}_2. \end{array}$$

It is easy to see that \bar{f}_1, \bar{f}_2 and \bar{i} are injective, and that this implies that \bar{f} is also injective. This completes the proof of (c) in the case $I = \{1, 2\}$.

- Second assume that I is **finite**. We prove the statement by induction.
- Third assume that I is **arbitrary**, and let N' be a finitely generated submodule of $N = \bigoplus_{i \in I} N_i$. Then there is a finite subset J of I such that $N' \subset \bigoplus_{j \in J} N_j$, and we have

$$N = \left(\bigoplus_{j \in J} N_j \right) \oplus \left(\bigoplus_{i \in I \setminus J} N_i \right).$$

We leave the rest of the proof to the reader.

From this point, the proof of Theorem 27 p. 52 is straightforward. The details are again left to the reader.

□

3.2.28 Page 35, flat modules (a digression)

Taken from Chapter 1, Section 2, Subsection 11 in Nicolas Bourbaki, **Algèbre commutative: Chapitres 1 à 4**, Masson, Paris 1985:

Let A be a [non necessarily commutative] associative ring with 1, let M be a right A -module, and let N be a left A -module. Then M is N -flat [see §3.2.27] if and only if the following condition holds:

For all finite families $(x_i) \subset M, (y_i) \subset N$ such that $\sum x_i \otimes y_i = 0$ there are finite families $(a_{ij}) \subset A$ and $(x'_j) \subset M$ such that $\sum_j x'_j a_{ij} = x_i$ for all i and $\sum_i a_{ij} y_i = 0$ for all j .

Proof. This follows easily from §3.1.3. □

We also have:

Let A be a [non necessarily commutative] associative ring with 1 and M a right A -module. Then M is flat if and only if the following condition holds:

For all finite families $(x_i) \subset M, (a_i) \subset A$ such that $\sum x_i a_i = 0$ there are finite families $(a'_{ij}) \subset A$ and $(x'_j) \subset M$ such that $\sum_j x'_j a'_{ij} = x_i$ for all i and $\sum_i a'_{ij} a_i = 0$ for all j .

Proof. We set $N := A$ in the previous statement and use §3.2.27. \square

There is also a statement and a proof of this result in the Stacks Project; see the tag <http://stacks.math.columbia.edu/tag/00HK>

3.2.29 Page 35, Exercise 2.27

Statement. A ring A is **absolutely flat** if every A -module is flat. Prove that the following are equivalent:

- (i) A is absolutely flat.
- (ii) Every principal ideal is idempotent.
- (iii) Every finitely generated ideal is a direct summand of A .

[(i) \implies (ii). Let $x \in A$. Then $A/(x)$ is a flat A -module, hence in the diagram

$$\begin{array}{ccc} (x) \otimes A & \xrightarrow{\beta} & (x) \otimes A/(x) \\ \downarrow & & \downarrow \alpha \\ A & \longrightarrow & A/(x) \end{array}$$

the mapping α is injective. Hence $\text{Im}(\beta) = 0$, hence $(x) = (x^2)$. (ii) \implies (iii). Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and we have $(e) = (x)$. Now if e, f are idempotents, then $(e, f) = (e + f - ef)$. Hence every finitely generated ideal is principal, and generated by an idempotent e , hence is a direct summand because $A = (e) \oplus (1 - e)$. (iii) \implies (i). Use the criterion of Exercise 26 (see Theorem 27 p. 52).]

Solution.

• (i) \implies (ii): Let x be in A . Tensoring the exact sequence $0 \rightarrow (x) \rightarrow A$ with the flat module $A/(x)$ yields the exact sequence $0 \rightarrow (x)/(x)^2 \xrightarrow{f} A/(x)$. We clearly have $\text{Im } f = 0$, hence $f = 0$, hence $(x)^2 = (x)$.

Note: On replacing the principal ideal (x) with an arbitrary ideal this shows that **all** ideals of an absolutely flat ring are idempotent. Here is a naive way of proving that “all **principal** ideals of A are idempotent” implies “**arbitrary** ideals of A are idempotent”: Otherwise there would be an x in $\mathfrak{a} \setminus \mathfrak{a}^2$ for some ideal \mathfrak{a} , which would yield $x \in (x^2) \subset \mathfrak{a}^2$, contradiction.

- (ii) \implies (iii): The hint given in the book is complete.
- (iii) \implies (i): Let M be an A -module and \mathfrak{a} a finitely generated ideal. By Theorem 27 p. 52 above it suffices to check that the natural morphism $\mathfrak{a} \otimes M \rightarrow M$ is injective [in this section tensor products

are taken over A]. This morphism is the composite of the obvious morphisms

$$\mathfrak{a} \otimes M \rightarrow (\mathfrak{a} \oplus \mathfrak{b}) \otimes M = A \otimes M \rightarrow M,$$

where \mathfrak{b} is an ideal such that $A = \mathfrak{a} \oplus \mathfrak{b}$ [such an ideal exists by assumption]. These morphisms are clearly injective.

Note 28. *The above arguments show that (i), (ii) and (iii) are also equivalent to*

(iv) every finitely generated ideal is generated by an idempotent,

and also to

(v) for all a in A there is an x in A such that $a = a^2x$.

In particular an absolutely flat ring has no nonzero nilpotent element.

Each of the following three conditions is equivalent to (i):

(vi) each ideal is idempotent,

(vii) each principal ideal is radical,

(viii) each ideal is radical.

Indeed, we already know that (i) and (vi) are equivalent [see the note after the proof that (i) implies (ii)], and that (viii) implies trivially (vii). To prove that (ii) implies (viii), assume $a^n \in \mathfrak{a}$ for some element a of A , some positive integer n and some ideal \mathfrak{a} of A , and observe that we get $a \in (a) = (a^n) \subset \mathfrak{a}$. To show that (vii) implies (ii), let a be in A and note that we have $a \in r(a^2) = (a^2)$.

3.2.30 Page 35, Exercise 2.28

Statement. A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 [Section 2.2.8 p. 21] is absolutely flat. Every homomorphic image of an absolutely flat ring is absolutely flat. If a local ring is absolutely flat, then it is a field.

If A is absolutely flat, every non-unit in A is a zero-divisor.

Solution. The first three sentences follow from Property (v) above [Section 3.2.29]. The fact that an absolutely flat local ring is a field follows immediately from Property (iv) above and Exercise 1.12 p. 11 of the book [see Section 2.2.13 p. 22 above].

4 About Chapter 3

4.1 Comments

4.1.1 Page 37

Recall that $f : A \rightarrow S^{-1}A$ is the canonical morphism.

It is written:

"Conversely, these three conditions determine the ring $S^{-1}A$ up to isomorphism. Precisely:

Corollary 3.2. *If $g : A \rightarrow B$ is a ring homomorphism such that*

- (i) $s \in S \implies g(s)$ is a unit in B ;
- (ii) $g(a) = 0 \implies as = 0$ for some $s \in S$;
- (iii) Every element of B is of the form $g(a)g(s)^{-1}$.

Then there is a unique isomorphism $h : S^{-1}A \rightarrow B$ such that $g = h \circ f$."

The following wording would be slightly better:

Conversely, these three conditions determine the ring $S^{-1}A$ up to unique isomorphism. Precisely:

Corollary 3.2. *If $g : A \rightarrow B$ is a ring homomorphism such that*

- (i) $s \in S \implies g(s)$ is a unit in B ;
- (ii) $g(a) = 0 \implies as = 0$ for some $s \in S$;
- (iii) Every element of B is of the form $g(a)g(s)^{-1}$.

Then there is a unique morphism $h : S^{-1}A \rightarrow B$ such that $g = h \circ f$. Moreover this morphism is an isomorphism.

4.1.2 Page 39, Proposition 3.4

It is easy to see that we have $S^{-1} \sum M_i = \sum S^{-1}M_i$. It is also easy to see that we have

$$S^{-1} \bigcap M_i \subset \bigcap S^{-1}M_i.$$

But the converse inclusion

$$\bigcap S^{-1}M_i \subset S^{-1} \bigcap M_i$$

is not true in general. Here is a counterexample. Let K be a field and x an indeterminate. Setting $A := K[x]$, $S := K[x] \setminus \{0\}$ we get

$$\bigcap_n S^{-1}(x^n) = K(x), \quad S^{-1} \bigcap_n (x^n) = (0).$$

4.1.3 Pages 39-40, Propositions 3.5 and 3.7

Recall the statements of these propositions:

Proposition 29 (Proposition 3.5 of the book). *Let M be an A -module. Then the $S^{-1}A$ modules $S^{-1}M$ and $S^{-1}A \otimes_A M$ are isomorphic; more precisely, there exists a unique isomorphism $f : S^{-1}A \otimes_A M \rightarrow S^{-1}M$ for which*

$$f((a/s) \otimes m) = am/s \text{ for all } a \in A, m \in M, s \in S.$$

Proposition 30 (Proposition 3.7 of the book). *If M, N are A -modules, there is a unique isomorphism of $S^{-1}A$ -modules*

$$f : S^{-1}M \otimes_{S^{-1}A} S^{-1}N \rightarrow S^{-1}(M \otimes_A N)$$

such that

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st.$$

In particular, if \mathfrak{p} is any prime ideal, then

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \simeq (M \otimes_A N)_{\mathfrak{p}}$$

as $A_{\mathfrak{p}}$ -modules.

The natural A -linear map $f : M \rightarrow S^{-1}M$ has the following universal property:

For all A -linear map $g : M \rightarrow N$ from M to an $S^{-1}A$ -module N such that $sm = 0$ for some s in S and some m in M implies $g(m) = 0$, there is a unique $S^{-1}A$ -linear map $h : S^{-1}M \rightarrow N$ such that $g = h \circ f$:

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ f \downarrow & \nearrow h & \\ S^{-1}M & & \end{array}$$

Using this universal property one can describe explicitly the respective inverses of the isomorphisms in Propositions 3.5 and 3.7.

4.1.4 Page 39, Proposition 3.5

In the setting of Proposition 3.5 p. 39 [Proposition 29] we have:

If $1 \otimes x = 0$ in $S^{-1}A \otimes_A M$ then we have $sx = 0$ for some $s \in S$.

Proof. We have $1 \otimes x = 0$ in $N \otimes_A M$ where N is some finitely generated sub- A -module of $S^{-1}A$. For any $s \in S$ set $A/s := \{a/s \mid a \in A\}$; this is also a sub- A -module of $S^{-1}A$. Then $N \subset A/s$ for some s , and we have $1 \otimes x = 0$ in $A/s \otimes_A M$. Defining $f : A \rightarrow A/s$ by $f(a) := a/s$ and letting B be the kernel of f , and $g : M \rightarrow A/s \otimes_A M$ the map induced by f , we get exact sequences

$$B \otimes_A M \rightarrow M \xrightarrow{g} A/s \otimes_A M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow BM \rightarrow M \xrightarrow{g} A/s \otimes_A M \rightarrow 0.$$

We have $g(sx) = 0$ and thus $sx = \sum_{i=1}^n b_i x_i$ with $b_i \in B$ and $x_i \in M$. As $b_i/s = 0$ in $S^{-1}A$ there is a $t \in S$ such that $tb_i = 0$ for all i . This gives $tsx = 0$. \square

4.1.5 Page 40, Proposition 3.8

Recall the statement of Proposition 3.8.

Proposition 31 (Proposition 3.8 p. 40). *Let M be an A -module. Then the following are equivalent:*

- (i) $M = 0$;
- (ii) $M_{\mathfrak{p}} = 0$ for all prime ideals \mathfrak{p} of A ;
- (iii) $M_{\mathfrak{m}} = 0$ for all maximal ideals \mathfrak{m} of A .

Here is a corollary:

Corollary 32. *For $a \in A$ and $\mathfrak{p} \in \text{Spec}(A)$ write $a_{\mathfrak{p}}$ for the element $a/1 \in A_{\mathfrak{p}}$. Then we have for $a \in A$:*

$$a = 0 \iff a_{\mathfrak{p}} = 0 \ \forall \ \mathfrak{p} \in \text{Spec}(A) \iff a_{\mathfrak{m}} = 0 \ \forall \ \mathfrak{m} \text{ maximal.}$$

This follows from Proposition 3.8 and the easy equality $(a)_{\mathfrak{p}} = (a_{\mathfrak{p}})$.

4.1.6 Page 40, Proposition 3.9

Here is a mild generalization:

The complex of A -modules $M \xrightarrow{f} N \xrightarrow{g} P$ is exact if and only if the complex $M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} P_{\mathfrak{m}}$ is exact for all maximal ideal \mathfrak{m} of A .

Proof. Use the isomorphism $\text{Ker}(g_{\mathfrak{m}})/\text{Im}(f_{\mathfrak{m}}) \simeq (\text{Ker}(g)/\text{Im}(f))_{\mathfrak{m}}$ and Proposition 3.8 of the book. \square

4.1.7 Pages 41-42, Proposition 3.11

Recall the proposition:

Proposition 33 (Proposition 3.11 p. 41 of the book). (i) *Every ideal in $S^{-1}A$ is an extended ideal.*

(ii) *If \mathfrak{a} is an ideal in A , then $\mathfrak{a}^{\text{ec}} = \bigcup_{s \in S} (\mathfrak{a} : s)$. Hence $\mathfrak{a}^e = (1)$ if and only if \mathfrak{a} meets S .*

(iii) $\mathfrak{a} \in C \iff$ *no element of S is a zero-divisor in A/\mathfrak{a} .*

(iv) *The prime ideals of $S^{-1}A$ are in one-to-one correspondence ($\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p}$) with the prime ideals of A which don't meet S .*

(v) *The operation S^{-1} commutes with formation of finite sums, products, intersections and radicals.*

Here are some additional comments about Parts (ii) and (iv).

Part (ii).

- Here is a more detailed proof of the statement

If \mathfrak{a} is an ideal in A , then $\mathfrak{a}^{\text{ec}} = \bigcup_{s \in S} (\mathfrak{a} : s)$.

We have

$$\begin{aligned}
x \in \mathfrak{a}^{\text{ec}} &= (S^{-1}\mathfrak{a})^c \\
&\iff \frac{x}{1} = \frac{a}{s'} \text{ for some } a \in \mathfrak{a}, s' \in S \\
&\iff (xs' - a)t' = 0 \text{ for some } a \in \mathfrak{a} \text{ and some } s', t' \in S \\
&\iff xst \in \mathfrak{a} \text{ for some } s, t \in S \\
&\iff xs \in \mathfrak{a} \text{ for some } s \in S \\
&\iff x \in \bigcup_{s \in S} (\mathfrak{a} : s).
\end{aligned}$$

To prove

$$xst \in \mathfrak{a} \text{ for some } s, t \in S \implies (xs' - a)t' = 0 \text{ for some } a \in \mathfrak{a} \text{ and some } s', t' \in S,$$

we set $a := xst$, $s' := st$, $t' := 1$.

- A particular case of the statement

$$\mathfrak{a}^e = (1) \text{ if and only if } \mathfrak{a} \text{ meets } S$$

is

$$(0)^e = (1) \text{ if and only if } 0 \in S,$$

that is

$$S^{-1}A = 0 \iff 0 \in S. \tag{17}$$

[It is easy to prove this particular case directly.]

Part (iv). Set $X := \text{Spec}(A)$, $Y := \text{Spec}(S^{-1}A)$ and let $c : Y \rightarrow X$ be the contraction map. In view of Proposition 1.17iii p. 10 of the book, it suffices to show: $c(Y) = \{\mathfrak{p} \in X \mid \mathfrak{p} \cap S = \emptyset\}$ and $\mathfrak{p} \in c(Y) \implies S^{-1}\mathfrak{p} \in X$. The conclusion is that

The contraction and extension maps are inverse inclusion preserving bijections between Y and $c(Y)$.

4.1.8 Page 43, Proposition 3.14

The inclusion $S^{-1}\text{Ann } M \subset \text{Ann } S^{-1}M$ holds even if M is not finitely generated. Here is a counterexample to the reverse inclusion:

Let K be a field and x an indeterminate. Set

$$A := K[x], \quad S := K[x] \setminus \{0\}, \quad M := \bigoplus A/(x^n).$$

We get $\text{Ann } M = (0)$, $S^{-1}M = 0$, $\text{Ann } S^{-1}M = (1)$.

4.1.9 Page 43, Proposition 3.16

Here is the statement:

Proposition 34 (Proposition 3.16). *Let $A \rightarrow B$ be a ring homomorphism and let \mathfrak{p} be a prime ideal of A . Then \mathfrak{p} is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{\text{ec}} = \mathfrak{p}$.*

Compare with Exercise 3.21iv p. 47 of the book [Section 4.2.21 p. 72 below].

4.1.10 Proof of Proposition 3.16

The commutative diagram

$$\begin{array}{ccc} \mathfrak{p} \subsetneq A & \xrightarrow{f} & B \supsetneq \mathfrak{p}B \\ \alpha \downarrow & & \downarrow \beta \\ \mathfrak{p}_{\mathfrak{p}} \subsetneq A_{\mathfrak{p}} & \xrightarrow{f_{\mathfrak{p}}} & B_{\mathfrak{p}} \supsetneq \mathfrak{m} \supset \mathfrak{p}B_{\mathfrak{p}} \end{array}$$

might help. Note that we have $f_{\mathfrak{p}}^{-1}(\mathfrak{m}) = \mathfrak{p}_{\mathfrak{p}}$.

4.1.11 Strengthening of Proposition 3.16

Proposition 3.16 can also be stated as follows:

Let $A \rightarrow B$ be a morphism of commutative rings, let \mathfrak{a} be a contracted ideal in A , and let Σ be the set of those ideals in B which contract to \mathfrak{a} . [In particular Σ is nonempty.] Order Σ by inclusion. Then we have

- (a) \mathfrak{a}^e is the least element of Σ , or, equivalently, \mathfrak{a}^e is the intersection of all the elements of Σ ,
- (b) Σ has one, or more, maximal elements,
- (c) if \mathfrak{a} is prime, then any maximal element of Σ is also prime.

The proofs of these statements are straightforward and elementary. We will prove (c), the proofs of (a) and (b) being similar and left to the reader.

To prove (c), let \mathfrak{q} be a maximal element of Σ . Assume by contradiction that \mathfrak{q} is not prime. Then there are ideals $\mathfrak{b}, \mathfrak{b}'$ in B such that $\mathfrak{q} \not\supset \mathfrak{b}$, $\mathfrak{q} \not\supset \mathfrak{b}'$, $\mathfrak{q} \supset \mathfrak{b}\mathfrak{b}'$. Replacing \mathfrak{b} and \mathfrak{b}' with $\mathfrak{b} + \mathfrak{q}$ and $\mathfrak{b}' + \mathfrak{q}$, we can assume that \mathfrak{q} is a proper sub-ideal of \mathfrak{b} and \mathfrak{b}' . By maximality of \mathfrak{q} , the prime ideal \mathfrak{a} is a proper sub-ideal of \mathfrak{b}^c and \mathfrak{b}'^c . By Exercise 1.18 p. 10 [Section 2.1.7 p. 16] we also have $\mathfrak{b}^c\mathfrak{b}'^c \subset (\mathfrak{b}\mathfrak{b}')^c \subset \mathfrak{q}^c = \mathfrak{a}$, in contradiction with the primality of \mathfrak{a} .

4.1.12 Related result

Here is a related result:

$$\mathfrak{p} \text{ is the contraction of a prime ideal if and only if } A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \otimes_A B \neq 0. \quad (18)$$

This will follow from Claim 1 and Claim 2 below.

Claim 1. Let C_1, \dots, C_5 be the five B -algebras

$$A_{\mathfrak{p}} \otimes_A B/\mathfrak{p}B, \quad (B/\mathfrak{p}B)_{\mathfrak{p}}, \quad B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}, \quad \frac{A_{\mathfrak{p}}}{\mathfrak{p}_{\mathfrak{p}}} \otimes_A B, \quad (A/\mathfrak{p})_{\mathfrak{p}} \otimes_A B.$$

Then for any $1 \leq i, j \leq 5$ there is a unique B -algebra morphism $C_i \rightarrow C_j$, and this morphism is bijective. ???

Proof of Claim 1. Note that C_i has a unique $B_{\mathfrak{p}}$ -algebra structure which extends its natural B -algebra structure. In other words, the image in C_i of any $s \in A \setminus \mathfrak{p}$ is a unit. Moreover, any element of C_i is equal to $\frac{b}{s} \cdot 1$ for some b in B and s in $A \setminus \mathfrak{p}$. This implies that there is at most one morphism of B -algebras $\phi_{ji} : C_i \rightarrow C_j$. To prove that such morphisms exist, we proceed as follows. If U is an additive group and V a subgroup, we denote the class in U/V of $u \in U$ by \bar{u} when V is clear from the context. For $i = 1, \dots, 5$ define the set theoretical map $f_i : A \times (A \setminus \mathfrak{p}) \times B \rightarrow C_i$ by

$$(f_1(a, s, b), \dots, f_5(a, s, b)) := \left(\frac{a}{s} \otimes \bar{b}, \frac{\overline{a \cdot b}}{s}, \left(\frac{a \cdot b}{s} \right)^-, \left(\frac{a}{s} \right)^- \otimes b, \frac{\bar{a}}{s} \otimes b \right).$$

We leave it to the reader to check that, for $i, j = 1, \dots, 5$, there is a B -algebra morphism $\phi_{ji} : C_i \rightarrow C_j$ such that $\phi_{ji}(f_i(a, s, b)) = f_j(a, s, b)$ for all a, s, b .

Note that C_i is an initial object in the category of those B -algebras C such that the image of $a \in A$ in C is zero if $a \in \mathfrak{p}$ and is a unit if $a \notin \mathfrak{p}$.

Claim 2. We have $(B/\mathfrak{p}^e)_{\mathfrak{p}} = 0 \iff \mathfrak{p}^{\text{ec}} \neq \mathfrak{p}$.

Proof of Claim 2.

$$(B/\mathfrak{p}^e)_{\mathfrak{p}} = 0 \iff \frac{1}{1} = \frac{0}{1} \text{ in } (B/\mathfrak{p}^e)_{\mathfrak{p}} \iff \exists s \in A \setminus \mathfrak{p} \mid f(s) \in \mathfrak{p}^e \iff \exists s \in \mathfrak{p}^{\text{ec}} \setminus \mathfrak{p} \iff \mathfrak{p}^{\text{ec}} \neq \mathfrak{p}.$$

Statement (18) follows also from Exercise 3.21iv p. 47 of the book [see Section 4.2.21 p. 72 below].

4.2 Exercises

4.2.1 Page 43, Exercise 3.1

Statement. Let S be a multiplicatively closed subset of a ring A , and let M be a finitely generated A -module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution. If there exists $s \in S$ such that $sM = 0$ and if $x \in M$, then we have $\frac{x}{1} = \frac{sx}{s} = 0$. If $S^{-1}M = 0$ and x_1, \dots, x_n generate M , then for all i there is an $s_i \in S$ such that $s_i x_i = 0$, and the product $s_1 \cdots s_n$ annihilates M .

4.2.2 Page 43, Exercise 3.2

Statement. Let \mathfrak{a} be an ideal of a ring A , and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma [Proposition 14 p. 34] to give a proof of (2.5) [Corollary 10 p. 10] which does not depend on determinants. [If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Hint. $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$: For $\alpha, \beta, \gamma \in \mathfrak{a}$ and $a \in A$ we have

$$1 - \frac{\alpha}{1 + \beta} \frac{a}{1 + \gamma} = \frac{1 + \beta + \gamma + \beta\gamma - \alpha a}{1 + \beta + \gamma + \beta\gamma}$$

with $1 + \beta + \gamma + \beta\gamma - \alpha a \in S$.

4.2.3 Page 43, Exercise 3.3

Statement. Let A be a ring, let S and T be two multiplicatively closed subsets of A , and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic.

Solution. We have $U = \{\frac{t}{1} \in S^{-1}A \mid t \in T\}$.

We define the A -algebra morphism

$$f : (ST)^{-1}A \rightarrow U^{-1}(S^{-1}A)$$

as follows. Let f' be the composite of the natural morphisms $A \rightarrow S^{-1}A \rightarrow U^{-1}(S^{-1}A)$, and let $s \in S, t \in T$. To define f it suffices to show that $f'(st)$ is a unit. Clearly the inverse of

$$f'(st) = \frac{\frac{st}{1}}{\frac{1}{1}}$$

is

$$\frac{\frac{1}{s}}{\frac{t}{1}}.$$

We define the A -algebra morphism

$$g : U^{-1}(S^{-1}A) \rightarrow (ST)^{-1}A$$

as follows. We first define the A -algebra morphism $g' : S^{-1}A \rightarrow (ST)^{-1}A$ by noting that the image of the element $s \in S$ under the natural A -algebra morphism $A \rightarrow (ST)^{-1}A$ is a unit. Let t be in T . To complete the definition of g it suffices to show that $g'(\frac{t}{1}) = \frac{t}{1}$ is a unit, which is clear.

We then have

$$f\left(\frac{a}{st}\right) = \frac{\frac{a}{s}}{\frac{t}{1}} \quad \text{and} \quad g\left(\frac{\frac{a}{s}}{\frac{t}{1}}\right) = \frac{a}{st}$$

for all $a \in A, s \in S, t \in T$, and it is clear that f and g are inverse A -algebra isomorphisms.

4.2.4 Page 44, Exercise 3.4

Statement. Let $f : A \rightarrow B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A . Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$ -modules.

Hint. $S^{-1}B$ is the quotient of $B \times S$ by a certain equivalence relation \sim , and $f(S)^{-1}B$ is the quotient of $B \times f(S)$ by a certain equivalence relation \approx . Define $g : B \times S \rightarrow B \times f(S)$ by $g(b, s) = (b, f(s))$. One easily checks that, for $b, c \in B$ and $s, t \in S$ the conditions $(b, s) \sim (c, t)$ and $(b, (f(s))) \approx (c, f(t))$ are equivalent. This implies that there is a bijection $h : S^{-1}B \rightarrow T^{-1}B$ such that $h(\frac{b}{s}) = \frac{b}{f(s)}$ for all $b \in B, s \in S$. It is straightforward to verify that h is $S^{-1}A$ -linear.

4.2.5 Page 44, Exercise 3.5

Statement. Let A be a ring. Suppose that, for each prime ideal \mathfrak{p} , the local ring $A_{\mathfrak{p}}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_{\mathfrak{p}}$ is an integral domain, is A necessarily an integral domain?

Hint. First part: Use Corollary 32 p. 58 above. Alternatively, use Proposition 31 p. 58 and Corollary 3.12 p. 42 of the book². Second part: take the zero ring [or a product of two fields].

4.2.6 Page 44, Exercise 3.6

Statement. Let A be a ring $\neq 0$ and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A \setminus S$ is a minimal prime ideal of A .

Hint. The union of a chain in Σ belongs to Σ . If $S \in \Sigma$ is maximal, then $S^{-1}A \neq 0$. In particular A has a prime ideal \mathfrak{p} disjoint from S , and $A \setminus \mathfrak{p}$ is an element of Σ containing S , hence equal to S by maximality of S .

4.2.7 Page 44, Exercise 3.7

Preliminaries. Here are a few comments one can make at the outset.

Note 35. Let S and T be two multiplicative subsets of A . Then there is at most one A -algebra morphism from $S^{-1}A \rightarrow T^{-1}A$. Moreover such a morphism exists if and only if for each $s \in S$ the image of s in $T^{-1}A$ is a unit.

Note 36. Let S be a multiplicative subset of A . Then the following five subsets of A are equal:

- the intersection of all saturated multiplicative subsets of A containing S ,
- the set of all those elements of A whose image in $S^{-1}A$ is a unit,
- the complement in A of the union of the prime ideals of A which are disjoint from S ,
- the set of all those elements $a \in A$ such that $ab \in S$ for some $b \in A$.

Moreover this set is the least saturated multiplicative subset of A containing S .

This set is called the **saturation** of S and is denoted by \overline{S} .

²Statement of the corollary: If \mathfrak{N} is the nilradical of A , the nilradical of $S^{-1}A$ is $S^{-1}\mathfrak{N}$.

Note 37. The unique A -algebra morphism $S^{-1}A \rightarrow \overline{S}^{-1}A$ is bijective. Moreover, if T is another multiplicative subset, then there is a [necessarily unique] A -algebra morphism $S^{-1}A \rightarrow T^{-1}A$ if and only if $\overline{S} \subset T$.

Note 38. Let A be a ring, let \mathcal{A} be the category whose objects are the A -algebras of the form $S^{-1}A$, where S is a multiplicative subset of A , and whose morphisms are the A -algebra morphisms, let \mathcal{B} be the category whose objects are the subsets of $\text{Spec}(A)$ and whose morphisms are defined by

$$\text{Hom}_{\mathcal{B}}(Y, Z) = \begin{cases} \{\emptyset\} & \text{if } Z \subset Y \\ \emptyset & \text{otherwise} \end{cases}$$

[that is \mathcal{B}^{op} is the poset given by power set of $\text{Spec}(A)$, poset viewed as a category], and let $F : \mathcal{A} \rightarrow \mathcal{B}$ the functor $S^{-1}A \mapsto \text{Spec}(S^{-1}A)$, where $\text{Spec}(S^{-1}A)$ is viewed as the set of those prime ideals of A disjoint from S . Then F is an equivalence.

Statement. A multiplicatively closed subset S of a ring A is said to be **saturated** if

$$xy \in S \iff x \in S \text{ and } y \in S.$$

Prove that

- (i) S is saturated $\iff A \setminus S$ is a union of prime ideals.
- (ii) If S is any multiplicatively closed subset of A , there is a unique smallest saturated multiplicatively closed subset \overline{S} containing S , and that \overline{S} is the complement in A of the union of the prime ideals which do not meet S . (\overline{S} is called the **saturation** of S .)

If $S = 1 + \mathfrak{a}$, where \mathfrak{a} is an ideal of A , find \overline{S} .

Solution. Let U be the union of the prime ideals which do not meet S .

- (i) Implication \Leftarrow is easy. Let us prove \Rightarrow . Assume by contradiction that there is an $a \in A \setminus S$ which is not in U . Then $a/1$ belongs to no prime ideal of $S^{-1}A$, and is therefore a unit of $S^{-1}A$. Hence there are $b \in A$ and $s \in S$ such that

$$\frac{ab}{s} = \frac{a}{1} \frac{b}{s} = \frac{1}{1}.$$

This implies $abt \in S$ for some $t \in S$, contradicting the saturation of S .

- (ii) Follows from Section 4.2.7 p. 63 above.

- (iii) \overline{S} is the complement of the union of the maximal ideals containing \mathfrak{a} .

Proof. Let M be the union of the maximal ideals containing \mathfrak{a} , and let P be the union of the prime ideals disjoint from $1 + \mathfrak{a}$:

$$M := \bigcup_{\mathfrak{m} \supset \mathfrak{a}} \mathfrak{m}, \quad P := \bigcup_{\mathfrak{p} \cap (1+\mathfrak{a}) = \emptyset} \mathfrak{p}.$$

It suffices to show $M = P$.

To prove $M \subset P$, assume $\mathfrak{m} \supset \mathfrak{a}$, \mathfrak{m} maximal. It is enough to check $\mathfrak{m} \cap (1 + \mathfrak{a}) = \emptyset$. If there was an x in $\mathfrak{m} \cap (1 + \mathfrak{a})$, there would be an a in \mathfrak{a} with $x = 1 + a$, which would imply $1 \in \mathfrak{m}$, contradiction.

Let us verify $P \subset M$. Assume $\mathfrak{p} \cap (1 + \mathfrak{a}) = \emptyset$, with \mathfrak{p} prime. It suffices to show $\mathfrak{p} \subset M$. We claim $\mathfrak{p} + \mathfrak{a} \neq (1)$. If not we would have $p + a = 1$ with $p \in \mathfrak{p}$ and $a \in \mathfrak{a}$, and thus

$$p = 1 - a \in \mathfrak{p} \cap (1 + \mathfrak{a}) = \emptyset.$$

As $\mathfrak{p} + \mathfrak{a} \neq (1)$, there is a maximal ideal containing \mathfrak{p} and \mathfrak{a} . This implies $\mathfrak{p} \subset M$, as announced.

4.2.8 Page 44, Exercise 3.8

Statement. Let S, T be multiplicatively closed subsets of A , such that $S \subset T$. Let $\phi : S^{-1}A \rightarrow T^{-1}A$ be the homomorphism which maps each $a/s \in S^{-1}A$ to a/s considered as an element of $T^{-1}A$. Show that the following statements are equivalent:

- (i) ϕ is bijective.
- (ii) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.
- (iii) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.
- (iv) T is contained in the saturation of S (Exercise 7).
- (v) Every prime ideal which meets T also meets S .

Solution.

- (i) \implies (ii): The element $x = t/1$ of $S^{-1}A$ is a unit if and only if $\phi(x)$ is a unit of $T^{-1}A$, which is true because $\phi(x)$ is equal to $t/1 \in T^{-1}A$.
- (ii) \implies (iii): If a/s is the inverse of $t/1 \in S^{-1}A$, we have $\frac{a}{s} \frac{t}{1} = \frac{1}{1}$ in $S^{-1}A$, that is $as't = ss' \in S$ for some $s' \in S$.
- (iii) \implies (iv): Obvious.
- (iv) \implies (v): This follows from Part (ii) of the previous exercise.
- (v) \implies (ii): Let $t \in T$ be such that $t/1 \in T$ is not a unit in $S^{-1}A$. Then $t/1$ belongs to some prime ideal of $S^{-1}A$, that is, t belongs to some prime ideal of A disjoint from S .
- (ii) \implies (i): There is a [necessarily unique] A -algebra morphism $\psi : T^{-1}A \rightarrow S^{-1}A$ such that $\psi(a/t) = \frac{a}{1} \left(\frac{t}{1}\right)^{-1}$, and ψ is the inverse of ϕ .

4.2.9 Page 44, Exercise 3.9

See Section 2.2.15 p. 23. — Actually there are two proofs of the fact that the set of zero-divisors is a union of prime ideals: one is Exercise 1.14 p. 12, the other is Exercise 3.7 p. 44 [see Section 4.2.7 p 64].

- To show that a minimal prime ideal \mathfrak{p} consists of zero-divisors, set $S := A \setminus \mathfrak{p}$. Then S is a maximal element of the set denoted Σ in Exercise 3.6 [see Section 4.2.6 p. 63]. It suffices to prove $S_0 \subset S$. Let s_0 be in S_0 . If s_0 was not in S , then s_0 and S would generate a multiplicative set not containing 0 which is larger than S , contradiction.

- (ii) Assume that $a/s \in S_0^{-1}A$ is neither a unit nor a zero-divisor. We have $a \notin S_0$. Then there is a nonzero b in A with $ab = 0$, and we get $\frac{a}{s} \frac{b}{1} = 0$, hence $\frac{b}{1} = 0$, that is $bs_0 = 0$ for some $s_0 \in S_0$. This implies $b = 0$, contradiction.

4.2.10 Page 44, Exercise 3.10

Statement. Let A be a ring.

- (i) If A is absolutely flat (Chapter 2, Exercise 27, Section 3.2.29 p. 54) and S is any multiplicatively closed subset of A , then $S^{-1}A$ is absolutely flat.
- (ii) A is absolutely flat $\iff A_{\mathfrak{m}}$ is a field for each maximal ideal \mathfrak{m} .

Solution.

- (i) We have, with obvious notation,

$$a = a^2x \implies \frac{a^2}{s^2} \frac{sx}{1} = \frac{a^2xs}{s^2} = \frac{a^2x}{s} = \frac{a}{s}.$$

- (ii) Note that the statement to prove is equivalent to any of the following two statements:

- A is absolutely flat if and only if for all maximal ideal \mathfrak{m} of A we have $\mathfrak{m}_{\mathfrak{m}} = (0)$.
- A is absolutely flat if and only if for all $\mu \in \mathfrak{m} \subset A$ with \mathfrak{m} maximal, there is an $s \in A \setminus \mathfrak{m}$ such that $s\mu = 0$.

Let us show that A is absolutely flat if and only if $A_{\mathfrak{m}}$ is a field for each maximal ideal \mathfrak{m} .

If A is absolutely flat, then so is $A_{\mathfrak{m}}$ by Part (i) of the same exercise, and $A_{\mathfrak{m}}$ is a field by Exercise 2.28 p. 35 of the book [see Section 3.2.30 p. 55 above].

Conversely, assume that $A_{\mathfrak{m}}$ is a field for each maximal ideal \mathfrak{m} of A , and let a be in A . We have $(a^2)_{\mathfrak{m}} = (a)_{\mathfrak{m}}$ for all maximal \mathfrak{m} , and thus $(a^2) = (a)$. This shows that A is absolutely flat by the implication (ii) \implies (i) in Exercise 2.27 p. 35 of the book [see Section 3.2.29 p. 54].

4.2.11 Page 44, Exercise 3.11

Statement. Let A be a ring. Prove that the following are equivalent:

- (i) A/\mathfrak{N} is absolutely flat [\mathfrak{N} being the nilradical of A].
- (ii) Every prime ideal of A is maximal.
- (iii) $\text{Spec}(A)$ is a T1-space [i.e., every subset consisting of a single point is closed].
- (iv) $\text{Spec}(A)$ is Hausdorff.

If these conditions are satisfied, show that $\text{Spec}(A)$ is compact and totally disconnected [i.e. the only connected subsets of $\text{Spec}(A)$ are those consisting of a single point].

Solution. We clearly have (iv) \implies (iii) \iff (ii).

- (ii) \implies (iv): We assume that each prime ideal of A is maximal and we show that $X = \text{Spec}(A)$ is Hausdorff. Let x and y be two distinct points of X . We claim:

(\star) There are elements $a \in \mathfrak{p}_y \setminus \mathfrak{p}_x$, $b \in \mathfrak{p}_x \setminus \mathfrak{p}_y$ such that $ab = 0$.

Statement (\star) implying that X_a and X_b are disjoint open neighborhoods of x and y respectively, it suffices to prove (\star).

Assume (\star) is false. Then $S := (A \setminus \mathfrak{p}_x)(A \setminus \mathfrak{p}_y)$ is a multiplicatively closed subset avoiding 0, hence $S^{-1}A$ is not the zero ring, hence there is a maximal ideal \mathfrak{m} in $S^{-1}A$. The contraction \mathfrak{m}^c of \mathfrak{m} in A is a prime, and thus maximal, ideal contained in $A \setminus S \subset \mathfrak{p}_x$ (because $A \setminus \mathfrak{p}_x \subset S$). This implies $\mathfrak{m}^c = \mathfrak{p}_x$. Similarly we have $\mathfrak{m}^c = \mathfrak{p}_y$. As $\mathfrak{p}_x \neq \mathfrak{p}_y$, this is a contradiction. \square

At this point we know that (ii), (iii) and (iv) are equivalent.

Introduce the following notation: For any ideal \mathfrak{a} of any ring R write $\bar{\mathfrak{a}}$ for the image of \mathfrak{a} in $\bar{R} := R/\mathfrak{N}(R)$, and define $\bar{r} \in \bar{R}$ for $r \in R$ similarly.

(i) \implies (ii): Let A/\mathfrak{N} be absolutely flat and assume by contradiction that there is a prime ideal \mathfrak{p} strictly contained in a maximal ideal \mathfrak{m} . Let us denote this situation by $\mathfrak{p} \subsetneq \mathfrak{m}$. Then we get firstly $\bar{\mathfrak{p}} \subsetneq \bar{\mathfrak{m}}$ with $\bar{\mathfrak{p}}$ prime and $\bar{\mathfrak{m}}$ maximal, and secondly $\bar{\mathfrak{p}}_{\bar{\mathfrak{m}}} \subsetneq \bar{\mathfrak{m}}_{\bar{\mathfrak{m}}}$ with $\bar{\mathfrak{p}}_{\bar{\mathfrak{m}}}$ prime and $\bar{\mathfrak{m}}_{\bar{\mathfrak{m}}}$ maximal. But Exercise 3.10ii p. 44 of the book [see Section 4.2.10 above] implies $\bar{\mathfrak{m}}_{\bar{\mathfrak{m}}} = (\bar{0})$. \square

(ii) \implies (i): We assume that the prime ideals of A are maximal and we show that A/\mathfrak{N} is absolutely flat. Let \mathfrak{m} be a maximal ideal of A . By Section 4.2.10 above it suffices to show $\bar{\mathfrak{m}}_{\bar{\mathfrak{m}}} = (\bar{0})$. But we have $\bar{\mathfrak{m}}_{\bar{\mathfrak{m}}} = \mathfrak{N}(\bar{A}_{\bar{\mathfrak{m}}}) = \mathfrak{N}(\bar{A})_{\bar{\mathfrak{m}}} = (0)$, the second equality following from Corollary 3.12 p. 42 of the book. \square

Proof that X is totally disconnected: Let x and y be two different points of X . We saw that there is an $a \in A$ such that $x \in X_a$ and $y \notin X_a$ [see Statement (\star) above]. It suffices to show that X_a is closed, but this follows from the fact that X is Hausdorff and X_a is compact. \square

Here is a related result:

The following conditions on a ring A are equivalent:

- (a) *the Krull dimension of A is at most zero,*
- (b) *A/\mathfrak{N} is absolutely flat, where \mathfrak{N} is the nilradical of A ,*
- (c) *for each a in A the descending chain $(a) \supset (a^2) \supset \cdots$ stabilizes.*

Proof. In view of Exercise 3.11 p. 44 of the book [see Section 4.2.11 p. 66], it suffices to prove (b) \implies (c) \implies (a).

(b) \implies (c): With obvious notation we have $\bar{a} = \bar{a}^2 \bar{b}$ for some b in A , that is $(a - a^2 b)^n = 0$ for some $n \geq 1$. This is easily seen to imply $a^n \in (a^{n+1})$ and thus $(a^{n+1}) = (a^n)$.

(c) \implies (a): Let \mathfrak{p} be a prime ideal of A and let a be in $A \setminus \mathfrak{p}$. We have $a^n(1 - ab) = 0$ for some b in A and some n in \mathbb{N} . In particular $a^n(1 - ab) \in \mathfrak{p}$, and thus $1 - ab \in \mathfrak{p}$. This implies that A/\mathfrak{p} is a field, and therefore that \mathfrak{p} is maximal. \square

4.2.12 Page 45, Exercise 3.12

Statement. Let A be an integral domain and M an A -module. An element $x \in M$ is a **torsion element** of M if $\text{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A . Show that the torsion elements of M form a submodule of M . This submodule is called the **torsion submodule**

of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

- (i) If M is any A -module, then $M/T(M)$ is torsion-free.
- (ii) If $f : M \rightarrow N$ is a module homomorphism, then $f(T(M)) \subset T(N)$.
- (iii) If $0 \rightarrow M' \rightarrow M \rightarrow M''$ is an exact sequence, then the sequence $0 \rightarrow T(M') \rightarrow T(M) \rightarrow T(M'')$ is exact.
- (iv) If M is any A -module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A .

[For iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 1, Exercise 15 and Exercise 20 [Exercise 2.2.16 p. 24 and Exercise 2.2.21 p. 27], show that if $1 \otimes X = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x = 0$.]

Hints.

- (i) Let $x + T(M) \in M/T(M)$ with $x \in M$, and let $a \in A$, $a \neq 0$, satisfy $a(x + T(M)) = 0$, that is $ax \in T(M)$. Then there is a nonzero $b \in A$ such that $ba x = 0$, and thus $x \in T(M)$.
- (ii) If $x \in T(M)$, then we get $ax = 0$ for some $a \in A$, $a \neq 0$, and thus $af(x) = f(ax) = 0$.
- (iii) The exactness at $T(M')$ is clear. To prove the exactness at $T(M)$, let x be in $T(M)$, and let x'' be its image in M'' ; in particular $x'' \in T(M'')$. Suppose $x'' = 0$. Then x is the image of some $x' \in M'$. We have $ax = 0$ for some nonzero $a \in A$, and thus $ax' = 0$. This shows that $x' \in T(M')$.
- (iv) [In the hint, “Chapter 1” should be “Chapter 2”.] By Exercise 2.20 p. 33 of the book we have

$$K \otimes_A M \simeq \left(\operatorname{colim}_{a \in A \setminus \{0\}} Aa^{-1} \right) \otimes_A M \simeq \operatorname{colim}_{a \in A \setminus \{0\}} (Aa^{-1} \otimes_A M).$$

If $1 \otimes x$ vanishes in $K \otimes_A M$, then Exercise 2.15 p. 33 of the book implies that $1 \otimes x$ already vanishes in $Aa^{-1} \otimes_A M$ for some $a \in A \setminus \{0\}$, and we get $0 = 1 \otimes x = a^{-1}a \otimes x = a^{-1} \otimes ax$ in $Aa^{-1} \otimes_A M$. This implies $ax = 0$ because the map $M \rightarrow Aa^{-1} \otimes_A M$, $y \mapsto a^{-1} \otimes y$ is an isomorphism.

4.2.13 Page 45, Exercise 3.13

Statement. Let S be a multiplicatively closed subset of an integral domain A . In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

- (i) M is torsion-free.
- (ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals \mathfrak{p} .
- (iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals \mathfrak{m} .

[Note the implicit assumption $0 \notin S$.]

Solution. We claim

$$\operatorname{Ann} \left(\frac{x}{s} \right) = S^{-1} \operatorname{Ann}(x). \quad (19)$$

Proof: We have $\operatorname{Ann}(\frac{x}{s}) = \operatorname{Ann}(\frac{x}{1}) = \operatorname{Ann}((S^{-1}A)\frac{x}{1}) = \operatorname{Ann}(S^{-1}(Ax)) = S^{-1} \operatorname{Ann}(Ax) = S^{-1} \operatorname{Ann}(x)$, the equality $\operatorname{Ann}(S^{-1}(Ax)) = S^{-1} \operatorname{Ann}(Ax)$ following from Proposition 3.14 p. 43 of the book. This proves (19).

To show

$$T(S^{-1}M) = S^{-1}(TM) \quad (20)$$

let first $\frac{x}{s} \in T(S^{-1}M)$. We have $(0) \neq \text{Ann}(\frac{x}{s}) = S^{-1}\text{Ann}(x)$ by (19). This implies $\text{Ann}(x) \neq (0)$, hence $x \in T(M)$, hence $\frac{x}{s} \in S^{-1}T(M)$. Suppose now $\frac{x}{s} \in S^{-1}T(M)$, that is $\frac{x}{s} = \frac{y}{t}$ with $y \in T(M)$, $t \in S$ and $tux = suy$ for some $u \in S$; and we get $\text{Ann}(\frac{x}{s}) = \text{Ann}(\frac{y}{t}) = S^{-1}\text{Ann}(y)$ by (19). Since $\text{Ann}(y)$ is nonzero, so is $S^{-1}\text{Ann}(y)$, hence $\frac{x}{s} \in T(S^{-1}M)$. This completes the proof of (20).

We now show that (i), (ii) and (iii) are equivalent:

(i) \implies (ii): Follows immediately from (20).

(ii) \implies (iii): Obvious.

(iii) \implies (i): Follows immediately from (20) and Proposition 3.8 p. 40 of the book.

4.2.14 Page 45, Exercise 3.14

Statement. Let M be an A -module and \mathfrak{a} an ideal of A . Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supset \mathfrak{a}$. Prove that $M = \mathfrak{a}M$.

Solution. Let \mathfrak{m} be a maximal ideal of A/\mathfrak{a} . It suffices to show $(M/\mathfrak{a}M)_{\mathfrak{m}} = 0$. Letting \mathfrak{m}^c be the contraction of \mathfrak{m} in A , we get $(M/\mathfrak{a}M)_{\mathfrak{m}} \simeq (M/\mathfrak{a}M)_{\mathfrak{m}^c} \simeq M_{\mathfrak{m}^c}/(\mathfrak{a}M)_{\mathfrak{m}^c} = 0$.

4.2.15 Page 45, Exercise 3.15

Statement. Let A be a ring. Show that every set of n generators of A^n is a basis of A^n . Deduce that every set of generators of A^n has at least n elements. [Hint. Let x_1, \dots, x_n be a set of generators and e_1, \dots, e_n the canonical basis of A^n . Define $\phi : A^n \rightarrow A^n$ by $\phi(e_i) = x_i$. Then ϕ is surjective and we have to prove that it is an isomorphism. By (3.9) we may assume that A is a local ring. Let N be the kernel of ϕ and let $k = A/\mathfrak{m}$ be the residue field of $A \dots$]

Solution. Use Corollary 13 p. 34.

4.2.16 Page 46, Exercise 3.16

Statement. Let B be a flat A -algebra. Then the following conditions are equivalent:

- (i) $\mathfrak{a}^{\text{ec}} = \mathfrak{a}$ for all ideals \mathfrak{a} of A .
- (ii) $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective.
- (iii) For every maximal ideal \mathfrak{m} of A we have $\mathfrak{m}^e \neq (1)$.
- (iv) If M is any non-zero A -module, then $M_B \neq 0$.
- (v) For every A -module M , the mapping $x \mapsto 1 \otimes x$ of M into M_B is injective.

[For (i) \implies (ii), use (3.16) [Proposition 34 p. 60]. (ii) \implies (iii) is clear.

(iii) \implies (iv): Let x be a non-zero element of M and let $M' = Ax$. Since B is flat over A it is enough to show that $M'_B \neq 0$. We have $M' \simeq A/\mathfrak{a}$ for some ideal $\mathfrak{a} \neq (1)$, hence $M'_B \simeq B/\mathfrak{a}^e$. Now

as $\mathfrak{a} \subset \mathfrak{m}$ for some maximal ideal \mathfrak{m} , hence $\mathfrak{a}^e \subset \mathfrak{m}^e \neq (1)$. Hence $M'_B \neq 0$.

(iv) \implies (v): Let M' be the kernel of $M \rightarrow M_B$. Since B is flat over A , the sequence $0 \rightarrow M'_B \rightarrow M_B \rightarrow (M_B)_B$ is exact. But (Chapter 2, Exercise 13 [Section 3.2.13 p. 40], with $N = M_B$) the mapping $M_B \rightarrow (M_B)_B$ is injective, hence $M'_B = 0$ and therefore $M' = 0$.

(v) \implies (i): Take $M = A/\mathfrak{a}$.]

B is said to be **faithfully flat** over A .

Hints. We claim that Property (vi) below is equivalent to Properties (i) to (v).

(vi) For any A -linear map $M' \rightarrow M$, if $M'_B \rightarrow M_B$ is injective, then so is $M' \rightarrow M$.

(vi) \implies (v): We must prove that $M \rightarrow M_B$ is injective. It suffices to show that $M_B \rightarrow M_{BB}$ is injective. But this follows from Exercise 2.13 p. 32 of the book.

(iv) \implies (vi): Let $0 \rightarrow M' \rightarrow M \rightarrow M''$ be an exact sequence such that $M_B \rightarrow M''_B$ is injective. It suffices to show $M' = 0$, or even $M'_B = 0$. But this follows from the fact that $0 \rightarrow M'_B \rightarrow M_B \rightarrow M''_B$ is exact.

4.2.17 Page 46, Exercise 3.17

Statement. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. If $g \circ f$ is flat and g is faithfully flat, then f is flat.

Hints. Let $M' \rightarrow M$ be injective. We must show that $M'_B \rightarrow M_B$ is injective. By Section 4.2.16 p. 69 and by the fact that $B \rightarrow C$ is faithfully flat, it suffices to check that $M'_{BC} \rightarrow M_{BC}$ is injective, i.e., that $M'_C \rightarrow M_C$ is injective. But this follows from the fact that $A \rightarrow C$ is flat.

4.2.18 Page 46, Exercise 3.18

Statement. Let $f : A \rightarrow B$ be a flat homomorphism of rings, let \mathfrak{q} be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \text{Spec}(B_{\mathfrak{q}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ is surjective. [For $B_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ by (3.10)³, and $B_{\mathfrak{q}}$ is a local ring of $B_{\mathfrak{p}}$, hence is flat over $B_{\mathfrak{p}}$. Hence $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ and satisfies condition (3) of Exercise 16.]

Hints. The phrase “ $B_{\mathfrak{q}}$ is a local ring of $B_{\mathfrak{p}}$ ” means “ $B_{\mathfrak{q}}$ is a localization of $B_{\mathfrak{p}}$ ”.

4.2.19 Page 46, Exercise 3.19

Statement. Let A be a ring, M an A -module. The support of M is defined to be the set $\text{Supp}(M)$ of prime ideals \mathfrak{p} of A such that $M_{\mathfrak{p}} \neq 0$. Prove the following results:

(i) $M \neq 0 \iff \text{Supp}(M) \neq \emptyset$.

(ii) $V(\mathfrak{a}) = \text{Supp}(A/\mathfrak{a})$.

³Proposition 3.10 p. 41 of the book says that for any A -module M , the following statements are equivalent: (i) M is a flat A -module, (ii) $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$ -module for each prime ideal \mathfrak{p} ; (iii) $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$ -module for each maximal ideal \mathfrak{m} .

- (iii) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, then $\text{Supp}(M) = \text{Supp}(M') \cup \text{Supp}(M'')$.
- (iv) If $M = \sum M_i$ then $\text{Supp}(M) = \bigcup \text{Supp}(M_i)$.
- (v) If M is finitely generated, then $\text{Supp}(M) = V(\text{Ann}(M))$ (and is therefore a closed subset of $\text{Spec}(A)$).
- (vi) If M, N are finitely generated, then $\text{Supp}(M \otimes_A N) = \text{Supp}(M) \cap \text{Supp}(N)$. [Use Chapter 2, Exercise 3, Section 3.2.3 p. 37.]
- (vii) If M is finitely generated and \mathfrak{a} is an ideal of A , then $\text{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \text{Ann}(M))$.
- (viii) If $f : A \rightarrow B$ is a ring homomorphism and M is a finitely generated A -module, then $\text{Supp}(B \otimes_A M) = f^{*-1}(\text{Supp}(M))$.

General Observation. If $P(M)$ is a property that an A -module M may or may not have, then to prove $P(M)$ for all finitely generated module, it suffices to prove that

- $P(A/\mathfrak{a})$ holds for all ideal \mathfrak{a} ,
- $P(M_1 + M_2)$ holds for all finitely generated submodules M_1 and M_2 of a module M whenever $P(M_1)$ and $P(M_2)$ hold.

Hints. It seems better to start by proving (iii) and (iv), and then (ii). For (iv), see (14) p. 46. For (v), use (ii) and the General Observation.

Hint for (vi): Use Proposition 3.7 p. 40 of the book [Proposition 30 p. 30].

Proof of (vii): Let \mathfrak{b} be the annihilator of M . Set $\bar{A} := A/\mathfrak{a}$ and $\bar{M} := M/\mathfrak{a}M$. Let $\pi : A \rightarrow \bar{A}$ be the canonical projection and put $\bar{\mathfrak{c}} := \pi(\mathfrak{c})$ for any ideal \mathfrak{c} of A . Let \mathfrak{p} be a prime ideal of A . We must show $\bar{M}_{\bar{\mathfrak{p}}} \neq 0 \iff \mathfrak{p} \supset \mathfrak{a} + \mathfrak{b}$. Since $\mathfrak{p} \not\supset \mathfrak{a}$ implies $\mathfrak{a}_{\mathfrak{p}} = (1)$ and thus

$$\bar{M}_{\bar{\mathfrak{p}}} \simeq M_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}M_{\mathfrak{p}} = M_{\mathfrak{p}}/M_{\mathfrak{p}} = 0,$$

we can assume $\mathfrak{p} \supset \mathfrak{a}$, and we get $\bar{M}_{\bar{\mathfrak{p}}} \neq 0 \iff \bar{M}_{\bar{\mathfrak{p}}} \neq 0 \iff \bar{\mathfrak{p}} \supset \bar{\mathfrak{b}} \iff \mathfrak{p} \supset \mathfrak{b}$ by (v).

Proof of (viii): Let $\mathfrak{q} \in \text{Spec}(B)$ and set $\mathfrak{p} := \mathfrak{q}^c$. It suffices to show $M_{B,\mathfrak{q}} = 0 \iff M_{\mathfrak{p}} = 0$. We claim

$$\frac{M_{B,\mathfrak{q}}}{\mathfrak{q}M_{B,\mathfrak{q}}} \simeq \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_{A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}} \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}} . \quad (21)$$

As M is finitely generated, (21) will imply

$$M_{B,\mathfrak{q}} = 0 \iff \frac{M_{B,\mathfrak{q}}}{\mathfrak{q}M_{B,\mathfrak{q}}} = 0 \iff \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}} = 0 \iff M_{\mathfrak{p}} = 0.$$

Let us prove (21). We have

$$\begin{aligned} \frac{M_{B,\mathfrak{q}}}{\mathfrak{q}M_{B,\mathfrak{q}}} &= \frac{(B \otimes_A M)_{\mathfrak{q}}}{\mathfrak{q}(B \otimes_A M)_{\mathfrak{q}}} \simeq \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_B B \otimes_A M \simeq \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_A M \\ &\simeq \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_{A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}} \frac{A_{\mathfrak{p}}}{\mathfrak{p}_{\mathfrak{p}}} \otimes_A M \simeq \frac{B_{\mathfrak{q}}}{\mathfrak{q}_{\mathfrak{q}}} \otimes_{A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}} \frac{M_{\mathfrak{p}}}{\mathfrak{p}M_{\mathfrak{p}}} . \end{aligned}$$

Let us show that the inclusion $f^{*-1}(\text{Supp}(M)) \subset \text{Supp}(M_B)$ holds even if M is not finitely generated. It suffices to prove $(B \otimes_A M)_{\mathfrak{q}} \simeq B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. We have

$$(B \otimes_A M)_{\mathfrak{q}} \simeq B_{\mathfrak{q}} \otimes_B B \otimes_A M \simeq B_{\mathfrak{q}} \otimes_A M \simeq B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \otimes_A M \simeq B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} .$$

4.2.20 Page 46, Exercise 3.20

Statement. Let $f : A \rightarrow B$ be a ring homomorphism, $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ the associated mapping. Show that

- (i) Every prime ideal of A is a contracted ideal $\iff f^*$ is surjective.
- (ii) Every prime ideal of B is an extended ideal $\implies f^*$ is injective.

Is the converse of (ii) true?

Hints. (i) See Proposition 34 p. 60 above.

Counterexample to the converse of (ii): $A := K$ a field, $B := K[\varepsilon]$ with $\varepsilon^2 = 0$, $f : K \rightarrow K[\varepsilon]$ the inclusion.

4.2.21 Page 46, Exercise 3.21

Statement. (i) Let A be a ring, S a multiplicatively closed subset of A , and $\phi : A \rightarrow S^{-1}A$ the canonical homomorphism. Show that $\phi^* : \text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$ is a homeomorphism of $\text{Spec}(S^{-1}A)$ onto its image in $X = \text{Spec}(A)$. Let this image be denoted by $S^{-1}X$.

In particular, if $f \in A$, the image of $\text{Spec}(A_f)$ in X is the basic open set X_f (Chapter 1, Exercise 17).

(ii) Let $f : A \rightarrow B$ be a ring homomorphism. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, and let $f^* : Y \rightarrow X$ be the mapping associated with f . Identifying $\text{Spec}(S^{-1}A)$ with its canonical image $S^{-1}X$ in X , and $\text{Spec}(S^{-1}B) (= \text{Spec}(f(S)^{-1}B))$ with its canonical image $S^{-1}Y$ in Y , show that $S^{-1}f^* : \text{Spec}(S^{-1}B) \rightarrow \text{Spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = f^{*-1}(S^{-1}X)$.

(iii) Let \mathfrak{a} be an ideal of A and let $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B . Let $\bar{f} : A/\mathfrak{a} \rightarrow B/\mathfrak{b}$ be the homomorphism induced by f . If $\text{Spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a})$ in X , and $\text{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in Y , show that \bar{f}^* is the restriction of f^* to $V(\mathfrak{b})$.

(iv) Let \mathfrak{p} be a prime ideal of A . Take $S = A \setminus \mathfrak{p}$ in (ii) and then reduce mod $S^{-1}\mathfrak{p}$ as in (iii). Deduce that the subspace $f^{*-1}(\mathfrak{p})$ of Y is naturally homeomorphic to $\text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) = \text{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p})$ is the residue field of the local ring $A_{\mathfrak{p}}$.

$\text{Spec}(k(\mathfrak{p}) \otimes_A B)$ is called the fiber of f^* over \mathfrak{p} .

Solution. (i) Set $S^{-1}X := \{\mathfrak{p} \in X \mid \mathfrak{p} \cap S = \emptyset\}$. By Proposition 3.11iv p. 41 of the book [Proposition 33 p. 58] the maps

$$\text{Spec}(S^{-1}A) \xrightleftharpoons[S^{-1}]{\phi^*} S^{-1}X$$

are inverse bijections. Let us equip $S^{-1}X \subset X$ with the induced topology. Then the closed subsets of $S^{-1}X$ are precisely the subsets of the form $V(\mathfrak{a}) \cap S^{-1}X$ where \mathfrak{a} is an ideal of A . By Proposition 3.11i p. 41 of the book [Proposition 33 p. 58], the closed subsets of $S^{-1}A$ are precisely the subsets of the form $V(S^{-1}\mathfrak{a})$ where \mathfrak{a} is an ideal of A . Thus it suffices to show that, given an ideal \mathfrak{a} of A and a prime ideal \mathfrak{p} of A , we have $S^{-1}\mathfrak{a} \subset S^{-1}\mathfrak{p} \iff \mathfrak{a} \subset \mathfrak{p}$. Implication \Leftarrow is clear. Conversely $S^{-1}\mathfrak{a} \subset S^{-1}\mathfrak{p}$ implies

$$\mathfrak{a} \subset \phi^* S^{-1}\mathfrak{a} \subset \phi^* S^{-1}\mathfrak{p} = \mathfrak{p}.$$

(ii) Since the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ S^{-1}A & \xrightarrow{S^{-1}f} & S^{-1}B \end{array}$$

commutes, so does

$$\begin{array}{ccc} X & \xleftarrow{f^*} & Y \\ \uparrow & & \uparrow \\ S^{-1}X & \xleftarrow{(S^{-1}f)^*} & S^{-1}Y. \end{array}$$

This proves the first claim. To show $S^{-1}Y = f^{*-1}(S^{-1}X)$, note that, for $\mathfrak{q} \in Y$, we have

$$\mathfrak{q} \in S^{-1}Y \iff f(S) \cap \mathfrak{q} = \emptyset \iff S \cap f^*(\mathfrak{q}) = \emptyset \iff f^*(\mathfrak{q}) \in S^{-1}X \iff \mathfrak{q} \in f^{*-1}(S^{-1}X).$$

(iii) Since the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/\mathfrak{a} & \xrightarrow{\bar{f}} & B/\mathfrak{b} \end{array}$$

commutes, so does

$$\begin{array}{ccc} X & \xleftarrow{f^*} & Y \\ \uparrow & & \uparrow \\ V(\mathfrak{a}) & \xleftarrow{\bar{f}^*} & V(\mathfrak{b}). \end{array}$$

(iv) We have the commuting diagrams

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ \downarrow & & \downarrow & & \\ A/\mathfrak{p} & \longrightarrow & B/\mathfrak{p}B & & \\ \downarrow & & \downarrow & & \\ k(\mathfrak{p}) & \longequal{\quad} & (A/\mathfrak{p})_{\mathfrak{p}} & \longrightarrow & (B/\mathfrak{p}B)_{\mathfrak{p}} \longequal{\quad} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \end{array}$$

and

$$\begin{array}{ccc}
X & \xleftarrow{f^*} & Y \\
\uparrow & & \uparrow \\
V(\mathfrak{p}) & \xleftarrow{\bar{f}^*} & V(\mathfrak{p}B) \\
\uparrow & & \uparrow \\
\{\mathfrak{p}\} & \xleftarrow{\bar{f}_{\mathfrak{p}}^*} & Z
\end{array}$$

with $Z := \text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. The second claim of (ii) implies $Z = \bar{f}^{*-1}(\mathfrak{p})$. Finally note that we have

$$\bar{f}^{*-1}(\mathfrak{p}) = \{\mathfrak{q} \in Y \mid \mathfrak{q} \supset f(\mathfrak{p}), f^*(\mathfrak{q}) = \mathfrak{p}\} = \{\mathfrak{q} \in Y \mid f^*(\mathfrak{q}) = \mathfrak{p}\} = f^{*-1}(\mathfrak{p}),$$

that is, $\text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) = f^{*-1}(\mathfrak{p})$, as required.

4.2.22 Page 47, Exercise 3.22

Statement. Let A be a ring and \mathfrak{p} a prime ideal of A . Then the canonical image of $\text{Spec}(A_{\mathfrak{p}})$ in $\text{Spec}(A)$ is equal to the intersection of all the open neighborhoods of \mathfrak{p} in $\text{Spec}(A)$.

Solution. Let $X := \text{Spec}(A)$. It suffices to show

$$\{\mathfrak{q} \in X \mid \mathfrak{q} \subset \mathfrak{p}\} = \bigcap_{f \notin \mathfrak{p}} X_f.$$

We have

$$\bigcap_{f \notin \mathfrak{p}} X_f = \{\mathfrak{q} \in X \mid f \notin \mathfrak{p} \implies f \notin \mathfrak{q}\} = \{\mathfrak{q} \in X \mid f \in \mathfrak{q} \implies f \in \mathfrak{p}\} = \{\mathfrak{q} \in X \mid \mathfrak{q} \subset \mathfrak{p}\}.$$

4.2.23 Page 47, Exercise 3.23

Statement. Let A be a ring, let $X = \text{Spec}(A)$ and let U be a basic open set in X (i.e., $U = X_f$ for some $f \in A$: Chapter 1, Exercise 17).

- (i) If $U = X_f$, show that the ring $A(U) = A_f$ depends only on U and not on f .
- (ii) Let $U' = X_g$ be another basic open set such that $U' \subset U$. Show that there is an equation of the form $g^n = uf$ for some integer $n > 0$ and some $u \in A$, and use this to define a homomorphism $\rho: A(U) \rightarrow A(U')$ (i.e., $A_f \rightarrow A_g$) by mapping a/f^m to au^m/g^{mn} . Show that ρ depends only on U and U' . This homomorphism is called the **restriction** homomorphism.
- (iii) If $U = U'$, then ρ is the identity map.
- (iv) If $U \supset U' \supset U''$ are basic open sets in X , show that the diagram

$$\begin{array}{ccc}
A(U) & \xrightarrow{\quad} & A(U'') \\
& \searrow & \nearrow \\
& A(U') &
\end{array}$$

(in which the arrows are restriction homomorphisms) is commutative.

(v) Let $x(=\mathfrak{p})$ be a point of X . Show that

$$\varinjlim_{U \ni x} A(U) \simeq A_{\mathfrak{p}}.$$

Solution. In this Section we freely use the Notes in Section 4.2.7 p. 63 above.

(i) - (iv) In the notation of Note 38 p. 64 above we have $F(A_f) = X_f$, and the statements follow from Note 38.

(v) More generally, if S is any multiplicative subset of A then there is a natural isomorphism

$$\operatorname{colim}_{f \in S} A_f \xrightarrow{\sim} S^{-1}A. \quad (22)$$

This can be proved as follows [using again Note 38]. For $f \in S$ let ϕ_f be the unique A -algebra morphism from A_f to $S^{-1}A$. Let B be an A -algebra, and for each $f \in S$ let $\psi_f : A_f \rightarrow B$ be an A -algebra morphism. The choice of such an A -algebra morphism ψ_f is equivalent to the choice of a unit b_f in B , the dictionary being given by $\psi_f(f) = b_f$. To prove (22) it suffices to show that there is a unique A -algebra morphism $\psi : S^{-1}A \rightarrow B$ such that $\psi \circ \phi_f = \psi_f$ for all $f \in S$, or, equivalently, such that $\psi(f) = b_f$ for all $f \in S$. Let us check that the formula $\psi(\frac{a}{f}) := \psi(a)b_f^{-1}$ does define our morphism ψ . If we have $\frac{a}{f} = \frac{u}{g}$, there is an $h \in S$ such that $agh = bfh$, hence $\psi(a)b_gb_h = \psi(u)b_fb_h$, hence $\psi(a)b_f^{-1} = \psi(u)b_g^{-1}$.

4.2.24 Page 47, Exercise 3.24

Statement. Show that the presheaf of Exercise 23 has the following property. Let $(U_i)_{i \in I}$ be a covering of X by basic open sets. For each $i \in I$ let $s_i \in A(U_i)$ be such that, for each pair of indices i, j , the images of s_i and s_j in $A(U_i \cap U_j)$ are equal. Then there exists a unique $s \in A (= A(X))$ whose image in $A(U_i)$ is s_i for all $i \in I$. (This essentially implies that the presheaf is a sheaf.)

Solution. We can assume that I is nonempty, and that $0 \in I$. Let $\alpha_i : A \rightarrow A_{f_i}$ and $\phi_{ij} : A_{f_i} \rightarrow A_{f_i f_j}$ be the natural morphisms, and consider the diagram

$$0 \rightarrow A \xrightarrow{\alpha} \prod_{i \in I} A_{f_i} \xrightarrow{\beta} \prod_{i, j \in I} A_{f_i f_j},$$

where α is induced by the α_i and β is defined by $\beta((a_i)) := (\phi_{ij}(a_i) - \phi_{ji}(a_j))$. This is clearly a complex. It suffices to show that it is exact. By Section 4.1.6 p. 58 above, we can assume that A is local. As the f_i generate the unit ideal, one of them is a unit, so that we can assume $f_0 = 1$. It is easy to see that α is injective. Assuming $\beta((a_i)) = 0$, it is straightforward to check that (a_i) is equal to $\alpha(a_0)$.

4.2.25 Page 48, Exercise 3.25

Statement. Let $f : A \rightarrow B$, $g : A \rightarrow C$ be ring homomorphisms and let $h : A \rightarrow B \otimes_A C$ be defined by $h(x) = f(x) \otimes g(x)$. Let X, Y, Z, T be the prime spectra of $A, B, C, B \otimes_A C$ respectively. Then $h^*(T) = f^*(Y) \cap g^*(Z)$.

[Let $\mathfrak{p} \in X$, and let $k = k(\mathfrak{p})$ be the residue field at \mathfrak{p} . By Exercise 21, the fiber $h^{*-1}(\mathfrak{p})$ is the spectrum of $(B \otimes_A C) \otimes_A k \simeq (B \otimes_A k) \otimes_k (C \otimes_A k)$. Hence

$$\mathfrak{p} \in h^*(T) \iff (B \otimes_A k) \otimes (C \otimes_A k) \neq 0 \iff B \otimes_A k \neq 0$$

and $C \otimes_A k \neq 0 \iff \mathfrak{p} \in f^*(Y) \cap g^*(Z)$.]

Note. For the hint see Exercise 3.21iv p. 47 of the book and Section 4.1.12 p. 60 above.

4.2.26 Page 48, Exercise 3.26

For the hint see Exercise 3.21iv p. 47 of the book and Section 4.1.12 p. 60 above.

4.2.27 Page 48, Exercise 3.27

In the hint to (i), “Examples 25 and 26” should be “Exercises 25 and 26”.

Part (ii): see Exercise 1.22 p. 13 of the book [Section 2.2.23 p. 28].

For (iv), see Section 3.2.24 p. 47.

4.2.28 Page 48, Exercise 3.28

Statement. (Continuation of Exercise 27.)

(i) For each $g \in A$, the set X_g (Chapter 1, Exercise 17) is both open and closed in the constructible topology.

(ii) Let C' denote the smallest topology on X for which the sets X_g are both open and closed, and let $X_{C'}$ denote the set X endowed with this topology. Show that $X_{C'}$ is Hausdorff.

(iii) Deduce that the identity mapping $X_C \rightarrow X_{C'}$ is a homeomorphism. Hence a subset E of X is of the form $f^*(\text{Spec}(B))$ for some $f : A \rightarrow B$ if and only if it is closed in the topology C' .

(iv) The topological space X_C is compact, Hausdorff and totally disconnected.

Solution. (i) More generally, for each multiplicative subset $S \subset A$ the subset

$$S^{-1}X := \{\mathfrak{p} \in X \mid \mathfrak{p} \cap S = \emptyset\}$$

is closed in the constructible topology. In particular, so is X_g . But X_g , being open in the Zariski topology, is also open in the constructible topology.

(ii) Let $\mathfrak{p}, \mathfrak{q} \in X$, $\mathfrak{p} \neq \mathfrak{q}$. We can assume that there is a $g \in A$ which is in \mathfrak{q} but not in \mathfrak{p} . Consider the subsets $X_g, V(g) \subset X$: they are disjoint and open, and we have $\mathfrak{p} \in X_g$, $\mathfrak{q} \in V(g)$.

(iii) Follows from (i) and (ii), coupled with Part (iv) of Exercise 26.

(iv) In the notation of the solution to Part (ii), the connected component of \mathfrak{p} [for the constructible topology], being contained in X_g , does not contain \mathfrak{q} .

4.2.29 Page 49, Exercise 3.30

Statement. Show that the Zariski topology and the constructible topology on $\text{Spec}(A)$ are the same if and only if A/\mathfrak{N} is absolutely flat (where \mathfrak{N} is the nilradical of A). [Use Exercise 11.]

Solution. By Exercise 3.27iii p. 48 of the book, the identity of X is a continuous map $X \rightarrow X_C$. By Exercise 3.28iv p. 48 of the book, this map is a homeomorphism if and only if X is Hausdorff. Thus the Zariski and constructible topologies coincide if and only if X is Hausdorff. By Exercise 3.11 pages 44 and 45 of the book, X is Hausdorff if and only if A/\mathfrak{N} is absolutely flat. Therefore the Zariski and constructible topologies coincide if and only if A/\mathfrak{N} is absolutely flat.

5 About Chapter 4

5.1 Comments

5.1.1 Contracted primary ideals

Section 4.1.11 p. 60 prompts the question: Is a contracted primary ideal the contraction of a primary ideal? We show that the answer is negative.

Let K be a field and set $A := K[\varepsilon]$ with $\varepsilon^2 = 0$ but $\varepsilon \neq 0$. Note that $(0) \subset A$ is primary. We will define a ring B which contains A . Then $(0) \subset A$ will be a contracted primary ideal, and we will show that $(0) \subset A$ is not the contraction of a primary ideal of B .

We define B by

$$B = K[X, Y_1, Y_2, \dots] / \mathfrak{b} = K[x, y_1, y_2, \dots]$$

[obvious notation] with

$$\mathfrak{b} = (X^2Y_1) + \sum_{i \geq 2} (X^nY_n - XY_1),$$

and we embed A in B by setting $\varepsilon := xy_1$. We get $\varepsilon x = 0$ and $\varepsilon = x^n y_n$ for all $n \geq 1$; in particular $\varepsilon \in (x^n)$ for all $n \geq 1$.

We claim that we have $XY_1 \notin \mathfrak{b}$, or, equivalently, $\varepsilon \neq 0$.

Proof of the claim: Assume by contradiction $XY_1 \in (X^2Y_1, X^2Y_2 - XY_1, \dots, X^nY_n - XY_1)$. Dividing by X we get

$$Y_1 \in (XY_1, XY_2 - Y_1, X^2Y_3 - Y_1, \dots, X^{n-2}Y_{n-1} - Y_1, X^{n-1}Y_n - Y_1).$$

Setting $Y_i := X^{n-i}Y_n$ for $1 \leq i \leq n-1$ we get $X^{n-1}Y_n \in (X^nY_n)$, which is false.

The above proof was explained to me by an anonymous user, whose user name is user26857, of the Mathematics Stackexchange forum; see <https://math.stackexchange.com/q/2389114/660>.

We prove that $(0) \subset A$ is not the contraction of a primary ideal of B .

Let \mathfrak{q} be a primary ideal of B . If we had $\varepsilon \notin \mathfrak{q}$ and $x^n \notin \mathfrak{q}$ for all $n \geq 1$, then x would be a non nilpotent zero-divisor in B/\mathfrak{q} , contradicting the assumption that \mathfrak{q} is primary. Thus we have $x^n \in \mathfrak{q}$ for some $n \geq 1$, or $\varepsilon \in \mathfrak{q}$. But, since $\varepsilon \in (x^n)$, we get $\varepsilon \in \mathfrak{q}$ in both cases, and \mathfrak{q} contracts to (1) instead of contracting to (0) .

5.1.2 Page 50

Just before Proposition 4.1: In fact, the contraction of a \mathfrak{p} -primary ideal is a \mathfrak{p}^c -primary ideal.

5.1.3 Page 52, Corollary to Theorem 4.5

(Theorem 4.5 is the First Uniqueness Theorem.)

Corollary 39. *In the notation of Theorem 4.5, if $(\mathfrak{a} : x)$ is prime, then $(\mathfrak{a} : x) = \mathfrak{p}_i$ for some i .*

5.1.4 Page 52, Proof of Theorem 4.5

The following statement, occurring in the proof of Theorem 4.5, is often used in the sequel of the book:

In the setting of Theorem 4.5 we have

$$r(\mathfrak{a} : x) = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j. \quad (23)$$

[See Note 2 p. 13.] Here are more details. Still in the setting and notation of Theorem 4.5, for x in A set $J(x) := \{j \mid x \notin \mathfrak{q}_j\}$. Then $(\mathfrak{q}_j : x)$ is \mathfrak{p}_j -primary for $j \in J(x)$, and we have

$$(\mathfrak{a} : x) = \bigcap_{j \in J(x)} (\mathfrak{q}_j : x).$$

In particular, we have $J(x_i) = \{i\}$, and thus $(\mathfrak{a} : x_i) = (\mathfrak{q}_i : x_i)$ and $r(\mathfrak{a} : x_i) = \mathfrak{p}_i$.

5.1.5 Page 52, Corollary to Proposition 4.6

Note 40. *If \mathfrak{a} is decomposable, then the set of prime ideals containing \mathfrak{a} has only finitely many minimal elements.*

5.1.6 Page 53, decomposable ideals

The purpose of this section is to prove the following statement:

An ideal having only finitely many minimal primes is not necessarily decomposable.

The statements in this section are due to user26857 of Mathematics Stackexchange.

Taken from <https://math.stackexchange.com/a/207468/660>:

In order to find an ideal which does not have a primary decomposition, the following construction is useful. Let R be a commutative ring and M an R -module. On the set $A = R \times M$ one defines the following two algebraic operations:

$$(a, x) + (b, y) = (a + b, x + y), \quad (a, x)(b, y) = (ab, ay + bx).$$

With these two operations A becomes a commutative ring with $(1, 0)$ as unit element. (A is called the **idealization** of the R -module M or the **trivial extension** of R by M .)

Let us list some important properties of this ring:

1. $\{0\} \times M$ is an ideal of A isomorphic to M (as R -modules) and there is a one-to-one correspondence between the ideals of R and the ideals of A containing $\{0\} \times M$, the ideal $\mathfrak{a} \subset R$ corresponding to $\mathfrak{a} \times M \subset A$.

2. A is a Noetherian ring if and only if R is Noetherian and M is finitely generated.
3. All prime (maximal) ideals of A have the form $\mathfrak{p} \times M$, where \mathfrak{p} is a prime (maximal) ideal of R .
4. If R is an integral domain and M is divisible, then all the ideals of A have the form $\mathfrak{a} \times M$ with \mathfrak{a} ideal of R , or $\{0\} \times N$ with N submodule of M .

Taken from <https://math.stackexchange.com/a/1679116/660>:

Note 41. *If A is the idealization of the \mathbb{Z} -module \mathbb{Q} , then the primary ideals of A are*

- $p^n\mathbb{Z} \times \mathbb{Q}$ with p prime, $n \geq 1$,
- $\{0\} \times \mathbb{Q}$,
- $\{0\} \times \{0\}$.

Moreover $\{0\} \times \mathbb{Q}$ is the only minimal prime of $\{0\} \times \mathbb{Z}$, and $\{0\} \times \mathbb{Z}$ has no primary decomposition [see Note 40].

5.1.7 Page 53, proof of Proposition 4.8.ii

Proposition 4.8 states the correspondence between primary ideals of A and $S^{-1}A$. We try to describe below this correspondence as precisely as possible.

Using the notation of Section 4.1.7 p. 59 and taking Section 5.1.2 p. 78 into account, we set

$$X' := \{\mathfrak{q} \mid \mathfrak{q} \text{ primary ideal of } A\}, \quad Y' := \{\mathfrak{q} \mid \mathfrak{q} \text{ primary ideal of } S^{-1}A\}.$$

We then have a contraction map $c : Y' \rightarrow X'$ compatible with radicals. We check that

$$c(Y') = \{\mathfrak{q} \in X' \mid r(\mathfrak{q}) \in c(Y)\}$$

and that $\mathfrak{q} \in c(Y') \implies S^{-1}\mathfrak{q} \in Y'$, and we observe the following facts:

The contraction and extension maps are inverse bijections between Y' and $c(Y')$. Moreover they preserve inclusions and are compatible with radicals.

Moreover, a primary ideal of A is contracted if and only if its radical is disjoint from S .

In particular, if \mathfrak{p} is a minimal prime ideal, then the kernel of the natural morphism $A \rightarrow A_{\mathfrak{p}}$ is a minimal **primary** ideal.

5.1.8 Page 54, Proposition 4.9

Recall the statement:

Proposition 42 (Proposition 4.9 p. 54 of the book). *Let S be a multiplicatively closed subset of A and let \mathfrak{a} be a decomposable ideal. Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition of \mathfrak{a} . Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$ and suppose the \mathfrak{q}_i numbered so that S meets $\mathfrak{p}_{m+1}, \dots, \mathfrak{p}_n$ but not $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then*

$$S^{-1}\mathfrak{a} = \bigcap_{i=1}^m S^{-1}\mathfrak{q}_i, \quad S(\mathfrak{a}) = \bigcap_{i=1}^m \mathfrak{q}_i.$$

5.1.9 Page 54, Theorem 4.10

Recall the statement:

Theorem 43 (Theorem 4.10 p. 54 of the book, Second Uniqueness Theorem). *Let \mathfrak{a} be a decomposable ideal, let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition of \mathfrak{a} , and let $\{\mathfrak{p}_{i_1}, \dots, \mathfrak{p}_{i_m}\}$ be an isolated set of prime ideals of \mathfrak{a} . Then $\mathfrak{q}_{i_1} \cap \dots \cap \mathfrak{q}_{i_m}$ is independent of the decomposition.*

Recall also Corollary 4.11:

Corollary 44 (Corollary 4.11 p. 54 of the book). *The isolated primary components (i.e., the primary components \mathfrak{q}_i corresponding to minimal prime ideals \mathfrak{p}_i) are uniquely determined by \mathfrak{a} .*

Here is a corollary to the corollary [see also Proposition 4 p. 14]:

Corollary 45. *Let $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ be a reduced primary decomposition.*

- (a) *If $\mathfrak{p}_i := r(\mathfrak{q}_i)$ is isolated, then \mathfrak{q}_i is the smallest \mathfrak{p}_i -primary ideal containing \mathfrak{a} .*
- (b) *If $r(\mathfrak{a}) = \mathfrak{p}$ for some prime ideal \mathfrak{p} , then \mathfrak{p} is the only isolated prime ideal of \mathfrak{a} , and the corresponding primary component is the smallest \mathfrak{p} -primary ideal containing \mathfrak{a} .*

Proof. (a) Assume $i = 1$ without loss of generality. Step 1: if $\mathfrak{q} \subset \mathfrak{q}_1$ is a \mathfrak{p}_1 -primary ideal containing \mathfrak{a} , then $\mathfrak{a} = \mathfrak{q} \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_n$ is a reduced primary decomposition; by Corollary 44 we have $\mathfrak{q} = \mathfrak{q}_1$. Step 2: if \mathfrak{q} is an arbitrary \mathfrak{p}_1 -primary ideal containing \mathfrak{a} , then $\mathfrak{q} \cap \mathfrak{q}_1$ is again \mathfrak{p}_1 -primary by Lemma 4.3 p. 51 of the book, and we get $\mathfrak{q} \cap \mathfrak{q}_1 = \mathfrak{q}_1$ by Step 1, hence $\mathfrak{q}_1 \subset \mathfrak{q}$.

(b) We have $\mathfrak{p} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$ and Proposition 4 p. 14 implies $\mathfrak{p} = \mathfrak{p}_i$ for some i , hence $\mathfrak{p}_i \subset \mathfrak{p}_j$ for all j . So \mathfrak{p}_i is isolated and the claim follows from (a). \square

5.2 Exercises

5.2.1 Page 55, Exercise 4.1

Statement. If an ideal \mathfrak{a} has a primary decomposition, then $\text{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Hint. Use Exercise 1.20iv p. 13 of the book [see Section 2.2.21 p. 27 above] and Proposition 4.6 p. 52 of the book.

5.2.2 Page 55, Exercise 4.2

Statement. If $\mathfrak{a} = r(\mathfrak{a})$, then \mathfrak{a} has no embedded prime ideals.

Solution. It is implicitly assumed that \mathfrak{a} is decomposable. In general, if $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ is a reduced primary decomposition of \mathfrak{a} such that the minimal prime ideals of \mathfrak{a} are $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ [with $\mathfrak{p}_i = r(\mathfrak{q}_i)$], then $r(\mathfrak{a}) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ is the unique reduced primary decomposition of $r(\mathfrak{a})$. In particular $r(\mathfrak{a})$ has no embedded prime ideals.

Summary: If \mathfrak{a} is decomposable, then so is $r(\mathfrak{a})$, and $r(\mathfrak{a})$ has no embedded prime ideals.

5.2.3 Page 55, Exercise 4.3

Statement. If A is absolutely flat, every primary ideal is maximal.

Solution. Assume A is absolutely flat and (0) is primary in A . It suffices to show that A is a field. Let $0 \neq a \in A$. There is an $x \in A$ such that $a = a^2x$, that is $a(ax - 1) = 0$. If $ax = 1$, then a is a unit and we are done. Otherwise a is a zero-divisor, hence a is nilpotent, and we get $(a) = (a)^2 = (a)^3 = \cdots = (0)$, contradiction.

5.3 Page 55, About Exercise 4.3

The only reference I know for the proposition below, which is the converse of Exercise 4.3, is the Mathematics Stackexchange post <https://math.stackexchange.com/a/569442/660> by user wxu.

Proposition 46. *Let A be a ring. If all the primary ideals of A are prime, then A is absolutely flat.*

Proof. Following wxu, say that a ring is **PP** if all its primary ideals of A are prime.

By the results of Section 4.2.10 p. 66 it suffices to show that each local PP ring is a field. Also note that the property of being PP is inherited by quotients and rings of fractions.

Throughout this proof we assume that A is a local PP ring with maximal ideal \mathfrak{m} , and that $\mathfrak{m} \neq (0)$. In view of the previous paragraph, it is enough to derive a contradiction.

Case 1: \mathfrak{m} is the only prime ideal of A . We have $r(0) = \mathfrak{m}$, hence (0) is primary, hence prime, hence equal to \mathfrak{m} , contradiction.

Case 2: A is a domain and \mathfrak{m} is minimal over (a) for some nonzero a in \mathfrak{m} . We get $r(a) = \mathfrak{m}$, hence, as before, $(a) = \mathfrak{m}$; in particular \mathfrak{m} is finitely generated. Since \mathfrak{m}^2 is \mathfrak{m} -primary, we have also $\mathfrak{m}^2 = \mathfrak{m}$, and Nakayama's Lemma [Proposition 14 p. 34] implies $\mathfrak{m} = (0)$, contradiction.

Case 3: A is a domain. Let a be a nonzero element of \mathfrak{m} and \mathfrak{p} a prime which is minimal over (a) . Then $\mathfrak{p}_{\mathfrak{p}}$ is minimal over $(\frac{a}{1})$, and Case 2 implies $\mathfrak{p}_{\mathfrak{p}} = (0)$, hence $\mathfrak{p} = (0)$, contradiction.

Case 4: this is the general case. In view of Case 1 we can assume that we have $\mathfrak{p} \subsetneq \mathfrak{m}$ for some prime ideal \mathfrak{p} . Then the ideal $(0) \subset A/\mathfrak{p}$ is a proper prime sub-ideal of $\mathfrak{m}/\mathfrak{p}$, and Case 3 implies $\mathfrak{p} = \mathfrak{m}$, contradiction. \square

5.3.1 Page 55, Exercise 4.4

Statement. In the polynomial ring $\mathbb{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal and the ideal $\mathfrak{q} = (4, t)$ is \mathfrak{m} -primary, but is not a power of \mathfrak{m} .

Hints. We have $\mathbb{Z}[t]/\mathfrak{q} \simeq \mathbb{Z}/(4)$ and $\mathfrak{m}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$.

5.3.2 Page 55, Exercise 4.5

Statement. In the polynomial ring $K[x, y, z]$ where K is a field and x, y, z are independent indeterminates, let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, $\mathfrak{m} = (x, y, z)$; \mathfrak{p}_1 and \mathfrak{p}_2 are prime, and \mathfrak{m} is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of \mathfrak{a} . Which components are isolated and which are embedded?

Hints. Let \mathfrak{a} be an ideal of $A := K[x, y, z]$. Then \mathfrak{a} is generated by monomials if and only if it has the following property:

A polynomial $f \in A$ is in \mathfrak{a} if and only if all the monomials occurring in f are in \mathfrak{a} .

In particular, if two ideals are generated by monomials, so is their intersection.

Here is a variant of the exercise: Let A be the K -algebra $[K \text{ a field}]$ generated by x, y, z with the relations $0 = x^2 = xy = xz = yz$, and set $\mathfrak{p}_1 := (x, y)$, $\mathfrak{p}_2 := (x, z)$, $\mathfrak{m} := (x, y, z)$. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a primary decomposition of (0) in A .

We have:

$$\begin{aligned} A &= K \oplus Kx \oplus yK[y] \oplus zK[z], & \mathfrak{p}_1 &= Kx \oplus yK[y], & \mathfrak{p}_2 &= Kx \oplus zK[z], \\ \mathfrak{m}^2 &= y^2K[y] \oplus z^2K[z], & \mathfrak{p}_1 \cap \mathfrak{p}_2 &= Kx, & \mathfrak{p}_1 \cap \mathfrak{m}^2 &= y^2K[y] & \mathfrak{p}_2 \cap \mathfrak{m}^2 &= z^2K[z]. \end{aligned}$$

5.3.3 Page 55, Exercise 4.6

Statement. Let X be an infinite compact Hausdorff space, $C(X)$ the ring of real-valued continuous functions on X (Chapter 1, Exercise 26). Is the zero ideal decomposable in this ring?

Hints. Claim: any prime ideal is contained in a **unique** maximal ideal.

Proof: Using Urysohn's Lemma it is easy to see that, given distinct points x and y in X and denoting by \mathfrak{m}_x and \mathfrak{m}_y the corresponding maximal ideals, there are $f, g \in C(X)$ such that $fg = 0$, $f \in \mathfrak{m}_x \setminus \mathfrak{m}_y$, $g \in \mathfrak{m}_y \setminus \mathfrak{m}_x$.

The claim implies that there are infinitely many minimal prime ideals.

5.3.4 Page 55, Exercise 4.7

Statement. Let A be a ring and let $A[x]$ denote the ring of polynomials in one indeterminate over A . For each ideal \mathfrak{a} of A , let $\mathfrak{a}[x]$ denote the set of all polynomials in $A[x]$ with coefficients in \mathfrak{a} .

(i) $\mathfrak{a}[x]$ is the extension of \mathfrak{a} to $A[x]$.

- (ii) If \mathfrak{p} is a prime ideal in A , then $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.
- (iii) If \mathfrak{q} is a \mathfrak{p} -primary ideal in A , then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$ -primary ideal in $A[x]$. [Use Chapter 1, Exercise 2.]
- (iv) If $\mathfrak{a} = \bigcup_{i=1}^n \mathfrak{q}_i$ is a minimal primary decomposition in A , then $\mathfrak{a}[x] = \bigcup_{i=1}^n \mathfrak{q}_i[x]$ is a minimal primary decomposition in $A[x]$.
- (v) If \mathfrak{p} is a minimal prime ideal of \mathfrak{a} , then $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$.

Hints. Observe that $A[x]$ is faithfully flat over A : see Exercise 3.16 p. 45 of the book and Section 4.2.16 p. 69 above.

Note that we have $M[x] \simeq A[x] \otimes_A M$ for any A -module M . In particular $M \mapsto M[x]$ is exact. [See Exercise 2.6 p. 32 of the book.]

- Part (i) is clear.
- Part (ii): In fact we have

Lemma 47. *If $f : A \rightarrow A[x]$ is the natural embedding, then the fiber $f^{*-1}(\mathfrak{p})$ of*

$$f^* : \operatorname{Spec}(A[x]) \rightarrow \operatorname{Spec}(A)$$

above a prime ideal \mathfrak{p} of A is order isomorphic to the spectrum of $k \otimes_A A[x] \simeq k[x]$, where k is the residue field at \mathfrak{p} . Moreover, the least element of $f^{-1}(\mathfrak{p})$ is $\mathfrak{p}[x]$, and $\mathfrak{p}[x] + (x)$ is a maximal element of $f^{*-1}(\mathfrak{p})$.*

See Exercise 3.21iv p. 47 of the book and Section 4.2.21 p. 72 above. Also note that, if $A \rightarrow B$ is a ring morphism and \mathfrak{a} a contracted ideal of A , then \mathfrak{a}^e is the least element of the set of ideals of B contracting to \mathfrak{a} .

- Part (iii): Use Exercises 1.2ii and 1.2iii p. 11.
- Part (iv) is clear.
- Part (v): We have

$$\mathfrak{a} \subset \mathfrak{b} \iff \mathfrak{a}[x] \subset \mathfrak{b}[x] \tag{24}$$

[obvious notation], and Part (v) follows from Lemma 47.

5.3.5 Page 55, Exercise 4.8

Statement. Let k be a field. Show that in the polynomial ring $k[x_1, \dots, x_n]$ the ideals $\mathfrak{p}_i = (x_1, \dots, x_i)$ ($1 \leq i \leq n$) are prime and all their powers are primary. [Use Exercise 7.]

Hints. Setting

$$B := k[x_1, \dots, x_r], \quad A := B[y_1, \dots, y_s], \quad \mathfrak{m} := (x_1, \dots, x_r) \subset B, \quad \mathfrak{p} := (x_1, \dots, x_r) \subset A,$$

we get $\mathfrak{p}^n = \mathfrak{m}^n[y_1, \dots, y_s]$, and we can use Exercise 4.7iii.

5.3.6 Page 55, Exercise 4.9

Statement. [I found it convenient to make some minor changes to the wording of the book.] In a ring A , let $D(A)$ denote the set of prime ideals \mathfrak{p} which satisfy the following condition: there exists $a \in A$ such that \mathfrak{p} is minimal in the set of prime ideals containing $(0 : a)$.

(a) Show that $b \in A$ is a zero-divisor if and only if $b \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.

(b) Let S be a multiplicatively closed subset of A , and write $S^{-1}D(A)$ for the set of all prime ideals of the form $S^{-1}\mathfrak{p}$ with $\mathfrak{p} \in D(A)$ [in particular such a \mathfrak{p} is disjoint from S]. Show that

$$D(S^{-1}A) = S^{-1}D(A).$$

(c) If the zero ideal has a primary decomposition, show that $D(A)$ is the set of associated prime ideals of (0) .

Hints.

(a) Let $a, b \in A$ and $\mathfrak{p} \in \text{Spec}(A)$ be such that $b \in \mathfrak{p}$ and \mathfrak{p} is minimal over $(0 : a)$.

Claim 1: b is zero-divisor.

Proof of Claim 1. Assume by contradiction that b is not a zero-divisor. Obviously $a \neq 0$ and $b \notin (0 : a)$. Let s be in $A \setminus \mathfrak{p}$ and n be in \mathbb{N} . To prove Claim 1, we first prove:

Claim 2: $b^n s \notin (0 : a)$.

Proof of Claim 2. Assume by contradiction that $b^n s \in (0 : a)$. We can suppose that n is minimum for this condition. We have $n \geq 1$ [because otherwise we would get $s \in (0 : a) \subset \mathfrak{p}$] and $b(b^{n-1}sa) = b^n sa = 0$. Since b is not a zero-divisor, this implies $b^{n-1}sa = 0$, that is $b^{n-1}s \in (0 : a)$, in contradiction with the minimality of n . \square

To complete the proof of Claim 1 set $T := \{b^n s \mid n \in \mathbb{N}, s \in A \setminus \mathfrak{p}\}$. This is a multiplicative set. Claim 2 implies that T is disjoint from $(0 : a)$. Thus there is a prime ideal \mathfrak{p}' which contains $(0 : a)$ and is disjoint from T , and we have $A \setminus \mathfrak{p} \subset T \subset A \setminus \mathfrak{p}'$, hence $(0 : a) \subset \mathfrak{p}' \subset \mathfrak{p}$. By minimality of \mathfrak{p} over $(0 : a)$ this forces $\mathfrak{p}' = \mathfrak{p}$, and thus $b \in \mathfrak{p} \cap T = \mathfrak{p}' \cap T = \emptyset$, contradiction. This completes the proof of Claim 1. \square

It remains to show that if b is a zero-divisor there is an element a of A and a prime ideal \mathfrak{p} of A such that $b \in \mathfrak{p}$ and \mathfrak{p} is minimal over $(0 : a)$. By Proposition 1.15 p. 9 of the book, there is an $a \in A$ such that $b \in r(0 : a)$, and it suffices to let \mathfrak{p} be any prime ideal which is minimal over $(0 : a)$.

(b) It is enough to prove any of the following two equivalent statements:

(b1) Let a be an element of A and \mathfrak{p} a prime ideal of A disjoint from S . Then $S^{-1}\mathfrak{p}$ is minimal over $(0 : \frac{a}{1})$ if and only if \mathfrak{p} is minimal over $(0 : a)$.

(b2) If a is an element of A and \mathfrak{p} a prime ideal of A disjoint from S , then

$$S^{-1}\mathfrak{p} \supset \left(0 : \frac{a}{1}\right) \iff \mathfrak{p} \supset (0 : a).$$

The proof is left to the reader.

(c) Let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a reduced decomposition of (0) in A and set $\mathfrak{p}_i := r(\mathfrak{q}_i)$. Each \mathfrak{p}_i is of the form $r(0 : a)$ by Theorem 4.5 p. 52 of the book [the First Uniqueness Theorem]. In particular \mathfrak{p}_i

is minimal over $r(0 : a)$, and thus $\mathfrak{p}_i \in D(A)$. Conversely, if $\mathfrak{p} \in D(A)$, then \mathfrak{p} is minimal over $(0 : a)$ for some $a \in A$. By (23) p. 79 we have

$$\bigcap_{\mathfrak{q}_i \not\supset a} \mathfrak{p}_i = r(0 : a) \subset \mathfrak{p}.$$

By Proposition 4 p. 14] this implies $\mathfrak{p} \supset \mathfrak{p}_j$ for some j . As $\mathfrak{p}_j \supset (0 : a)$, we get $\mathfrak{p} = \mathfrak{p}_j$ by minimality of \mathfrak{p} .

5.3.7 Page 55, Exercise 4.10

Statement. For any prime ideal \mathfrak{p} in a ring A , let $S_{\mathfrak{p}}(0)$ denote the kernel of the homomorphism $A \rightarrow A_{\mathfrak{p}}$. Prove that

- (i) $S_{\mathfrak{p}}(0) \subset \mathfrak{p}$,
- (ii) \mathfrak{p} is a minimal prime ideal of $A \iff r_A(S_{\mathfrak{p}}(0)) = \mathfrak{p}$,
- (iii) if $\mathfrak{p} \supset \mathfrak{p}'$, then $S_{\mathfrak{p}}(0) \subset S_{\mathfrak{p}'}(0)$,
- (iv) $\bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = (0)$, where $D(A)$ is defined in Exercise 9 [Section 5.3.6 p. 85].

Solution. Note that $S_{\mathfrak{p}}(0) = (0)^c$, where the contraction is taken with respect to $A \rightarrow A_{\mathfrak{p}}$. We also have

$$S_{\mathfrak{p}}(0) = \bigcup_{s \in A \setminus \mathfrak{p}} (0 : s) \tag{25}$$

and $r_A(S_{\mathfrak{p}}(0)) = r_A((0)^c) = (r_{A_{\mathfrak{p}}}(0))^c$ [Exercise 1.18 p. 10 and Proposition 3.11ii p. 41; see Section 2.1.7 p. 16 and Proposition 33 p. 58].

- (i) Follows from (25) above.
- (ii) If \mathfrak{p} is a minimal prime ideal of A we have $r_{A_{\mathfrak{p}}}(0) = \mathfrak{p}_{\mathfrak{p}}$ [because $\mathfrak{p}_{\mathfrak{p}}$ is the unique prime ideal of $A_{\mathfrak{p}}$] and thus $r_A(S_{\mathfrak{p}}(0)) = (r_{A_{\mathfrak{p}}}(0))^c = (\mathfrak{p}_{\mathfrak{p}})^c = \mathfrak{p}$.

If \mathfrak{p} is **not** minimal, there is a prime ideal \mathfrak{p}' such that $\mathfrak{p}' \subsetneq \mathfrak{p}$, and we get $r_{A_{\mathfrak{p}}}(0) \subset \mathfrak{p}'_{\mathfrak{p}} \subsetneq \mathfrak{p}_{\mathfrak{p}}$, hence $r_A((0)^c) \subset \mathfrak{p}' \subsetneq \mathfrak{p}$ and thus $(r_{A_{\mathfrak{p}}}(0))^c \neq \mathfrak{p}$.

- (iii) We have

$$S_{\mathfrak{p}}(0) = \bigcup_{s \in A \setminus \mathfrak{p}} (0 : s) \subset \bigcup_{s' \in A \setminus \mathfrak{p}'} (0 : s') = S_{\mathfrak{p}'}(0).$$

- (iv) Let $0 \neq a \in A$. There is a prime ideal \mathfrak{p} which is minimal over $(0 : a)$. In particular $\mathfrak{p} \in D(A)$. Then $a \in S_{\mathfrak{p}}(0) = \bigcup_{s \in A \setminus \mathfrak{p}} (0 : s)$ would imply $as = 0$ for some $s \in A \setminus \mathfrak{p}$, and thus $s \in (0 : a) \subset \mathfrak{p}$, contradiction.

5.3.8 Page 56, Exercise 4.11

Statement.

- (a) If \mathfrak{p} is a minimal prime ideal of a ring A , show that $S_{\mathfrak{p}}(0)$ [Exercise 10] is the smallest \mathfrak{p} -primary ideal.

(b) Let \mathfrak{a} be the intersection of the ideals $S_{\mathfrak{p}}(0)$ as \mathfrak{p} runs through the minimal prime ideals of A . Show that \mathfrak{a} is contained in the nilradical of A .

(c) Suppose that the zero ideal is decomposable. Prove that $\mathfrak{a} = (0)$ if and only if every prime ideal of (0) is isolated.

Solution.

(a) The ideal $\mathfrak{p}_{\mathfrak{p}}$, being the unique prime ideal of $A_{\mathfrak{p}}$, coincides with the nilradical: $\mathfrak{p}_{\mathfrak{p}} = r_{A_{\mathfrak{p}}}(0)$. Proposition 4.2 p. 51 of the book [see reminder below] implies that (0) is $\mathfrak{p}_{\mathfrak{p}}$ -primary, and is thus the smallest $\mathfrak{p}_{\mathfrak{p}}$ -primary ideal of $A_{\mathfrak{p}}$. As $\mathfrak{p}_{\mathfrak{p}}$ contracts to \mathfrak{p} , Section 5.1.7 p. 80 above entails that $S_{\mathfrak{p}}(0) = (0)^c$ is the smallest \mathfrak{p} -primary ideal of A .

Recall Proposition 4.2:

Proposition 48 (Proposition 4.2 p. 51 of the book). *If $r(\mathfrak{a})$ is maximal, then \mathfrak{a} is primary. In particular, the powers of a maximal ideal \mathfrak{m} are \mathfrak{m} -primary.*

(b) This follows from Exercise 4.10i p. 55, see Section 5.3.7 p. 86 above.

(c) Let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a reduced decomposition. We can assume that there is an m such that $1 \leq m \leq n$ and $\mathfrak{p}_i := r(\mathfrak{q}_i)$ is isolated if and only if $i \leq m$. The Second Uniqueness Theorem [Theorem 4.10 p. 54 of the book] and (a) above imply

$$\begin{aligned} (0) &= S_{\mathfrak{p}_1}(0) \cap \cdots \cap S_{\mathfrak{p}_m}(0) \cap \mathfrak{q}_{m+1} \cap \cdots \cap \mathfrak{q}_n \\ &= \mathfrak{a} \cap \mathfrak{q}_{m+1} \cap \cdots \cap \mathfrak{q}_n, \end{aligned} \tag{26}$$

and we must show $\mathfrak{a} = (0) \iff m = n$. Implication \Leftarrow is clear. Conversely if $\mathfrak{a} = (0)$, then $m = n$ because (26) is a **reduced** decomposition.

5.3.9 Page 56, Exercise 4.12

Statement. Let A be a ring, S a multiplicatively closed subset of A . For any ideal \mathfrak{a} , let $S(\mathfrak{a})$ denote the contraction of $S^{-1}\mathfrak{a}$ in A . The ideal $S(\mathfrak{a})$ is called the **saturation** of \mathfrak{a} with respect to S . Prove that

(i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$,

(ii) $S(r(\mathfrak{a})) = r(S(\mathfrak{a}))$,

(iii) $S(\mathfrak{a}) = (1) \iff \mathfrak{a}$ meets S ,

(iv) $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$.

If \mathfrak{a} has a primary decomposition, prove that the set of ideals $S(\mathfrak{a})$ (where S runs through all multiplicatively closed subsets of A) is finite.

Solution. Follows from statements 1.18, 3.11ii and 3.11v and 4.9 in the book [see Section 2.1.7 p. 16, Proposition 33 p. 58 and Proposition 42 p. 81].

5.3.10 Page 56, Exercise 4.13

Statement. Let A be a ring and \mathfrak{p} a prime ideal of A . The n -th **symbolic power** of \mathfrak{p} is defined to be the ideal (in the notation of Exercise 12)

$$\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$$

where $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$. Show that

- (i) $\mathfrak{p}^{(n)}$ is a \mathfrak{p} -primary ideal;
- (ii) if \mathfrak{p}^n has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its \mathfrak{p} -primary component;
- (iii) if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its \mathfrak{p} -primary component;
- (iv) $\mathfrak{p}^{(n)} = \mathfrak{p}^n \iff \mathfrak{p}^n$ is \mathfrak{p} -primary.

Note. $\mathfrak{p}^{(n)}$ is the contraction in A of $\mathfrak{p}_{\mathfrak{p}}^n \subset A_{\mathfrak{p}}$.

Solution. (i) We must show that $(\mathfrak{p}_{\mathfrak{p}}^n)^c$ is \mathfrak{p} -primary. This results from the following facts: $\mathfrak{p}_{\mathfrak{p}}$ is a maximal ideal of $A_{\mathfrak{p}}$, a power of a maximal ideal is primary [Proposition 48 p. 87], the contraction of a \mathfrak{p}' -primary ideal is \mathfrak{p}'^c -primary, the contraction of $\mathfrak{p}_{\mathfrak{p}}$ is \mathfrak{p} [Proposition 33iv p 58].

We shall use Claims (a) and (b) below.

- (a) If $\mathfrak{a} \subset \mathfrak{p}$ is an ideal such that $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}^k$, then $\mathfrak{p}^{(k)}$ is the smallest \mathfrak{p} -primary ideal containing \mathfrak{a} .

Proof. Assuming $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$ with \mathfrak{q} a \mathfrak{p} -primary ideal, we get $\mathfrak{p}_{\mathfrak{p}}^k = \mathfrak{a}_{\mathfrak{p}} \subset \mathfrak{q}_{\mathfrak{p}} \subset \mathfrak{p}_{\mathfrak{p}}$, and thus $\mathfrak{p}^{(k)} \subset \mathfrak{q} \subset \mathfrak{p}$.

- (b) If \mathfrak{a} and k are as above and if in addition $r(\mathfrak{a}) = \mathfrak{p}$ and \mathfrak{a} has a primary decomposition, then $\mathfrak{p}^{(k)}$ is its \mathfrak{p} -primary component.

Proof. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ be a reduced primary decomposition. Taking radical we get $\mathfrak{p} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$ with $\mathfrak{p}_i = r(\mathfrak{q}_i)$. By Corollary 45b p. 81 this implies $\mathfrak{p} = \mathfrak{p}_i$ for some i , and \mathfrak{p} is isolated. The statement now follows from (a) and Corollary 45a p. 81.

- (ii) Follows from (b) with $\mathfrak{a} = \mathfrak{p}^n$ and $k = n$.
- (iii) Since we have $r(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = r(\mathfrak{p}^{(m)}) \cap r(\mathfrak{p}^{(n)}) = \mathfrak{p}$ by (i), the statement follows from (b) with $\mathfrak{a} = \mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ and $k = m + n$.
- (iv) We must show $\mathfrak{p}^{(n)} = \mathfrak{p}^n \iff \mathfrak{p}^n$ is \mathfrak{p} -primary. Implication \implies follows from (i). If \mathfrak{p}^n is \mathfrak{p} -primary, then $\mathfrak{p}^n = \mathfrak{p}^n$ is a reduced primary decomposition, and (ii) implies $\mathfrak{p}^{(n)} = \mathfrak{p}^n$.

5.3.11 Page 56, Exercise 4.14

Statement. Let \mathfrak{a} be a decomposable ideal in a ring A and let \mathfrak{p} be a maximal element of the set of ideals $(\mathfrak{a} : x)$, where $x \in A$ and $x \notin \mathfrak{a}$. Show that \mathfrak{p} is a prime ideal belonging to \mathfrak{a} .

Solution. By Corollary 39 p. 78 above [which is a corollary to the First Uniqueness Theorem, that is to Theorem 4.5 p. 52 of the book], it suffices to show that $(\mathfrak{a} : x)$ is prime. If not there would be $y, z \in A$ such that

$$y \notin (\mathfrak{a} : x), \quad z \notin (\mathfrak{a} : x), \quad yz \in (\mathfrak{a} : x),$$

that is

$$xy \notin \mathfrak{a}, \quad xz \notin \mathfrak{a}, \quad xyz \in \mathfrak{a}.$$

This implies

$$y \in (\mathfrak{a} : xz) \supset (\mathfrak{a} : x) \not\ni y,$$

contradicting the maximality of $(\mathfrak{a} : x)$.

5.3.12 Page 56, Exercise 4.15

Statement. Let \mathfrak{a} be a decomposable ideal in a ring A , let Σ be an isolated set of prime ideals belonging to \mathfrak{a} , and let \mathfrak{q}_Σ be the intersection of the corresponding primary components. Let f be an element of A such that, for each prime ideal \mathfrak{p} belonging to \mathfrak{a} , we have $f \in \mathfrak{p} \iff \mathfrak{p} \notin \Sigma$, and let S_f be the set of all powers of f . Show that $\mathfrak{q}_\Sigma = S_f(\mathfrak{a}) = (a : f^n)$ for all large n .

Solution. We can assume that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ is a minimal primary decomposition, and that, setting $\mathfrak{p}_i := r(\mathfrak{q}_i)$, we have $\Sigma = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. The equality $\mathfrak{q}_\Sigma = S_f(\mathfrak{a})$ follows from Proposition 4.9 p. 54 of the book [Proposition 42 p. 81]. Proposition 3.11ii p. 41 of the book [Proposition 33 p. 58] implies $S_f(\mathfrak{a}) = \bigcup_{k \geq 0} (\mathfrak{a} : f^k)$. We have

$$(\mathfrak{a} : f^k) = (\mathfrak{q}_1 : f^k) \cap \cdots \cap (\mathfrak{q}_n : f^k). \quad (27)$$

Using Lemma 4.4 p. 51 of the book we see that

- if $1 \leq i \leq m$, then $(\mathfrak{q}_i : f^k) = \mathfrak{q}_i$,
- if $m + 1 \leq i \leq n$ and k is large enough, then $f^k \in \mathfrak{q}_i$, and thus $(\mathfrak{q}_i : f^k) = (1)$.

In view of (27) this entails $(\mathfrak{a} : f^k) = \mathfrak{q}_\Sigma$ for k large enough.

5.3.13 Page 56, Exercise 4.16

Statement. If A is a ring in which every ideal has a primary decomposition, show that every ring of fractions $S^{-1}A$ has the same property.

Solution. This follows from Proposition 3.11i p. 41 [Proposition 33 p. 58] and Proposition 4.9 p. 54 of the book [Proposition 42 p. 81].

5.3.14 Page 56, Exercise 4.17

Statement. Let A be a ring with the following property.

(L1) For every ideal $\mathfrak{a} \neq (1)$ in A and every prime ideal \mathfrak{p} , there exists $x \notin \mathfrak{p}$ such that $S_{\mathfrak{p}}(\mathfrak{a}) = (\mathfrak{a} : x)$, where $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$.

Then every ideal in A is an intersection of (possibly infinitely many) primary ideals.

First solution. Let \mathfrak{a}' be the intersection of all the primary ideals containing a given ideal \mathfrak{a} , and let us assume

$$\mathfrak{a} \subsetneq \mathfrak{a}'. \quad (28)$$

It suffices to reach a contradiction.

Definition 49. An *admissible quadruple* is a quadruple $(\mathfrak{b}, \mathfrak{p}, \mathfrak{q}, x)$ such that

- \mathfrak{b} is an ideal $\neq (1)$ in A ,
- \mathfrak{p} is a minimal element of the set of prime ideals containing \mathfrak{b} ,
- $\mathfrak{q} = S_{\mathfrak{p}}(\mathfrak{b})$,
- $x \in A \setminus \mathfrak{p}$ satisfies $\mathfrak{q} = (\mathfrak{b} : x)$.

Lemma 50. (a) For all $\mathfrak{b} \neq (1)$ there is an admissible quadruple $(\mathfrak{b}, \mathfrak{p}, \mathfrak{q}, x)$.

(b) If $(\mathfrak{b}, \mathfrak{p}, \mathfrak{q}, x)$ is an admissible quadruple, then

- (b1) \mathfrak{q} is \mathfrak{p} -primary,
- (b2) $\mathfrak{b} = \mathfrak{q} \cap (\mathfrak{b} + (x))$.

Proof. Part (a) follows from Assumption (L1). Part (b1) follows from Exercise 4.11 p. 56 of the book [see Section 5.3.8 p. 86 above]. Let us prove (b2). Set $\mathfrak{b}' := \mathfrak{q} \cap (\mathfrak{b} + (x))$. The inclusion $\mathfrak{b} \subset \mathfrak{b}'$ is obvious. To prove the other inclusion, let $\beta + ax \in \mathfrak{q}$ with $\beta \in \mathfrak{b}, a \in A$. It suffices to show $ax \in \mathfrak{b}$. Recall that $\mathfrak{q} = S_{\mathfrak{p}}(\mathfrak{b}) = (\mathfrak{b} : x)$. As $ax \in \mathfrak{q}$, we have $ax^2 \in \mathfrak{b}$, hence

$$a \in (\mathfrak{b} : x^2) \subset \bigcup_{s \notin \mathfrak{p}} (\mathfrak{b} : s) = S_{\mathfrak{p}}(\mathfrak{b}) = (\mathfrak{b} : x)$$

by Proposition 3.11ii p. 41 of the book [Proposition 33 p. 58], and thus $ax \in \mathfrak{b}$, as desired. \square

Note that we have $\mathfrak{b} \subset \mathfrak{q} \subset \mathfrak{p} \not\ni x$. Let κ be a cardinal larger than the cardinality of the power set of A , and let W be the set of all ordinals $\leq \kappa$. We will define, by induction on $\alpha \in W$, a map

$$\alpha \mapsto (\mathfrak{a}_{\alpha}, \mathfrak{p}_{\alpha}, \mathfrak{q}_{\alpha}, x_{\alpha}) \quad (29)$$

from W to the set of admissible quadruples, such that

- (a) $\mathfrak{a}_0 = \mathfrak{a}$,
- (b) the map

$$\alpha \mapsto \mathfrak{a}_{\alpha} \text{ is increasing,} \quad (30)$$

- (c) each ordinal $\alpha \in W$ satisfies

$$\mathfrak{a} = \mathfrak{a}_{\alpha} \cap \bigcap_{\beta < \alpha} \mathfrak{q}_{\beta}. \quad (31)$$

Definition 51. We call (31) *Condition C*(α).

Lemma 50b1 will imply that \mathfrak{q}_α is primary for all $\alpha \in W$.

The existence of (29) satisfying (30) will give the desired contradiction.

Here is the key point:

Assume that $(\mathfrak{a}_\beta, \mathfrak{p}_\beta, \mathfrak{q}_\beta, x_\beta)$ has been constructed for $\beta < \alpha$, and that \mathfrak{a}_α has been defined and satisfies Condition C(α). Then Assumption (28) and Lemma 50b1 imply $\mathfrak{a}_\alpha \neq (1)$. In particular there is, by Lemma 50a, an admissible quadruple $(\mathfrak{a}_\alpha, \mathfrak{p}_\alpha, \mathfrak{q}_\alpha, x_\alpha)$.

We embark on the construction of (29).

- The case of the zero ordinal. The ideal \mathfrak{a}_0 is the ideal \mathfrak{a} given in the statement of the exercise, and we choose $\mathfrak{p}_0, \mathfrak{q}_0, x_0$ in such a way that $(\mathfrak{a}_0, \mathfrak{p}_0, \mathfrak{q}_0, x_0)$ is admissible. Condition C(0) holds trivially.
- From α to $\alpha + 1$. Let $\alpha \in W$ be such that $\alpha + 1 \in W$, and assume that $(\mathfrak{a}_\beta, \mathfrak{p}_\beta, \mathfrak{q}_\beta, x_\beta)$ has already been constructed for $\beta \leq \alpha$ and that Condition C(α) holds. Let $\alpha \in W$. We have

$$\mathfrak{a} \subset (\mathfrak{a}_\alpha + (x_\alpha)) \cap \bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta \quad (32)$$

because $\mathfrak{a} \subset \mathfrak{a}_\beta \subset (\mathfrak{a}_\beta : x_\beta) = \mathfrak{q}_\beta$ for all $\beta \in W$, and Lemma 50b2 implies

$$(\mathfrak{a}_\alpha + (x_\alpha)) \cap \bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta \subset (\mathfrak{a}_\alpha + (x_\alpha)) \cap \mathfrak{q}_\alpha = \mathfrak{a}_\alpha. \quad (33)$$

We also have

$$(\mathfrak{a}_\alpha + (x_\alpha)) \cap \bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta \subset \mathfrak{a}_\alpha \cap \left(\bigcap_{\beta < \alpha} \mathfrak{q}_\beta \right) \cap \mathfrak{q}_\alpha = \mathfrak{a} \cap \mathfrak{q}_\alpha \subset \mathfrak{a} \quad (34)$$

by (33) and C(α). Now (32) and (34) imply

$$(\mathfrak{a}_\alpha + (x_\alpha)) \cap \bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta = \mathfrak{a}. \quad (35)$$

By (35), the ideal $\mathfrak{a}_{\alpha+1} := \mathfrak{a}_\alpha + (x_\alpha)$ satisfies C($\alpha+1$). In particular $\mathfrak{a}_{\alpha+1} \neq (1)$ by Assumption (28) and Lemma 50b1. We define $\mathfrak{p}_{\alpha+1}, \mathfrak{q}_{\alpha+1}, x_{\alpha+1}$ in such a way that $(\mathfrak{a}_{\alpha+1}, \mathfrak{p}_{\alpha+1}, \mathfrak{q}_{\alpha+1}, x_{\alpha+1})$ is admissible [see Lemma 50a].

- The case of a limit ordinal. Assume now that $\gamma \in W$ is a limit ordinal, and set

$$\mathfrak{a}_\gamma := \sum_{\alpha < \gamma} \mathfrak{a}_\alpha = \bigcup_{\alpha < \gamma} \mathfrak{a}_\alpha.$$

We claim that Condition C(γ) holds.

Proof of the claim: We must show

$$\left(\bigcup_{\alpha < \gamma} \mathfrak{a}_\alpha \right) \cap \left(\bigcap_{\beta < \gamma} \mathfrak{q}_\beta \right) = \mathfrak{a}.$$

Inclusion \supset being clear, it suffices to prove \subset . Let x be in the left side. There is an $\alpha < \gamma$ such that

$$x \in \mathfrak{a}_\alpha \cap \bigcap_{\beta < \gamma} \mathfrak{q}_\beta \subset \mathfrak{a}_\alpha \cap \bigcap_{\beta < \alpha} \mathfrak{q}_\beta = \mathfrak{a},$$

the equality following from $C(\alpha)$. This proves Condition $C(\gamma)$.

In particular $\mathfrak{a}_\gamma \neq (1)$. We define $\mathfrak{p}_\gamma, \mathfrak{q}_\gamma, x_\gamma$ in such a way that $(\mathfrak{a}_\gamma, \mathfrak{p}_\gamma, \mathfrak{q}_\gamma, x_\gamma)$ is admissible [Lemma 50a].

Now the map (29) is defined. The map $\alpha \mapsto \mathfrak{a}_\alpha$ is clearly increasing. As already indicated, this is a contradiction.

This completes the solution to Exercise 4.17.

Second solution. The following variant of the above proof will be used to solve Exercise 4.18.

We stop assuming (28).

Let Q be the set of admissible quadruples. We define a map $f : W \rightarrow Q \cup \{\emptyset\}$ [note that the union is disjoint] as follows:

- We assume $\mathfrak{a} \neq (1)$ and define $(\mathfrak{a}_0, \mathfrak{p}_0, \mathfrak{q}_0, x_0)$ as before, and we set $f(0) := (\mathfrak{a}_0, \mathfrak{p}_0, \mathfrak{q}_0, x_0)$.
- In the passage from α to $\alpha + 1$, we define $f(\alpha + 1)$ as follows [assuming that $f(\alpha)$ has already been defined]:

If $f(\alpha) = \emptyset$ we set $f(\alpha + 1) := \emptyset$.

Otherwise there is an ideal $\mathfrak{a}_{\alpha+1}$ containing $\mathfrak{a}_\alpha + (x_\alpha)$ which is maximal subject to the constraint $C(\alpha + 1)$ [see Definition 51 p. 90].

If $\mathfrak{a}_{\alpha+1} \neq (1)$ we define $\mathfrak{p}_{\alpha+1}, \mathfrak{q}_{\alpha+1}, x_{\alpha+1}$ in such a way that $(\mathfrak{a}_{\alpha+1}, \mathfrak{p}_{\alpha+1}, \mathfrak{q}_{\alpha+1}, x_{\alpha+1})$ is admissible, as before, and we set $f(\alpha + 1) := (\mathfrak{a}_{\alpha+1}, \mathfrak{p}_{\alpha+1}, \mathfrak{q}_{\alpha+1}, x_{\alpha+1})$.

If $\mathfrak{a}_{\alpha+1} = (1)$ we set $f(\alpha + 1) := \emptyset$.

- In the case of a limit ordinal γ , we define $f(\gamma)$ as follows (assuming that $f(\alpha)$ has already been defined for $\alpha < \gamma$):

If $f(\alpha) = \emptyset$ for some $\alpha < \gamma$ we set $f(\gamma) = \emptyset$.

Otherwise we define \mathfrak{a}_γ as before.

If $\mathfrak{a}_\gamma \neq (1)$ we define $\mathfrak{p}_\gamma, \mathfrak{q}_\gamma, x_\gamma$ in such a way that $(\mathfrak{a}_\gamma, \mathfrak{p}_\gamma, \mathfrak{q}_\gamma, x_\gamma)$ is admissible, as before, and we set $f(\gamma) := (\mathfrak{a}_\gamma, \mathfrak{p}_\gamma, \mathfrak{q}_\gamma, x_\gamma)$.

If $\mathfrak{a}_\gamma = (1)$ we set $f(\gamma) = \emptyset$.

This ends the definition of f .

The subset $\{\alpha \in W \mid f(\alpha) = \emptyset\}$ of W being nonempty, it has a least element, which we denote by ζ :

$$\zeta := \min \{\alpha \in W \mid f(\alpha) = \emptyset\}, \quad (36)$$

and we get

$$\mathfrak{a} = \bigcap_{\alpha < \zeta} \mathfrak{q}_\alpha. \quad (37)$$

Note 52. The map $\alpha \mapsto \mathfrak{a}_\alpha$ from the set $\{\alpha \in W \mid \alpha \leq \zeta\}$ to the set of ideals of A is increasing.

The following lemma will be used to solve Exercise 4.18.

Lemma 53. The map $n \mapsto \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n$ from the set $\{n \in \mathbb{N} \mid n < \zeta\}$ to the set of ideals of A is decreasing.

Proof. Suppose by contradiction that we have $\mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_{n-1} = \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_{n-1} \cap \mathfrak{q}_n$ for some $0 < n < \zeta$. Recall that \mathfrak{a}_n is maximal for

$$\mathfrak{a}_n \supset \mathfrak{a}_{n-1} + (x_{n-1}) \text{ and } \mathfrak{a} = \mathfrak{a}_n \cap \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_{n-1}. \quad (38)$$

Similarly \mathfrak{a}_{n+1} is maximal for

$$\mathfrak{a}_{n+1} \supset \mathfrak{a}_n + (x_n) \text{ and } \mathfrak{a} = \mathfrak{a}_{n+1} \cap \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_{n-1}.$$

As \mathfrak{a}_{n+1} satisfies (38) and contains \mathfrak{a}_n , these two ideals coincide, in contradiction with Note 52. \square

5.3.15 Page 57, Exercise 4.18

Statement. Consider the following condition on a ring A :

(L2) Given an ideal \mathfrak{a} and a descending chain $S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$ of multiplicatively closed subsets of A , there exists an integer n such that $S_n(\mathfrak{a}) = S_{n+1}(\mathfrak{a}) = \cdots$

Prove that the following are equivalent:

- (i) Every ideal in A has a primary decomposition;
- (ii) A satisfies (L1) and (L2).

[See Section 5.3.14 p. 89 for the definition of (L1).]

Solution.

(i) \implies (L1): Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition. Set $\mathfrak{p}_i := r(\mathfrak{q}_i)$ and let \mathfrak{p} be a prime ideal. We can assume that $\mathfrak{p}_i \subset \mathfrak{p}$ if and only if $i \leq m$. Proposition 4.9 p. 54 of the book [Proposition 42 p. 81] entails $S_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$. For $m+1 \leq i \leq n$ there is an element x_i which is in \mathfrak{q}_i but not in \mathfrak{p} . Then $x := x_{m+1} \cdots x_n$ is in $\mathfrak{q}_{m+1} \cap \cdots \cap \mathfrak{q}_n$ but not in \mathfrak{p} , and we get

$$\begin{aligned} & (\mathfrak{a} : x) \\ &= (\mathfrak{q}_1 : x) \cap \cdots \cap (\mathfrak{q}_m : x) \cap (\mathfrak{q}_{m+1} : x) \cap \cdots \cap (\mathfrak{q}_n : x) \\ &= \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m \cap (1) \cap \cdots \cap (1) \\ &= \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m = S_{\mathfrak{p}}(\mathfrak{a}), \end{aligned}$$

the second equality following from Lemma 4.4 p. 51 of the book.

(i) \implies (L2): Follows from Proposition 4.9 p. 54 of the book [Proposition 42 p. 81].

(ii) \implies (i): Consider the map $\alpha \mapsto (\mathfrak{a}_\alpha, \mathfrak{p}_\alpha, \mathfrak{q}_\alpha, x_\alpha)$ from W to the set of admissible quadruples defined in Section 5.3.14 above, and let ζ be defined as in (36).

In view of (37) it suffices to show that ζ is finite.

Assume by contradiction that ζ is infinite.

Recall that (31) p. 90 is called Condition $C(\alpha)$, and that \mathfrak{q}_α is primary for all $\alpha < \zeta$. For the reader's convenience let us rewrite $C(\alpha)$:

$$\mathfrak{a} = \mathfrak{a}_\alpha \cap \bigcap_{\beta < \alpha} \mathfrak{q}_\beta.$$

This holds for all $\alpha < \zeta$.

Setting $\mathfrak{p}_n := r(\mathfrak{q}_n)$ and $S_n := S_{\mathfrak{p}_0} \cap \cdots \cap S_{\mathfrak{p}_n}$ for $n \in \mathbb{N}$, we get, in view of $C(n+1)$ and Exercise 4.12i p. 56 of the book,

$$S_n(\mathfrak{a}) = S_n(\mathfrak{a}_{n+1}) \cap S_n(\mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n).$$

We claim $S_n(\mathfrak{a}_{n+1}) = (1)$, and thus $S_n(\mathfrak{a}) = S_n(\mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n)$.

To prove this it suffices to show $S_n \cap \mathfrak{a}_{n+1} \neq \emptyset$. Assume by contradiction $S_n \cap \mathfrak{a}_{n+1} = \emptyset$, that is $\mathfrak{a}_{n+1} \subset \mathfrak{p}_0 \cup \cdots \cup \mathfrak{p}_n$. Then Proposition 1.11i p. 8 of the book implies $\mathfrak{a}_{n+1} \subset \mathfrak{p}_i$ for some $0 \leq i \leq n$. This yields $x_i \in \mathfrak{a}_{i+1} \subset \mathfrak{a}_{n+1} \subset \mathfrak{p}_i$, which contradicts Definition 49 p. 90. This proves the equality $S_n(\mathfrak{a}_{n+1}) = (1)$.

Proposition 4.9 p. 54 of the book [Proposition 42 p. 81] implies

$$S_n(\mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n) = \bigcap_{S_n \cap \mathfrak{p}_i = \emptyset} \mathfrak{q}_i = \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n,$$

and we get

$$S_n(\mathfrak{a}) = \mathfrak{q}_0 \cap \cdots \cap \mathfrak{q}_n. \tag{39}$$

Recall that we are assuming that ζ is infinite and that we only need to derive a contradiction. Since (39) holds for all $n \in \mathbb{N}$, this contradiction is furnished by Condition (L2) and Lemma 53 p. 93.

5.3.16 Page 57, Exercise 4.19

Statement.

(a) Let A be a ring and \mathfrak{p} a prime ideal of A . Show that every \mathfrak{p} -primary ideal contains $S_{\mathfrak{p}}(0)$, the kernel of the canonical homomorphism $A \rightarrow A_{\mathfrak{p}}$.

(b) Suppose that A satisfies the following condition: for every prime ideal \mathfrak{p} , the intersection of all \mathfrak{p} -primary ideals of A is equal to $S_{\mathfrak{p}}(0)$. (Noetherian rings satisfy this condition: see Chapter 10⁴.) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be distinct prime ideals, none of which is a minimal prime ideal of A . Then there exists an ideal \mathfrak{a} in A whose associated prime ideals are $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

⁴See Corollary 10.21 p. 111 of the book.

[Proof of (b) by induction on n . The case $n = 1$ is trivial (take $\mathfrak{a} = \mathfrak{p}_1$). Suppose $n > 1$ and let \mathfrak{p}_n be maximal in the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. By the inductive hypothesis there exists an ideal \mathfrak{b} and a minimal primary decomposition $\mathfrak{b} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{n-1}$, where each \mathfrak{q}_i is \mathfrak{p}_i -primary. If $\mathfrak{b} \subset S_{\mathfrak{p}_n}(0)$ let \mathfrak{p} be a minimal prime ideal of A contained in \mathfrak{p}_n . Then $S_{\mathfrak{p}_n}(0) \subset S_{\mathfrak{p}}(0)$, hence $\mathfrak{b} \subset S_{\mathfrak{p}}(0)$. Taking radicals and using Exercise 10, we have $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{n-1} \subset \mathfrak{p}$, hence some $\mathfrak{p}_i \subset \mathfrak{p}$, hence $\mathfrak{p}_i = \mathfrak{p}$ since \mathfrak{p} is minimal. This is a contradiction since no \mathfrak{p}_i is minimal. Hence $\mathfrak{b} \not\subset S_{\mathfrak{p}_n}(0)$ and therefore⁵ there exists a \mathfrak{p}_n -primary ideal \mathfrak{q}_n such that $\mathfrak{b} \not\subset \mathfrak{q}_n$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ has the required properties.]

End of the statement.

Solution.

Proof of (a): If \mathfrak{q} is \mathfrak{p} -primary, then we have $(0) \subset \mathfrak{q}_{\mathfrak{p}} \subset A_{\mathfrak{p}}$, and thus $S_{\mathfrak{p}}(0) = (0)^c \subset (\mathfrak{q}_{\mathfrak{p}})^c = \mathfrak{q} \subset A$ [see Section 5.1.7 p. 80 above].

Proof of (b): The following argument was explained to me by user withoutfeather over on Mathematics Stackexchange: <https://math.stackexchange.com/a/3338211/660>.

Assume that $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ is **not** reduced.

Since $\mathfrak{q}_n \not\supset \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{n-1}$, there is an index i with $1 \leq i \leq n-1$ such that \mathfrak{q}_i contains the intersection of the other \mathfrak{q}_j . We can assume that $i = 1$, i.e. $\mathfrak{q}_1 \supset \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_n$. By induction hypothesis, $\mathfrak{q}_1 \not\supset \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_{n-1}$, so there exists

$$x \in (\mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_{n-1}) \setminus \mathfrak{q}_1.$$

On the other hand, we have $\mathfrak{p}_1 = r(\mathfrak{q}_1) \not\supset \mathfrak{q}_n$. Indeed, $\mathfrak{p}_1 \supset \mathfrak{q}_n$ would imply $\mathfrak{p}_1 \supset \mathfrak{p}_n$ and, since \mathfrak{p}_n is maximal in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, we would have $\mathfrak{p}_1 = \mathfrak{p}_n$, contradiction. Hence there exists

$$y \in \mathfrak{q}_n \setminus \mathfrak{p}_1.$$

Then the product xy is in $\mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_n \subset \mathfrak{q}_1$. But this is impossible since x is not in \mathfrak{q}_1 and y is not in \mathfrak{p}_1 .

5.3.17 Page 57, Exercise 4.20

Statement. Let M be a fixed A -module, N a submodule of M . The **radical** of N in M is defined to be

$$r_M(N) = \{x \in A \mid x^q M \subset N \text{ for some } q > 0\}.$$

Show that $r_M(N) = r(N : M) = r(\text{Ann}(M/N))$. In particular, $r_M(N)$ is an ideal.

State and prove the formulas for r_M analogous to (1.13).

Solution. Analogue of (ii): $r_A(r_M(N)) = r_M(N)$.

Analogue of (iii): $r_M(N \cap P) = r_M(N) \cap r_M(P)$.

Analogue of (iv): $r_M(N) = (1) \iff N = M$.

⁵By the condition in the first sentence of (b).

Analogue of (v): $r_A(r_M(N) + r_M(P)) \subset r_M(N + P)$. As pointed out by Jeffrey Daniel Kasik Carlson in <https://spaces.ac.cn/usr/uploads/2017/07/4208763092.pdf> p. 70, the converse is false: set

$$A \neq 0, \quad M = A \oplus A, \quad N = A \oplus (0), \quad P = (0) \oplus A.$$

This yields $M = N + P$, so $r_M(N + P) = (1)$, but $r_M(N) = r_M(P) = 0$.

5.3.18 Page 57, Exercise 4.21

Statement. An element $x \in A$ defines an endomorphism ϕ_x of M , namely $m \mapsto xm$. The element x is said to be a **zero-divisor** (resp. **nilpotent**) in M if ϕ_x is not injective (resp. is nilpotent). A submodule Q of M is **primary** in M if $Q \neq M$ and every zero-divisor in M/Q is nilpotent.

Show that if Q is primary in M , then $(Q : M)$ is a primary ideal and hence $r_M(Q)$ is a prime ideal \mathfrak{p} . We say that Q is **\mathfrak{p} -primary** (in M).

Prove the analogues of (4.3) and (4.4).

Solution. Analogue of (4.3): In the statement and the proof of (4.3) one can replace each occurrence of \mathfrak{q} with Q and each occurrence of r with r_M .

Analogue of (4.4): For the analogues of (i) and (ii), replace \mathfrak{q} with Q and $x \in A$ with $x \in M$. I see no analogue of (iii). For the proof of (ii), one can modify the text of the book according to the following table:

$y \in (\mathfrak{q} : x)$	$xy \in \mathfrak{q}$	$x \notin \mathfrak{q}$	$\mathfrak{q} \subset (\mathfrak{q} : x) \subset \mathfrak{p}$	$r(\mathfrak{q} : x) = \mathfrak{p}$
$y \in (Q : x)$	$yx \in Q$	$x \notin Q$	$(Q : M) \subset (Q : x) \subset \mathfrak{p}$	$r(Q : x) = \mathfrak{p}$

Continuation of the table:

$yz \in (\mathfrak{q} : x)$	$xyz \in \mathfrak{q}$	$xz \in \mathfrak{q}$	$z \in (\mathfrak{q} : x)$
$yz \in (Q : x)$	$yzx \in Q$	$zx \in Q$	$z \in (Q : x)$

5.3.19 Page 58, Exercise 4.22

Statement. A primary decomposition of N in M is a representation of N as an intersection

$$N = Q_1 \cap \cdots \cap Q_n$$

of primary submodules of M ; it is a minimal primary decomposition if the ideals $\mathfrak{p}_i = r_M(Q_i)$ are all distinct and if none of the components Q_i can be omitted from the intersection, that is if $Q_i \not\supseteq_{j \neq i} Q_j$ ($1 \leq i \leq n$).

Prove the analogue of (4.5), that the prime ideals \mathfrak{p}_i depend only on N (and M). They are called the **prime ideals belonging to N** in M . Show that they are also the prime ideals belonging to 0 in M/N .

Solution. Analogue of (4.5):

\mathfrak{a}	\mathfrak{q}_i	$r(\mathfrak{q}_i)$	$x \in A$	$x_i \notin \mathfrak{q}_i$
N	Q_i	$r_M(Q_i)$	$x \in M$	$x_i \notin Q_i$

5.3.20 Page 58, Exercise 4.23

Statement. State and prove the analogues of (4.6)-(4.11) inclusive. (There is no loss of generality in taking $N = 0$.)

Solution. Analogues of (4.6)-(4.11), assuming $N = 0$:

- (4.6):

$\mathfrak{a} \subset A$	$\mathfrak{p} \supset \mathfrak{a}$	$\mathfrak{p} \supset \mathfrak{a} = \bigcap \mathfrak{q}_i$	$r(\mathfrak{q}_i)$
$0 \subset M$	$\mathfrak{p} \supset (0 : M)$	$\mathfrak{p} \supset (0 : M) = \bigcap (Q_i : M)$	$r(Q_i : M)$

- (4.7): Replace

“if the zero ideal is decomposable, the set D of zero-divisors of A is the union of the prime ideals belonging to 0 ”

with

“if the zero submodule of M is decomposable, the set D of zero-divisors of A in M is the union of the prime ideals belonging to $0 \subset M$ ”

- (4.8): Let S be a multiplicative subset of A , let M be an A -module and Q a \mathfrak{p} -primary submodule of M .

(i) If $S \cap \mathfrak{p} \neq \emptyset$, then $S^{-1}Q = S^{-1}M$. Proof: Let $s \in S \cap \mathfrak{p}$, $t \in S$ and $x \in M$. We get $s^n \in (Q : M)$, that is $s^n M \subset Q$, for some n , and thus

$$\frac{x}{t} = \frac{s^n x}{s^n t} \in S^{-1}Q.$$

(ii) Assume $S \cap \mathfrak{p} = \emptyset$. The analogue of (4.8ii) is:

$S^{-1}Q$ is $S^{-1}\mathfrak{p}$ -primary and its contraction in M is Q . Hence primary submodules correspond to primary submodules in the correspondence between submodules in $S^{-1}M$ and contracted submodules in M .

We can adapt the proof in the book by adding the following three observations:

(a) For any submodule N of M we have $S^{-1}(N : M) \subset (S^{-1}N : S^{-1}M)$. Proof: straightforward.

(b) We have $(S^{-1}Q : S^{-1}M) \subset S^{-1}(Q : M)$. Proof: straightforward.

(c) If N is a contracted submodule N of M such that $S^{-1}N$ is primary, then N is primary. Proof: Note that $sx \in N$ with $s \in S$ and $x \in M$ imply $x \in N$. Indeed, we have

$$\frac{x}{1} = \frac{1}{s} \frac{sx}{1} \in S^{-1}N.$$

Now if $a \in A$ and $x \in M \setminus N$ satisfy $ax \in N$, we get $\frac{x}{1} \notin S^{-1}N$ and $\frac{a}{1} \frac{x}{1} = \frac{ax}{1} \in S^{-1}N$, hence

$$\frac{a^n}{1} S^{-1}M = \left(\frac{a}{1}\right)^n S^{-1}M \subset S^{-1}N$$

for some n , and thus $a^n M \subset N$.

- (4.9) [Proposition 42 p. 81], (4.10) and (4.11): We use the table

$\mathfrak{a} \subset A$	\mathfrak{q}_i	$r(\mathfrak{q}_i)$
$0 \subset M$	Q_i	$r(Q_i : M)$

5.3.21 Primary decomposition of a submodule after Bourbaki

We follow closely Bourbaki's **Algèbre commutative**, Chapter IV. Unless otherwise stated, A is a commutative ring with one and M is an A -module. For any element a of A and any A -module M write a_M for the map $x \mapsto ax$, $M \rightarrow M$

Definition 54. Let M be an A -module. We say that a prime ideal \mathfrak{p} of A is **associated** to M if there is an $x \in M$ such that \mathfrak{p} is the annihilator of x . We write $\text{Ass}(M)$ for the set of those prime ideals of A which are associated to M .

The annihilator of the zero module being the unit ideal, an element $x \in M$ whose annihilator is prime is nonzero.

Lemma 55. Let \mathfrak{p} be a prime ideal of A . Then \mathfrak{p} is associated to M if and only if M contains a submodule N isomorphic to A/\mathfrak{p} , in which case we can take $N := Ax$ for any $x \in M$ whose annihilator is \mathfrak{p} .

Proof. This is clear. □

If M is the union of a family $(M_i)_{i \in I}$ of submodules, we clearly have

$$\text{Ass}(M) = \bigcup_{i \in I} \text{Ass}(M_i). \quad (40)$$

Proposition 56. For all prime ideal \mathfrak{p} of A and all nonzero submodule M of A/\mathfrak{p} we have $\text{Ass}(M) = \{\mathfrak{p}\}$.

Proof. As A/\mathfrak{p} is a domain, the annihilator in A of any of its nonzero element is \mathfrak{p} . □

Proposition 57. Let \mathfrak{a} be an ideal of A which is maximal among all ideals of the form $\text{Ann}(x)$, $x \in M \setminus \{0\}$. Then \mathfrak{a} is prime, and thus $\mathfrak{a} \in \text{Ass}(M)$.

Proof. Let x and \mathfrak{a} be as above. It suffices to check that \mathfrak{a} is prime. As $x \neq 0$, we have $\mathfrak{a} \neq (1)$. Let b, c be elements of A such that $bc \in \mathfrak{a}$ and $c \notin \mathfrak{a}$. We get $cx \neq 0$, $b \in \text{Ann}(cx)$ and $\mathfrak{a} \subset \text{Ann}(cx)$. As \mathfrak{a} is maximal, this implies $\text{Ann}(cx) = \mathfrak{a}$, whence $b \in \mathfrak{a}$. This shows that \mathfrak{a} is prime. □

Corollary 58. If A is a noetherian ring and M an A -module, then the conditions $M = 0$ and $\text{Ass}(M) = \emptyset$ are equivalent.

Proof. If $M = 0$, then $\text{Ass}(M)$ is empty (even if A were not noetherian). If $M \neq 0$, the set S of all ideals of the form $\text{Ann}(x)$, $x \in M \setminus \{0\}$, is nonempty and $(1) \notin S$; as A is noetherian, this set has a maximal element; and Proposition 57 gives the desired conclusion. □

Corollary 59. *Let A be noetherian, let a be in A and let M be an A -module. Then a_M is injective if and only if a belongs to no prime ideal associated to M .*

Proof. If $a \in \mathfrak{p} \in \text{Ass}(M)$, we have $\mathfrak{p} = \text{Ann}(x)$ for some $x \in M \setminus \{0\}$, and the equality $ax = 0$ shows that a_M is not injective. Conversely, if $ax = 0$ for some $x \in M \setminus \{0\}$, we get $Ax \neq 0$, whence $\text{Ass}(Ax) \neq \emptyset$ (Corollary 58). Let $\mathfrak{p} \in \text{Ass}(Ax)$; we obviously have $\mathfrak{p} \in \text{Ass}(M)$ and $\mathfrak{p} = \text{Ann}(bx)$ for some $b \in A$; whence $a \in \mathfrak{p}$ since $abx = 0$. \square

Corollary 60. *The set of all zero-divisors in a noetherian ring A is the union of the ideals $\mathfrak{p} \in \text{Ass}(A)$.*

Proposition 61. *If N is a submodule of M , we have $\text{Ass}(N) \subset \text{Ass}(M) \subset \text{Ass}(N) \cup \text{Ass}(M/N)$.*

Proof. The inclusion $\text{Ass}(N) \subset \text{Ass}(M)$ is obvious. Let $\mathfrak{p} \in \text{Ass}(M)$. By Lemma 55 there is a submodule E of M isomorphic to A/\mathfrak{p} . Set $F := E \cap N$. If $F = 0$, then E is isomorphic to a submodule of M/N , and (again by Lemma 55) $\mathfrak{p} \in \text{Ass}(M/N)$. If $F \neq 0$, the annihilator of any nonzero element of F is \mathfrak{p} (Proposition 56), hence $\mathfrak{p} \in \text{Ass}(N)$. \square

Lemma 62. *If A is a noetherian ring, and if $(N_i)_{i \in I}$ is a family of submodules of an A -module M such that the intersection of the $\text{Ass}(N_i)$ is empty, then the intersection of the N_i is the zero submodule.*

Proof. This follows from Corollary 58 and Proposition 61. \square

Corollary 63. *If M is the direct sum of a family $(M_i)_{i \in I}$ of submodules, then $\text{Ass}(M)$ is the union of the $\text{Ass}(M_i)$.*

Proof. By (40) we can assume that I is finite. Arguing by induction, we see that it suffices to handle the case $I = \{1, 2\}$. But this case follows from Proposition 61. \square

Corollary 64. *Let M be an A -module and Q_1, \dots, Q_n submodules of M . If the intersection of the Q_i is 0, then $\text{Ass}(M)$ is contained in the union of the $\text{Ass}(Q_i)$, and thus $\text{Ass}(M)$ coincides with the union of the $\text{Ass}(Q_i)$.*

Proof. The canonical map $M \rightarrow \bigoplus (M/Q_i)$ being injective, it suffices to apply Proposition 61 and Corollary 63. \square

Proposition 65. *Given $\Psi \subset \text{Ass}(M)$ there is a submodule N of M such that*

$$\text{Ass}(N) = \text{Ass}(M) \setminus \Psi \text{ and } \text{Ass}(M/N) = \Psi.$$

Proof. Let Σ be the set of those submodules P of M such that $\text{Ass}(P) \subset \text{Ass}(M) \setminus \Psi$. By (40) the set Σ , ordered by inclusion, is inductive; moreover the zero submodule belongs to Σ , so Σ is nonempty. Let N be a maximal element of Σ . We have $\text{Ass}(N) \subset \text{Ass}(M) \setminus \Psi$. By Proposition 61, it suffices to prove $\text{Ass}(M/N) \subset \Psi$. Let $\mathfrak{p} \in \text{Ass}(M/N)$; then, by Lemma 55, M/N contains a submodule F/N isomorphic to A/\mathfrak{p} . By Propositions 56 and 61 we have $\text{Ass}(F) \subset \text{Ass}(N) \cup \{\mathfrak{p}\}$. The maximality of N implies $F \notin \Sigma$, and thus $\mathfrak{p} \in \Psi$. \square

Proposition 66. *If A is a noetherian ring and if $(\Psi_i)_{i \in I}$ is a family of subsets of $\text{Ass}(M)$ whose union is $\text{Ass}(M)$, then there is a family $(N_i)_{i \in I}$ of submodules of M such that $\text{Ass}(M/N_i) = \Psi_i$ for all i , and the intersection of the N_i is the zero submodule.*

Proof. Write $M(\Psi)$ for the submodule denoted by N in the proof of Proposition 65 and apply Lemma 62 to the $M(\Psi_i)$. \square

Definition 67. *Let A be a noetherian ring. A submodule N of an A -module M is **primary** if $\text{Ass}(M/N) = \{\mathfrak{p}\}$ for some prime ideal \mathfrak{p} of A , in which case one also says that N is **\mathfrak{p} -primary**.*

Note that Bourbaki's definition given above is **not** equivalent to Atiyah and MacDonald's, as shown by the following example.

Let K be a field, and x and y indeterminates. Set $A := K[[x]]$, $M := K[y]$, and define an A -module structure on M by $(\sum a_n x^n)f := \sum a_n f^{(n)}$, where $f^{(n)}$ is the n -th derivative of f . Then the zero submodule of M is primary in Bourbaki's sense but is **not** an intersection of primary submodules in the sense of Atiyah and MacDonald.

The following consequence of Proposition 66 is stated as Theorem 2.2.1 in Somaya Muiny's thesis https://scholarworks.gsu.edu/math_theses/70/

Theorem 68 (Somaya Muiny). *If M is a module over a noetherian ring A , then the intersection of its primary submodules is zero. More generally, if N is submodule of an A -module M , then N is the intersection of the primary submodules of M which contain N . In fact, N is the intersection of a family $(M(\mathfrak{p}))_{\mathfrak{p} \in \text{Ass}(M/N)}$ of submodules containing N such that each $M(\mathfrak{p})/N$ is \mathfrak{p} -primary.*

Proof. Assuming $N = 0$ without loss of generality, we apply Proposition 66 to the family

$$(\{\mathfrak{p}\})_{\mathfrak{p} \in \text{Ass}(M)}.$$

\square

Proposition 69. *Let A be a noetherian ring and M an A -module. Then the zero submodule of M is the intersection of a finite family (Q_1, \dots, Q_n) of primary submodules of M if and only if $\text{Ass}(M)$ is finite, in which case $\text{Ass}(M)$ is the disjoint union of the $\text{Ass}(Q_i)$. There is a similar statement for an arbitrary submodule of M [instead of the zero submodule].*

Proof. This follows from Corollary 64 and Theorem 68. \square

6 About Chapter 5

6.1 Comments

6.1.1 Page 61, Corollary 5.9

Here is a slightly stronger statement:

Let A be a subring of a ring B such that B is integral over A ; let $\mathfrak{q}, \mathfrak{b}$ be ideals of B such that \mathfrak{q} is prime, $\mathfrak{q} \subset \mathfrak{b}$ and $\mathfrak{q}^c = \mathfrak{b}^c = \mathfrak{p}$ say. Then $\mathfrak{q} = \mathfrak{b}$.

Proof. Arguing as in the book, we see that $\mathfrak{p}_{\mathfrak{p}} \subset A_{\mathfrak{p}}$ and $\mathfrak{q}_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ are maximal, and that we have $\mathfrak{q}_{\mathfrak{p}} \subset \mathfrak{b}_{\mathfrak{p}} \subset B_{\mathfrak{p}}$. Proposition 3.11ii p. 41 of the book [Proposition 33 p. 58] implies $\mathfrak{b}_{\mathfrak{p}} \neq B_{\mathfrak{p}}$, and thus $\mathfrak{q}_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}$. If b is in \mathfrak{b} , we get $b/1 = q/s$ for some q in \mathfrak{q} and some s in $A \setminus \mathfrak{p}$, then $tb \in \mathfrak{q}$ for some t in $A \setminus \mathfrak{p}$, and thus $b \in \mathfrak{q}$. \square

6.1.2 Page 62, proof of Corollary 5.9

Let $A \subset B$ be rings, let \mathfrak{q} be a prime ideal of B and set $\mathfrak{p} := A \cap \mathfrak{q}$:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \alpha \downarrow & & \downarrow \beta \\ A_{\mathfrak{p}} & \hookrightarrow & B_{\mathfrak{p}}. \end{array}$$

Then Corollary 3.4ii p. 39 of the book implies $A_{\mathfrak{p}} \cap \mathfrak{q}_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$.

6.1.3 Page 62, Theorem 5.10

Let \mathfrak{b} be an ideal of B , let \mathfrak{a} be its contraction in A , and assume $\mathfrak{a} \subset \mathfrak{p}$. Then it is easy to see that \mathfrak{q} can be chosen among the prime ideals of B containing \mathfrak{b} .

Also note that Corollary 5.8 and Theorem 5.10 imply that an ideal of A is maximal if and only if it is the contraction of a maximal ideal of B .

6.1.4 Page 62, $\dim A = \dim B$

If $A \subset B$ is an integral extension, then we have $\dim A = \dim B$.

More precisely, we have $\dim A \geq \dim B$ by Corollary 5.9 p. 61 and $\dim A \leq \dim B$ by Theorem 5.11 p. 62.

6.1.5 Page 62, integrally closed domain

Here is an example of a domain which is **not** integrally closed. Let k be a field, let x be an indeterminate and set $A := k[x^2, x^3] \subset k[x]$. Then x is in the field of fractions of A , and is integral over A , but is not in A .

6.1.6 Page 62, Proposition 5.12

Here is a corollary to Proposition 5.12 p. 62:

Let A be a domain and S a multiplicative subset with $0 \notin S$. If A is integrally closed, so is $S^{-1}A$.

6.1.7 Page 63

If $A \subset D$ are domains, and if \mathfrak{a} is an ideal of A , then we denote the extension of \mathfrak{a} in D by $D\mathfrak{a}$, and the integral closure of \mathfrak{a} in D by $D * \mathfrak{a}$.

Lemma 5.14 says

$$D * \mathfrak{a} = r\left((D * A)\mathfrak{a}\right). \quad (41)$$

In particular $D * \mathfrak{a}$ is an ideal of $D * A$.

In the proof of Proposition 5.15, we have the inclusions $\mathfrak{a} \subset A \subset L$. Let

$$x^m + a_1x^{m-1} + \cdots + a_m$$

be the minimal polynomial of x over K . Each a_j being in the ideal (x_1, \dots, x_n) of $L * A$ generated by the x_i , and each x_i being in the set $L * \mathfrak{a}$, which is an ideal of $L * A$ by (41), we get

$$a_j \in (L * \mathfrak{a}) \cap K = K * \mathfrak{a} = r\left((K * A)\mathfrak{a}\right) = r(A\mathfrak{a}) = r(\mathfrak{a})$$

for all j [the second equality following from (41)].

6.1.8 Page 64, proof of Theorem 5.16

Last line of the first paragraph of the proof: It suffices to show $B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A \subset \mathfrak{p}_2$.

6.1.9 Page 65

I would change the sentence

“The conditions of Zorn’s lemma are clearly satisfied and therefore the set Σ has at least one maximal element”

to

“Assuming $(A, f) \in \Sigma$, the conditions of Zorn’s lemma are clearly satisfied and therefore Σ has at least one maximal element $(B, g) \geq (A, f)$ ”.

6.1.10 Page 65, Lemma 5.20

The proof of Lemma 5.20 shows:

Let K be a field, let B be a local subring of K with maximal ideal \mathfrak{m} , let x be a non-zero element of K , let $B[x]$ be the subring of K generated by x over B , and let $\mathfrak{m}[x]$ be the extension of \mathfrak{m} in $B[x]$. Then either $\mathfrak{m}[x] \neq B[x]$ or $\mathfrak{m}[x^{-1}] \neq B[x]$.

6.1.11 Page 66, Theorem 5.21

The statement of the Theorem is:

Theorem 70 (Theorem 5.21 p. 66). *Let (B, g) be a maximal element of Σ . Then B is a valuation ring of the field K .*

Here is a partial converse:

Proposition 71. *Let (B, g) be an element of Σ such that B is a valuation ring of K and $\text{Ker } g$ is the maximal ideal of B . Then (B, g) is maximal.*

The following Lemma will be handy:

Lemma 72. *Let A be a valuation ring of a field K with maximal ideal \mathfrak{m} , and let B be a ring satisfying $A \subsetneq B \subset K$. Then $\mathfrak{m}B = B$.*

Proof. If b is in $B \setminus A$, then b^{-1} , being a non-unit of A , is in \mathfrak{m} , and we get $\mathfrak{m}B \ni b^{-1}b = 1$. \square

Proof of Proposition 71. If we had $(B, g) < (C, h)$ for some $(C, h) \in \Sigma$, we would get $C = (\text{Ker } g)C$ by Lemma 72, and $(\text{Ker } g)C \subset \text{Ker } h$ by assumption. \square

6.1.12 Page 66, proof of Theorem 5.21

The proof shows:

In the setting of Section 6.1.10 above, let x be a non-zero element of K . If $\mathfrak{m}[x] \neq B[x]$, then there is a maximal ideal \mathfrak{m}' of $B[x]$ such that $B \cap \mathfrak{m}' = \mathfrak{m}$ and $B[x]/\mathfrak{m}'$ is algebraic over B/\mathfrak{m} . If $\mathfrak{m}[x^{-1}] \neq B[x^{-1}]$, then there is a maximal ideal \mathfrak{m}' of $B[x^{-1}]$ such that $B \cap \mathfrak{m}' = \mathfrak{m}$ and $B[x]/\mathfrak{m}'$ is algebraic over B/\mathfrak{m} .

6.1.13 Page 66, proof of Corollary 5.22

Here is the statement of Corollary 5.22:

Corollary 73 (Corollary 5.22). *Let A be a subring of a field K . Then the integral closure \bar{A} of A in K is the intersection of all the valuation rings of K which contain A .*

In the proof given in the book I would change the sentences

“Then the restriction to A of the natural homomorphism $A' \rightarrow k'$ defines a homomorphism of A into Ω . By (5.21) this can be extended to some valuation ring $B \supseteq A$.”

to

“By (5.21) the natural homomorphism $A' \rightarrow \Omega$ can be extended to some valuation ring $B \supseteq A'$.”

Also, it would be slightly better to change “let $x \notin \overline{A}$ ” to “let $x \in K \setminus \overline{A}$ ” on the third line of the proof.

6.2 Exercises

6.2.1 Page 67, Exercise 5.1

Statement. Let $f : A \rightarrow B$ be an integral homomorphism of rings. Show that $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a closed mapping, i.e. that it maps closed sets to closed sets. (This is a geometrical equivalent of (5.10).)

Solution. Use Exercise 3.21iii p. 47, Proposition 5.6i p. 61 and Theorem 5.10 p. 62 of the book.

6.2.2 Page 67, Exercise 5.2

Statement. Let A be a subring of a ring B such that B is integral over A , and let $f : A \rightarrow \Omega$ be a homomorphism of A into an algebraically closed field Ω . Show that f can be extended to a homomorphism of B into Ω . [Use (5.10).]

Solution. Set $\mathfrak{p} := \text{Ker } f$ and let $\mathfrak{q} \subset B$ be given by Theorem 5.10 p. 62. Our problem can be summarized as follows:

$$\begin{array}{ccc} A/\mathfrak{p} & \xrightarrow{\quad} & B/\mathfrak{q} \\ \downarrow & \swarrow \text{dashed} & \\ \Omega & & \end{array}$$

Writing K and L for the respective fields of fractions of A/\mathfrak{p} and B/\mathfrak{q} , our problem becomes

$$\begin{array}{ccc} K & \xrightarrow{\quad} & L \\ \downarrow & \swarrow \text{dashed} & \\ \Omega & & \end{array}$$

As L/K is algebraic and Ω algebraically closed, this problem has a solution.

6.2.3 Page 67, Exercise 5.3

Statement. Let $f : B \rightarrow B'$ be a homomorphism of A -algebras, and let C be an A -algebra. If f is integral, prove that $f \otimes 1 : B \otimes_A C \rightarrow B' \otimes_A C$ is integral. [This includes (5.6ii) as a special case.]

Solution. Set $D := (f \otimes 1)(B \otimes_A C)$ and let E be the integral closure of D in $B' \otimes_A C$. We have $1 \otimes c \in D \subset E$ for $c \in C$ and it is easy to see that $b' \otimes 1 \in E$ for $b' \in B'$. Then E contains $b' \otimes c$ for $b' \in B', c \in C$, hence E contains $B' \otimes_A C$.

A more general question. Let $C \leftarrow A \rightarrow B \rightarrow B'$ be morphisms of rings. In this section tensor products are taken over A , and, for any morphism of rings $f : X \rightarrow Y$, the notation \overline{X}^Y means “integral closure of $f(X)$ in Y ”.

The purpose of this section is to show that the morphism $\overline{B}^{B'} \otimes C \rightarrow \overline{B \otimes C}^{B' \otimes C}$ induced by the natural morphism $\overline{B}^{B'} \otimes C \rightarrow B' \otimes C$ is not always surjective.

Let K be a field of characteristic $\neq 2$ and x an indeterminate, and set

$$\begin{aligned} A &:= K[x^2], & B &:= K[x^2, x\sqrt{x^2-1}], \\ B' &:= K(x^2, x\sqrt{x^2-1}), & C &:= K[x]. \end{aligned}$$

We claim

$$B'' := \overline{K[x^2, x\sqrt{x^2-1}]^{K(x^2, x\sqrt{x^2-1})}} = K[x^2, x\sqrt{x^2-1}].$$

Note that the two-element set

$$\{1, x\sqrt{x^2-1}\}$$

is a $K[x^2]$ -basis of $K[x\sqrt{x^2-1}]$, as well as a $K(x^2)$ -basis of $K(x\sqrt{x^2-1})$. Using this fact it is easy to see that $B'' \cap K(x^2) = K[x^2]$. To prove the claim, let u be in B'' . We can write

$$u = f(x^2) + x\sqrt{x^2-1} g(x^2)$$

with $f(x^2), g(x^2) \in K(x^2)$. It suffices to check that $f(x^2)$ and $g(x^2)$ are in $K[x^2]$, or equivalently, that they are in B'' . We have $v := f(x^2) - x\sqrt{x^2-1} g(x^2) \in B''$, and we successively see that the following elements of $K(x^2)$ are in $K[x^2]$:

$$\begin{aligned} \frac{u+v}{2} &= f(x^2), \\ uv &= f(x^2)^2 - x^2(x^2-1) g(x^2)^2, \\ x^2(x^2-1) g(x^2)^2 &. \end{aligned}$$

As $x^2(x^2-1)$ is square-free in $K[x^2]$, we conclude that $g(x^2)$ is also in $K[x^2]$. This completes the proof of the claim.

Using the claim we get firstly

$$\begin{aligned} \overline{B}^{B'} \otimes C &= \overline{K[x^2, x\sqrt{x^2-1}]^{K(x^2, x\sqrt{x^2-1})}} \otimes K[x] \\ &= K[x^2, x\sqrt{x^2-1}] \otimes K[x] \simeq K[x, x\sqrt{x^2-1}], \end{aligned}$$

and secondly

$$\begin{aligned} \overline{B \otimes C}^{B' \otimes C} &= \overline{K[x^2, x\sqrt{x^2-1}] \otimes K[x]^{K(x^2, x\sqrt{x^2-1}) \otimes K[x]}} \\ &\simeq \overline{K[x, x\sqrt{x^2-1}]^{K(x, \sqrt{x^2-1})}} \subset K(x, \sqrt{x^2-1}). \end{aligned}$$

The element $\sqrt{x^2-1} \in K(x, \sqrt{x^2-1})$ is integral over $K[x, x\sqrt{x^2-1}]$, but does **not** belong to this ring.

In fact $K[x, \sqrt{x^2-1}]$ is integrally closed, and thus Dedekind.

6.2.4 Page 67, Exercise 5.4

Statement. Let A be a subring of a ring B such that B is integral over A . Let \mathfrak{n} be a maximal ideal of B and let $\mathfrak{m} = \mathfrak{n} \cap A$ be the corresponding maximal ideal of A . Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?

[Consider the subring $k[x^2 - 1]$ of $k[x]$, where k is a field, and let $\mathfrak{n} = (x - 1)$. Can the element $1/(x + 1)$ be integral?]

Solution. The last sentence of the hint should be “Can the element $1/(x + 1)$ be integral over $k[x^2 - 1]_{(x^2 - 1)}$?”

Note that

$$k[x^2 - 1]_{(x^2 - 1)} = \left\{ \frac{f(x^2 - 1)}{g(x^2 - 1)} \mid f, g \in k[t], g(0) \neq 0 \right\},$$

and, if $1/(x + 1)$ was integral over $k[x^2 - 1]_{(x^2 - 1)}$, we would get

$$\frac{1}{(x + 1)^n} + \frac{f_1(x^2 - 1)}{g_1(x^2 - 1)} \frac{1}{(x + 1)^{n-1}} + \cdots + \frac{f_n(x^2 - 1)}{g_n(x^2 - 1)} = 0$$

[obvious notation]. Multiplying through by $(x + 1)^n$ and setting $x = -1$ yields $1 = 0$.

6.2.5 Page 67, Exercise 5.5

Statement. Let $A \subset B$ be rings, B integral over A .

- (i) If $a \in A$ is a unit in B then it is a unit in A .
- (ii) The Jacobson radical of A is the contraction of the Jacobson radical of B .

Solution.

(i) Let a be an element of A which has an inverse in B . If a was not a unit of A , there would be a prime ideal \mathfrak{p} of A containing a , and, by Theorem 5.10 p. 62 of the book, a prime ideal \mathfrak{q} of B containing a , contradicting the invertibility of a in B .

(ii) In the lines below \mathfrak{m} and \mathfrak{n} run respectively over the maximal ideals of A and B . It suffices to show $A \cap (\bigcap \mathfrak{n}) = \bigcap \mathfrak{m}$, that is $\bigcap (A \cap \mathfrak{n}) = \bigcap \mathfrak{m}$. But this follows immediately from the second paragraph of Section 6.1.3 p. 101.

6.2.6 Page 67, Exercise 5.6

Statement. Let B_1, \dots, B_n be integral A -algebras. Show that $\prod B_i$ is an integral A -algebra.

Solution. It suffices to show that $(0, \dots, 0, b_i, 0, \dots, 0)$ is integral over A , which is easy.

6.2.7 Page 67, Exercise 5.7

Statement. Let A be a subring of a ring B , such that the set $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B .

Solution. Assume by contradiction that $b \in B \setminus A$ is integral over A . We have

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$, $n \geq 1$, and we can suppose that n is minimum. Setting

$$a' := b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-1}$$

we get $a'b = -a_n \in A$, hence $a' \in A$, and the equality

$$b^{n-1} + a_1 b^{n-2} + \cdots + a_{n-2} b + (a_{n-1} - a') = 0$$

contradicts the minimality of n .

6.2.8 Page 67, Exercise 5.8

Statement. (i) Let A be a subring of an integral domain B , and let C be the integral closure of A in B . Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then f, g are in $C[x]$. [Take a field containing B in which the polynomials f, g split into linear factors: say $f = \prod (x - \xi_i)$, $g = \prod (x - \eta_j)$. Each ξ_i , and each η_j is a root of fg , hence is integral over C . Hence the coefficients of f and g are integral over C .]

(ii) Prove the same result without assuming that B (or A) is an integral domain.

Solution. It suffices to prove (ii). To do this we use the argument in the hint: Take a **ring** D containing B in which the polynomials f, g split into linear factors: say $f = \prod (x - \xi_i)$, $g = \prod (x - \eta_j)$. Each ξ_i , and each η_j is a root of fg , hence is integral over C . Hence the coefficients of f and g are integral over C . [As suggested in the book, an obvious argument proves the existence of such a ring D if B is a domain, but this argument does not really use the assumption that B is a domain.]

6.2.9 Page 68, Exercise 5.9

Statement. Let A be a subring of a ring B and let C be the integral closure of A in B . Prove that $C[x]$ is the integral closure of $A[x]$ in $B[x]$. [If $f \in B[x]$ is integral over $A[x]$, then

$$f^m + g_1 f^{m-1} + \cdots + g_m = 0 \quad (g_i \in A[x]).$$

Let r be an integer larger than the degrees of f, g_1, \dots, g_m , and let $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1 \cdot (f_1 + x^r)^{m-1} + \cdots + g_m = 0$$

or say

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0$$

with $h_i \in A[x]$ and

$$h_m = (x^r)^m + (x^r)^{m-1} g_1 + \cdots + g_m \in A[x].$$

Now apply Exercise 8 to the polynomials $-f_1$ and $f_2 := f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}$. [I have made minor changes to the statement in the book.]

Solution. The hint given in the book is complete. Note that, in view of the definition of r , the polynomials $-f_1$ and h_m are monic. Since $-f_1 f_2 = h_m$, the polynomial f_2 is also monic.

6.2.10 Page 68, Exercise 5.10

Statement. A ring homomorphism $f : A \rightarrow B$ is said to have the **going-up property** (resp. the **going-down property**) if the conclusion of the going-up theorem (5.11) (resp. the going-down theorem (5.16)) holds for B and its subring $f(A)$.

Let $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the mapping associated with f .

(i) Consider the following three statements:

(a) f^* is a closed mapping.

(b) f has the going-up property.

(c) Let \mathfrak{q} be any prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c$. Then $f^* : \text{Spec}(B/\mathfrak{q}) \rightarrow \text{Spec}(A/\mathfrak{p})$ is surjective.

Prove that (a) \iff (b) \implies (c). (See also Chapter 6, Exercise 11.)

(ii) Consider the following three statements:

(a') f^* is an open mapping.

(b') f has the going-down property.

(c') For any prime ideal \mathfrak{q} of B , if $\mathfrak{p} = \mathfrak{q}^c$, then $f^* : \text{Spec}(B_{\mathfrak{q}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ is surjective.

Prove that (a') \iff (b') \implies (c'). (See also Chapter 7, Exercise 23.)

[To prove that (a') \iff (c'), observe that $B_{\mathfrak{q}}$ is the direct limit of the rings B_t where $t \in B \setminus \mathfrak{q}$; hence, by Chapter 3, Exercise 26, we have

$$f^*(\text{Spec}(B_{\mathfrak{q}})) = \bigcap_t f^*(\text{Spec}(B_t)) = \bigcap_t f^*(Y_t).$$

Since Y_t is an open neighborhood of \mathfrak{q} in Y , and since f^* is open, it follows that $f^*(Y_t)$ is an open neighborhood of \mathfrak{p} in X and therefore contains $\text{Spec}(A_{\mathfrak{p}})$.]

Solution. Let us just note that, firstly, it is easy to show that (a) implies (c), and that (b) and (c) are equivalent, and, secondly, for the phrase “and therefore contains $\text{Spec}(A_{\mathfrak{p}})$ ” at the end of the hint, see Section 4.2.22 p. 74 above.

6.2.11 Page 68, Exercise 5.11

Statement. Let $f : A \rightarrow B$ be a flat homomorphism of rings. Then f has the going-down property. [Chapter 3, Exercise 18.]

Hint. The proof of Theorem 5.16 p. 64 of the book shows that the going-down property for $f : A \rightarrow B$ is equivalent to the condition that, for all $\mathfrak{q} \in \text{Spec}(B)$, the natural map $\text{Spec}(B_{\mathfrak{q}}) \rightarrow \text{Spec}(A_{\mathfrak{q}^c})$ is surjective.

6.2.12 Page 68, Exercise 5.12

Statement. Let G be a finite group of automorphisms of a ring A , and let A^G denote the subring of G -invariants, that is of all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$. Prove that A is integral over

A^G . [If $x \in A$, observe that x is a root of the polynomial $\prod_{\sigma \in G} (t - \sigma(x))$.]

Let S be a multiplicatively closed subset of A such that $\sigma(S) \subset S$ for all $\sigma \in G$, and let $S^G = S \cap A^G$. Show that the action of G on A extends to an action on $S^{-1}A$, and that $(S^G)^{-1}A^G \simeq (S^{-1}A)^G$.

Hints. The main point is to show that an invariant fraction is equal to a fraction with invariant numerator and denominator. Let $x = \frac{a}{s}$ be our fraction [obvious notation].

Step 1. Set $t := \prod_{\sigma \neq 1} \sigma(s)$. The product st is invariant, and we have

$$x = \frac{a}{s} = \frac{at}{st}.$$

In other words we can assume that the denominator s of our fraction $x = \frac{a}{s}$ is invariant.

Step 2. For $\sigma \in G$ we have $\frac{\sigma(a)}{s} = \frac{a}{s}$, that is $\sigma(a)st_\sigma = ast_\sigma$ for some $t_\sigma \in S$. Arguing as in Step 1 we can assume that the t_σ are invariant.

Details. Set $u_\sigma := \prod_{\tau \neq 1} \tau(t_\sigma)$. Then $t_\sigma u_\sigma$ is invariant, and we get $\sigma(a)st_\sigma u_\sigma = ast_\sigma u_\sigma$, and we can indeed assume that the t_σ are invariant.

Step 3. By a similar trick we can assume that the t_σ are all equal to some $t \in S^G$, and we get $ast = \sigma(a)st = \sigma(ast)$, and thus

$$\frac{a}{s} = \frac{ast}{s^2 t}.$$

Details. Setting $t := \prod_{\sigma} t_\sigma$ and $v_\sigma := \prod_{\tau \neq \sigma} t_\tau$, we get $t = t_\sigma v_\sigma$ and

$$\sigma(a)st = \sigma(a)st_\sigma v_\sigma = ast_\sigma v_\sigma = ast.$$

6.2.13 Page 68, Exercise 5.13

Statement. In the situation of Exercise 12, let \mathfrak{p} be a prime ideal of A^G , and let P be the set of prime ideals of A whose contraction is \mathfrak{p} . Show that G acts transitively on P . In particular, P is finite.

[Let $\mathfrak{p}_1, \mathfrak{p}_2 \in P$ and let $x \in \mathfrak{p}_1$. Then $\prod_{\sigma} \sigma(x) \in \mathfrak{p}_1 \cap A^G = \mathfrak{p} \subset \mathfrak{p}_2$, hence $\sigma(x) \in \mathfrak{p}_2$ for some $\sigma \in G$. Deduce that \mathfrak{p}_1 is contained in $\bigcup_{\sigma} \sigma(\mathfrak{p}_2)$, and then apply (1.11) and (5.9).]

Solution. We follow the hint given in the book.

Let $\mathfrak{q}, \mathfrak{q}' \in P$ and $a \in \mathfrak{q}'$. We have

$$\prod_{\sigma} \sigma(a) \in \mathfrak{q}' \cap A^G = \mathfrak{p} = \mathfrak{q} \cap A^G \subset \mathfrak{q}.$$

Hence there is a $\sigma_a \in G$ such that $\sigma_a(a) \in \mathfrak{q}$, and thus $a \in \sigma_a^{-1}(\mathfrak{q})$. This implies $\mathfrak{q}' \subset \bigcup_{\sigma} \sigma(\mathfrak{q})$, and thus, by Proposition 3 p. 14, $\mathfrak{q}' \subset \sigma(\mathfrak{q})$ for some $\sigma \in G$.

It suffices to prove $\mathfrak{q}' = \sigma(\mathfrak{q})$.

As

$$\sigma(\mathfrak{q}) \cap A^G = \sigma(\mathfrak{q} \cap \sigma^{-1}(A^G)) = \sigma(\mathfrak{q} \cap A^G) = \sigma(\mathfrak{p}) = \mathfrak{p} = \mathfrak{q}' \cap A^G,$$

Corollary 5.9 p. 61 of the book [see Section 6.1.1 p. 101] implies $\mathfrak{q}' = \sigma(\mathfrak{q})$, as was to be shown.

6.2.14 Page 69, Exercise 5.14

Statement. Let A be an integrally closed domain, K its field of fractions and L a finite normal separable extension of K . Let G be the Galois group of L over K and let B be the integral closure of A in L . Show that $\sigma(B) = B$ for all $\sigma \in G$, and that $A = B^G$.

Hints. We clearly have $\sigma(B) \subset B$ for all $\sigma \in G$, hence $B \subset \sigma^{-1}(B)$ for all $\sigma \in G$, hence $B \subset \sigma(B)$ for all $\sigma \in G$, hence $\sigma(B) = B$ for all $\sigma \in G$. The inclusions $A \subset B^G$ and $B^G \subset L^G \cap B = K \cap B$ are obvious. If x is in $K \cap B$, then x is in K and is integral over A , hence x is in A .

6.2.15 Page 69, Exercise 5.15

Statement. Let A be an integrally closed domain, K its field of fractions and L any finite extension field of K , and let B be the integral closure of A in L . Show that, if \mathfrak{p} is any prime ideal of A , then the set of prime ideals \mathfrak{q} of B which contract to \mathfrak{p} is finite [in other words, that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ has finite fibers].

[Reduce to the two cases (a) L separable over K and (b) L purely inseparable over K . In case (a), embed L in a finite normal separable extension of K , and use Exercises 13 and 14. In case (b), if \mathfrak{q} is a prime ideal of B such that $\mathfrak{q} \cap A = \mathfrak{p}$, show that \mathfrak{q} is the set of all $x \in B$ such that $x^{p^m} \in \mathfrak{p}$ for some $m \geq 0$, where p is the characteristic of K , and hence that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is bijective in this case.]

Hints. (a) $\text{Spec}(B) \rightarrow \text{Spec}(A)$ has finite **nonempty** fibers.

(b) Let L/K be purely inseparable of characteristic $p > 0$, and let \mathfrak{p} be a prime ideal of A . By Theorem 5.10 p. 62 of the book, there is a prime ideal \mathfrak{q} of B lying above \mathfrak{p} . If $b \in B$, $n \in \mathbb{N}$ and $b^{p^n} \in \mathfrak{p}$, then $b \in \mathfrak{q}$. If $x \in \mathfrak{q}$, then $x^{p^n} \in K$ for some $n \in \mathbb{N}$, and thus $x^{p^n} \in K \cap \mathfrak{q} = \mathfrak{p}$. We conclude that \mathfrak{q} is the set of those $b \in B$ such that $b^{p^n} \in \mathfrak{p}$ for some $n \in \mathbb{N}$ [even if the extension is of infinite degree]. In particular the map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is bijective.

6.2.16 Page 69, Exercise 5.16, NNT

NNT stands for **Noether's Normalization Theorem**.

The purpose of this section is to review the following closely related statements: Noether's Normalization Theorem, Zariski's Lemma, and the Nullstellensatz.

Recall the general notation $A_s := A[s^{-1}]$.

Noether's Normalization Theorem.

Theorem 74 (Noether's Normalization Theorem). *Let $A \subset B$ be an inclusion of nonzero rings such that B a finitely generated A -algebra. Then there exist a nonzero element s in A , a nonnegative integer n , and elements x_1, \dots, x_n in B_s which are algebraically independent over A_s , such that B_s is a finitely generated module over $A_s[x_1, \dots, x_n]$.*

Proof. Let $y_1, \dots, y_m \in B$ generate B as an A -algebra: $B = A[y_1, \dots, y_m]$. We argue by induction on m . If $m = 0$ or if the y_i are algebraically independent over A , there is nothing to prove.

Thus we can assume that $m \geq 1$; that the statement holds with m replaced by $m - 1$; and that there is a non-constant polynomial $f \in A[Y_1, \dots, Y_m]$, where the Y_i are indeterminates, such that $f(y_1, \dots, y_m) = 0$.

We claim

(\star) There exist a nonzero t in A and elements z_1, \dots, z_{m-1} in B such that B_t is a finitely generated $A_t[z_1, \dots, z_{m-1}]$ -module.

We start the proof of (\star).

Let r be an integer larger than the degree of f , and, for any monomial $u \in A[Y_1, \dots, Y_m]$ occurring in f (with a nonzero coefficient), write u' for the monomial u viewed as a polynomial in the ring

$$\left(A \left[Y_2 - Y_1^r, \dots, Y_m - Y_1^{r^{m-1}} \right] \right) [Y_1].$$

If $u = Y_1^{\alpha_1} \dots Y_m^{\alpha_m}$, then u' is monic of degree $\alpha_1 + \alpha_2 r + \dots + \alpha_m r^{m-1}$. This implies that u' is monic, and that we have $\deg(u') \neq \deg(v')$ for any two distinct such monomials u and v .

As a result, there is a nonzero t in A , an integer $d \geq 1$, and a polynomial

$$g \in \left(A \left[Y_2 - Y_1^r, \dots, Y_m - Y_1^{r^{m-1}} \right] \right) [Y_1]$$

of degree less than d , such that $f = tY_1^d + g$.

In particular, y_1 is integral over the ring $C := A_t[z_1, \dots, z_{m-1}]$ with $z_i := y_{i+1} - y_1^{r^i}$, and $B_t = C[y_1]$ is a finitely generated C -module. This proves (\star).

Let us prove the Theorem.

We can assume that A_t is nonzero. By the inductive hypothesis applied to the inclusion

$$A_t \subset A_t[z_1, \dots, z_{m-1}],$$

there exist a nonzero element u in A_t , a nonnegative integer n , and elements x_1, \dots, x_n in

$$A_t[z_1, \dots, z_{m-1}]_u = (A_t)_u[z_1, \dots, z_{m-1}]$$

which are algebraically independent over $(A_t)_u$, such that $(A_t)_u[z_1, \dots, z_{m-1}]$ is a finitely generated $(A_t)_u[x_1, \dots, x_n]$ -module.

Now $u = v/t^i$ for some nonzero v in A and some integer $i \geq 0$, and we have $(A_t)_u = A_{tv}$.

Setting $s := tv$, we see that $s \neq 0$, that the elements $x_1, \dots, x_n \in A_s[z_1, \dots, z_{m-1}]$ are algebraically independent over A_s , and that $A_s[z_1, \dots, z_{m-1}]$ is a finitely generated $A_s[x_1, \dots, x_n]$ -module.

Recall that B_t is a finitely generated $A_t[z_1, \dots, z_{m-1}]$ -module [see (\star)].

In particular B_s is a finitely generated module over the ring $A_s[z_1, \dots, z_{m-1}]$, ring which is itself, as we have just seen, a finitely generated module over the ring $A_s[x_1, \dots, x_n]$.

This implies that B_s is a finitely generated $A_s[x_1, \dots, x_n]$ -module, as desired. \square

Zariski's Lemma. Here is a statement of **Zariski's Lemma**:

(ZL) *Let k be a field, A a finitely generated k -algebra. Let \mathfrak{m} be a maximal ideal of A . Then the field A/\mathfrak{m} is a finite algebraic extension of k . In particular, if k is algebraically closed then $A/\mathfrak{m} \simeq k$, that is, the natural morphism $k \rightarrow A/\mathfrak{m}$ is an isomorphism.*

In view of Proposition 5.7 p. 61 of the book, Zariski's Lemma follows immediately from Noether's Normalization Theorem.

Here is an easy consequence of (ZL):

(ZL2) *Set $A := k[t_1, \dots, t_n]$ where k is an algebraically closed field and the t_i are indeterminates, let \mathfrak{m} be a maximal ideal of A , regard k as a subring of A , identify k to A/\mathfrak{m} via the isomorphism in (ZL), let $\pi : A \rightarrow k$ be the canonical projection, and set $x_i := \pi(t_i) \in k \subset A$, that is $x_i \in k$ and $\pi(x_i) = \pi(t_i)$. Also write $x := (x_1, \dots, x_n) \in k^n$. Then we have:*

- (a) $\pi(f) = f(x)$ for all f in A ,
- (b) $A = k[t_1 - x_1, \dots, t_n - x_n]$,
- (c) $\mathfrak{m} = (t_1 - x_1, \dots, t_n - x_n)$.

Proof. Since Statement (a) holds for $f = t_i$ and the t_i generate A as a k -algebra, it holds in general. Statement (b) is clear. To prove (c) note that we have $\mathfrak{m} = \{f \in A \mid f(x) = 0\} = (t_1 - x_1, \dots, t_n - x_n)$, the two equalities following respectively from (a) and (b). \square

Nullstellensatz. Here is a statement of the **Nullstellensatz** [see Exercise 14 of Chapter 7, Section 8.2.12 p. 134]:

(N) *Let k be an algebraically closed field, let A denote the polynomial ring $k[t_1, \dots, t_n]$ and let \mathfrak{a} be an ideal in A . Let X be the variety in k^n defined by the ideal \mathfrak{a} , so that X is the set of all $x = (x_1, \dots, x_n) \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$. Let $I(X)$ be the ideal of X , i.e. the ideal of all polynomials $g \in A$ such that $g(x) = 0$ for all $x \in X$. Then $I(X) = r(\mathfrak{a})$.*

Proof. The inclusion $r(\mathfrak{a}) \subset I(X)$ is clear. Let us prove $I(X) \subset r(\mathfrak{a})$.

(ZL2) implies that $I(X)$ is the intersection of all the maximal ideals of A containing \mathfrak{a} . As $r(\mathfrak{a})$ is the intersection of all the prime ideals of A containing \mathfrak{a} , we can assume that \mathfrak{a} is a prime ideal \mathfrak{p} . Then the statement follows immediately from Exercises 5.23 and 5.24 p. 71 of the book. Here is a slightly different argument:

Let f be in $A \setminus \mathfrak{p}$ and set $A_f := A[1/f]$.

It suffices to prove the claim below.

Claim: *There is a maximal ideal of A which does not contain f .*

We give two proofs of the claim.

Proof 1. The extension \mathfrak{p}^e of \mathfrak{p} in A_f being clearly a proper ideal, there is a maximal ideal $\mathfrak{m} \subset A_f$ containing \mathfrak{p}^e .

Note successively that $f \notin \mathfrak{m}$; that $\mathfrak{q} := \mathfrak{m} \cap A$ is a prime ideal of A which does not contain f ; that A_f/\mathfrak{m} is isomorphic to k by (ZL); and that the monomorphism of k -algebras $0 \neq A/\mathfrak{q} \hookrightarrow A_f/\mathfrak{m} \simeq k$

implies that \mathfrak{q} is a maximal ideal of A .

Proof 2. By Noether's Normalization Theorem applied to the k -algebra A_f , there are elements u_1, \dots, u_m of A_f which are algebraically independent over k , such that A_f is integral over

$$k[u_1, \dots, u_m].$$

By Exercise 5.2 p. 67 of the book [see Section 6.2.2 p. 104 above] the k -algebra morphism

$$\varphi : k[u_1, \dots, u_m] \rightarrow k$$

which maps u_i to 0 extends to a k -algebra morphism $\Phi : A_f \rightarrow k$. As $\Phi(f) \neq 0$, we see that $A \cap \text{Ker } \Phi$ is a maximal ideal of A which does not contain f . \square

6.2.17 Page 69, Exercise 5.17

The Weak Nullstellensatz is the statement that the affine algebraic variety attached to a **proper** ideal of $k[t_1, \dots, t_n]$ is nonempty. It is **not** the trivial statement $I(X) \neq (1) \implies X \neq \emptyset$. The Weak Nullstellensatz follows immediately from (ZL2) above [Section 6.2.16].

6.2.18 Page 70, Exercise 5.18

The purpose of this Exercise is to give an alternate proof of Zariski's Lemma [Section 6.2.16]. The hint given in the book is fairly complete.

6.2.19 Page 70, Exercise 5.19

The purpose of this exercise is to deduce the result of Exercise 17 from Exercise 18, that is to deduce the Weak Nullstellensatz from Zariski's Lemma. As indicated in Section 6.2.17 above, the Weak Nullstellensatz follows immediately from (ZL2) [see Section 6.2.16].

6.2.20 Page 70, Exercise 5.20

Statement. Let A be a subring of an integral domain B such that B is finitely generated over A . Show that there exists $t \neq 0$ in A and elements y_1, \dots, y_n in B , algebraically independent over A and such that B_t is integral over $A[y_1, \dots, y_n]_t$.

Solution. By Noether's Normalization Theorem [Theorem 74 p. 110] above there exist a nonzero element s in A and elements x_1, \dots, x_n in B_s which are algebraically independent over A_s , such that B_s is a finitely generated module over $A_s[x_1, \dots, x_n]$. There is a power t of s and elements y_1, \dots, y_n in B such that $x_i = y_i/t$. Then the y_i are algebraically independent over A and we have $B_t = B_s$ and $A[y_1, \dots, y_n]_t = A_s[x_1, \dots, x_n]$.

6.2.21 Page 70, Exercise 5.21

Statement. Let A be a subring of an integral domain B such that B is finitely generated over A . Show that there exists $t \neq 0$ in A such that, if Ω is an algebraically closed field and $f : A \rightarrow \Omega$ is a homomorphism for which $f(t) \neq 0$, then f can be extended to a homomorphism $B \rightarrow \Omega$. [With the notation of Exercise 20, f can be extended first of all to $A[y_1, \dots, y_n]$, for example by mapping each y_i to 0; then to B_t (because $f(t) \neq 0$), and finally to B_t (by Exercise 2, because B_t is integral over $A[y_1, \dots, y_n]_t$).]

Solution. Omitted. [The hint is fairly complete.]

Here is a corollary:

In the setting of the exercise, if $i : A \rightarrow B$ denotes the inclusion, then we have $X_t \subset i^(\text{Spec}(B))$. In particular the interior of $i^*(\text{Spec}(B))$ is nonempty.*

Proof: given $\mathfrak{p} \in X_t$ pick an algebraically closed field Ω containing A/\mathfrak{p} .

6.2.22 Page 70, Exercise 5.22

Statement. Let A, B be as in Exercise 20. If the Jacobson radical of A is zero, then so is the Jacobson radical of B .

[Let $v \neq 0$ be an element of B . We have to show that there is a maximal ideal of B which does not contain v . By applying Exercise 21 to the ring B_v and its subring A , we obtain an element $t \neq 0$ in A . Let \mathfrak{m} be a maximal ideal of A such that $t \notin \mathfrak{m}$, and let $k = A/\mathfrak{m}$. Then the canonical mapping $A \rightarrow k$ extends to a homomorphism g of B_v into an algebraic closure Ω of k . Show that $g(v) \neq 0$ and that $\text{Ker}(g) \cap B$ is a maximal ideal of B .]

Solution. Note that we have $k \subset g(B) \subset \Omega$. By Proposition 5.7 p. 61 of the book, $g(B)$ is a field.

6.2.23 Page 71, Exercise 5.23

Statement. Let A be a ring. Show that the following are equivalent:

- (i) Every prime ideal in A is an intersection of maximal ideals.
- (ii) In every homomorphic image of A the nilradical is equal to the Jacobson radical.
- (iii) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

[The only hard part is (iii) \implies (i). Suppose (i) false, then there is a prime ideal which is not an intersection of maximal ideals. Passing to the quotient ring, we may assume that A is an integral domain whose Jacobson radical \mathfrak{R} is not zero. Let f be a non-zero element of \mathfrak{R} . Then $A_f \neq 0$, hence A_f has a maximal ideal, whose contraction in A is a prime ideal \mathfrak{p} such that $f \notin \mathfrak{p}$ and which is maximal with respect to this property. Then \mathfrak{p} is not maximal and is not equal to the intersection of the prime ideals strictly containing \mathfrak{p} .]

A ring A with the three equivalent properties above is called a **Jacobson ring**.

The hint is fairly complete.

6.2.24 Page 71, Exercise 5.24

Statement. Let A be a Jacobson ring (Exercise 23) and B an A -algebra. Show that if B is either (i) integral over A or (ii) finitely generated as an A -algebra, then B is Jacobson. [Use Exercise 22 for (ii).]

In particular, every finitely generated ring, and every finitely generated algebra over a field, is a Jacobson ring.

Solution. Part (i). Let us prove

If $A \subset B$ is an integral extension of rings and if A is Jacobson, then B is Jacobson.

Proof. By Proposition 5.6ii p. 61 of the book we can assume that B is a domain. Let $\mathfrak{b} \subset B$ be the Jacobson radical of B . It suffices to prove $\mathfrak{b} = (0)$. We have $\mathfrak{b}^c = (0)$ by Exercise 5.5ii p. 67 of the book [see Section 6.2.5 p. 106 above], and the version of Corollary 5.9 p. 61 of the book proved in Section 6.1.1 p. 101, implies $\mathfrak{b} = (0)$. \square

Part (ii) follows from Exercise 5.22 p. 70 of the book [Section 6.2.22 p. 114 above].

6.2.25 Page 71, Exercise 5.25

Statement. Let A be a ring. Show that the following are equivalent:

- (i) A is a Jacobson ring;
- (ii) Every finitely generated A -algebra B which is a field is finite over A .

[(i) \implies (ii). Reduce to the case where A is a subring of B , and use Exercise 21. If $s \in A$ is as in Exercise 21, then there exists a maximal ideal \mathfrak{m} of A not containing s , and the homomorphism $A \rightarrow A/\mathfrak{m} = k$ extends to a homomorphism g of B into an algebraic closure of k . Since B is a field, g is injective, and $g(B)$ is algebraic over k , hence finite algebraic over k .

(ii) \implies (i). Use criterion (iii) of Exercise 23. Let \mathfrak{p} be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let f be a non-zero element of B . Then B_f is a finitely generated A -algebra. If it is a field it is finite over B , hence integral over B and therefore B is a field by (5.7). Hence B_f is not a field and therefore has a non-zero prime ideal, whose contraction in B is a non-zero ideal \mathfrak{p}' such that $f \notin \mathfrak{p}'$.]

Solution. The hint is fairly complete. [There is a typo in the book: “the algebraic closure of k ” instead of “an algebraic closure of k ”.]

6.2.26 Page 71, Exercise 5.26

The first two sentences are

“Let X be a topological space. A subset of X is **locally closed** if it is the intersection of an open set and a closed set, or equivalently if it is open in its closure.”

Let us prove this equivalence.

Let U be open and C closed. It suffices to show $(\star) U \cap C = U \cap \overline{U \cap C}$.

Proof of (\star) . We have:

- $U \cap C \subset U \cap \overline{U \cap C}$ because $U \cap C \subset U$ and $U \cap C \subset \overline{U \cap C}$,
- $U \cap \overline{U \cap C} \subset U \cap C$ because $\overline{U \cap C} \subset \overline{C} = C$. \square

Part 1.

Statement. Show that the three conditions below on a subset X_0 of a topological space X are equivalent:

- (1) Every non-empty locally closed subset of X meets X_0 ;
- (2) For every closed set C in X we have $\overline{X_0 \cap C} = C$;
- (3) The mapping $U \mapsto X_0 \cap U$ of the collection of open sets of X onto the collection of open sets of X_0 is bijective.

A subset X_0 satisfying these conditions is said to be **very dense** in X .

Solution. Condition (3) is clearly equivalent to

- (3') The map $C \mapsto X_0 \cap C$ from the set of closed subsets of X to the set of closed subsets of X_0 is bijective.

This is also equivalent to

- (3'') The map $C \mapsto X_0 \cap C$ from the set of closed subsets of X to the set of closed subsets of X_0 is injective.

(1) \implies (2): If there is a closed subset C of X such that $\overline{X_0 \cap C} \neq C$, then $L := C \setminus \overline{X_0 \cap C}$ is a nonempty locally closed subset satisfying $X_0 \cap L = \emptyset$.

(2) \implies (3''): (2) means that $C' \mapsto \overline{C'}$ is a left inverse to the map in (3'').

(3) \implies (1): for U, V open and $V \subset U$ the equality $X_0 \cap (U \setminus V) = \emptyset$ implies $X_0 \cap U = X_0 \cap V$.

Part 2.

Statement. If A is a ring, show that the following are equivalent:

- (i) A is a Jacobson ring;
- (ii) The set of maximal ideals of A is very dense in $\text{Spec}(A)$;
- (iii) Every locally closed subset of $\text{Spec}(A)$ consisting of a single point is closed.

[(ii) and (iii) are geometrical formulations of conditions (ii) and (iii) of Exercise 23.]

Solution.

- Proof of (i) \iff (ii).

Hint: Let $M \subset \text{Spec}(A)$ be the set of maximal ideals. In view of Condition (2) above, it suffices to show that (a) and (b) below are equivalent.

- (a) For all ideal \mathfrak{a} of A we have $\mathfrak{R}(A/\mathfrak{a}) \subset \mathfrak{R}(A/\mathfrak{a})$.
- (b) For all ideal \mathfrak{a} of A we have $V(\mathfrak{a}) \subset \overline{V(\mathfrak{a}) \cap M}$.

Let $\mathfrak{b}(\mathfrak{a})$ be the intersection of the maximal ideals containing \mathfrak{a} . In fact we have

$$\begin{aligned} (a) &\iff [\mathfrak{b}(\mathfrak{a}) \subset r(\mathfrak{a})] \forall \mathfrak{a} \\ &\iff [\mathfrak{p} \supset \mathfrak{a} \implies \mathfrak{p} \supset \mathfrak{b}(\mathfrak{a})] \forall \mathfrak{a} \forall \mathfrak{p} \iff (b), \end{aligned}$$

where “ $\forall \mathfrak{a}$ ” means “for all ideal \mathfrak{a} of A ” and “ $\forall \mathfrak{p}$ ” means “for all prime ideal \mathfrak{p} of A ”.

• Proof of (i) \iff (iii).

It suffices to show that (c) and (d) below are equivalent.

(c) Every non-maximal prime ideal \mathfrak{p} is the intersection of the strictly larger prime ideals.

(d) If \mathfrak{p} is a prime ideal and if the singleton $\{\mathfrak{p}\}$ is locally closed, then \mathfrak{p} is maximal.

(c) \implies (d): Assume by contradiction that the singleton $\{\mathfrak{p}\}$ is locally closed and \mathfrak{p} is not maximal, and let Q be the set of all those prime ideals of A which are strictly larger than \mathfrak{p} . We have

$$\mathfrak{p} = \bigcap_{\mathfrak{q} \in Q} \mathfrak{q} \quad (42)$$

and

$$\{\mathfrak{p}\} = V(\mathfrak{a}) \setminus V(\mathfrak{b}) \quad (43)$$

for some ideals \mathfrak{a} and \mathfrak{b} . Let \mathfrak{q} be in Q . Clearly \mathfrak{q} contains \mathfrak{a} . Equality (43) implies that \mathfrak{q} contains \mathfrak{b} . As \mathfrak{q} is an arbitrary element of Q , (42) entails that \mathfrak{p} contains \mathfrak{b} , contradicting (43).

(d) \implies (c): Let \mathfrak{p} be a non-maximal prime ideal, and let \mathfrak{b} be the intersection of those prime ideals which are strictly larger than \mathfrak{p} , and assume by contradiction that \mathfrak{b} is strictly larger than \mathfrak{p} . Let L be the locally closed subset $V(\mathfrak{p}) \setminus V(\mathfrak{b})$ of $\text{Spec}(A)$. Then \mathfrak{p} is in L , that is $\{\mathfrak{p}\} \subset L$. Since $\{\mathfrak{p}\}$ is not locally closed by (d), we have $\{\mathfrak{p}\} \neq L$, that is $\{\mathfrak{p}\} \subsetneq L$. Thus there is a \mathfrak{q} in $L \setminus \{\mathfrak{p}\}$, and we get $\mathfrak{q} \supsetneq \mathfrak{p}$ and $\mathfrak{q} \not\supset \mathfrak{b}$, and the definition of \mathfrak{b} implies $\mathfrak{q} \supset \mathfrak{b}$, contradiction.

6.2.27 Page 72, Exercise 5.27

Statement. Let A, B be two local rings. B is said to **dominate** A if A is a subring of B and the maximal ideal \mathfrak{m} of A is contained in the maximal ideal \mathfrak{n} of B [or, equivalently, if $\mathfrak{m} = \mathfrak{n} \cap A$]. Let K be a field and let Σ be the set of all local subrings of K . If Σ is ordered by the relation of domination, show that Σ has maximal elements and that $A \in \Sigma$ is maximal if and only if A is a valuation ring of K . [Use (5.21).]

Solution. For any algebraically closed field Ω we write $\Phi(K, \Omega)$ for the poset denoted by Σ on page 65 of the book [see Section 6.1.9 p. 102]. [More precisely, $\Phi(K, \Omega)$ is the set of all pairs (A, f) , where A is a subring of K and f is a homomorphism of A into Ω , and we endow $\Phi(K, \Omega)$ with the partial order defined in the book.] Let $\Sigma(K)$ be the set of all local subrings of K . We equip $\Sigma(K)$ with the domination partial order. We leave it to the reader to check that $\Sigma(K)$ does have maximal elements.

Let (A, \mathfrak{m}) be in $\Sigma(K)$. It suffices to show that (A, \mathfrak{m}) is maximal in $\Sigma(K)$ if and only if A is a valuation ring of K .

Assume that A is a valuation ring of K and that (B, \mathfrak{n}) is a maximal element of $\Sigma(K)$ with $(A, \mathfrak{m}) \leq (B, \mathfrak{n})$. Then we have $\mathfrak{m}B \subset \mathfrak{n}$, and Lemma 72 p. 103 implies $B = A$.

Assume now that (A, \mathfrak{m}) is maximal in $\Sigma(K)$. Let Ω be an algebraic closure of A/\mathfrak{m} , let $f : A \rightarrow \Omega$ be the obvious morphism, and let (B, \mathfrak{g}) be a maximal element of $\Phi(K, \Omega)$ such that $(A, f) \leq (B, \mathfrak{g})$. As B is a valuation ring of K by Theorem 5.21 p. 66 of the book, it suffices to show $B = A$. Lemma 5.19 p. 65 of the book implies that B is local and that $\text{Ker } g$ is its maximal ideal. As $\mathfrak{m} = \text{Ker } f \subset \text{Ker } g$, we see that B dominates A , and thus that $B = A$, as desired.

6.2.28 Page 72, Exercise 5.28

Statement. Let A be an integral domain, K its field of fractions. Show that the following are equivalent:

- (1) A is a valuation ring of K .
- (2) If $\mathfrak{a}, \mathfrak{b}$ are any two ideals of A , then either $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$.

Deduce that if A is a valuation ring and \mathfrak{p} is a prime ideal of A , then $A_{\mathfrak{p}}$ and A/\mathfrak{p} are valuation rings of their fields of fractions.

Hint. It suffices to show that the following statements are equivalent:

- (a) A is **not** a valuation ring of K ,
- (b) there are incomparable principal ideals of A ,
- (c) there are incomparable ideals of A .

Clearly (b) implies (c). There are $a, b \in A \setminus \{0\}$ such that $\frac{a}{b}$ and $\frac{b}{a}$ are not in A if and only if the principal ideals (a) and (b) are incomparable. This implies that (a) and (b) are equivalent, and it only remains to show that (c) implies (b). If \mathfrak{a} and \mathfrak{b} are incomparable ideals, if a is in $\mathfrak{a} \setminus \mathfrak{b}$ and if b is in $\mathfrak{b} \setminus \mathfrak{a}$, then the principal ideals (a) and (b) are incomparable.

6.2.29 Page 72, Exercise 5.29

Statement. Let A be a valuation ring of a field K . Show that every subring of K which contains A is a local ring of A [i.e. is of the form $A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset A$].

Solution. Let B be the unnamed ring in the statement. By Proposition 5.18 (i) and (ii) p. 65 of the book, B is local. Let $\mathfrak{n} \subset B$ be the maximal ideal and set $\mathfrak{p} := A \cap \mathfrak{n}$. Then we have $A_{\mathfrak{p}} = B$. Indeed, the inclusion $A_{\mathfrak{p}} \subset B$ is clear. Let $b \in B \setminus A$. It suffices to show $b \in A_{\mathfrak{p}}$. We have $b^{-1} \in A \subset B$. As b^{-1} is a unit of B , it is not in \mathfrak{n} , and thus not in \mathfrak{p} , hence $b = 1/b^{-1} \in A_{\mathfrak{p}}$.

6.2.30 Page 72, Exercise 5.30

Statement. Let A be a valuation ring of a field K . The group U of units of A is a subgroup of the multiplicative group K^* of K .

Let $\Gamma = K^*/U$. If $\xi, \eta \in \Gamma$ are represented by $x, y \in K$, define $\xi \geq \eta$ to mean $xy^{-1} \in A$.

Show that this defines a total ordering on Γ which is compatible with the group structure (i.e., $\xi \geq \eta \implies \xi\omega \geq \eta\omega$ for all $\omega \in \Gamma$). In other words, Γ is a totally ordered abelian group. It is called the **value group** of A .

Let $v : K^* \rightarrow \Gamma$ be the canonical homomorphism. Show that $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in K^*$ such that $x + y \in K^*$.

Hint. We can assume $v(x) \geq v(y)$, that is $xy^{-1} \in A$. We get $A \ni xy^{-1} + 1 = (x + y)y^{-1}$, and thus $v(x + y) \geq v(y) = \min(v(x), v(y))$.

6.2.31 Page 72, Exercise 5.31

Statement. Conversely, let Γ be a totally ordered abelian group (written additively), and let K be a field. A valuation of K with values in Γ is a mapping $v : K^* \rightarrow \Gamma$ such that

- (1) $v(xy) = v(x) + v(y)$,
- (2) $v(x + y) \geq \min(v(x), v(y))$ or $x + y = 0$,

for all $x, y \in K^*$. Show that the set of elements $x \in K$ such that $v(x) \geq 0$ if $x \neq 0$ is a valuation ring of K . This ring is called the valuation ring of v , and the subgroup $v(K^*)$ of Γ is the **value group** of v .

Thus the concepts of valuation ring and valuation are essentially equivalent.

Comments. There are typos in the statement given in the book. The proof is left to the reader.

6.2.32 Page 72, About Exercise 5.31

Let J be a set; for each $j \in J$ let A_j and B_j be two valuation domains with same value group G_j ; let $G_{j,\infty}$ be the totally ordered commutative monoid obtained by adjoining to G_j a largest element, denoted ∞ , such that $\infty + x = \infty$ for all $x \in G_{j,\infty}$; let $v_j : A_j \rightarrow G_{j,\infty}$ and $w_j : B_j \rightarrow G_{j,\infty}$ be the respective valuations; let $\mathfrak{a}_j \subset A_j$ and $\mathfrak{b}_j \subset B_j$ be ideals such that $v_j(\mathfrak{a}_j) = w_j(\mathfrak{b}_j)$ for all j , let A and B be the respective products of the A_j/\mathfrak{a}_j and the B_j/\mathfrak{b}_j .

Proposition 75. *In the above setting, the spectra of A and B are homeomorphic.*

Proof. The proof will actually give a description of the spectra.

Let j be in J . Let $G_{j,\infty,\geq 0}$ be the totally ordered commutative submonoid of nonnegative elements in $G_{j,\infty}$; let M_j be the totally ordered commutative monoid $G_{j,\infty,\geq 0}/\sim$, where \sim is the congruence defined by $x \sim y$ if and only if $x, y \in v_j(\mathfrak{a}_j)$; let M be the product of the M_j ; and let $v : A \rightarrow M$ be the map induced by the v_j .

Let \mathcal{I} be the set of those subsets $I \subset M$ which are upward closed and satisfy $x \wedge y \in I$ whenever $x, y \in I$. Then \mathcal{I} has an obvious structure of lattice for which we have $I \wedge I' = I \cap I'$ and $I \leq I' \iff I \subset I'$ for all $I, I' \in \mathcal{I}$. Let \mathcal{P} be the sub-poset of \mathcal{I} formed by the sets $P \in \mathcal{I}$ such that $x, y \in M \setminus P$ implies $x + y \notin P$.

We leave it to the reader to check that v induces a lattice isomorphism from the lattice of ideals of A to \mathcal{I} , and that $v(\mathfrak{a}) \in \mathcal{P}$ if and only if \mathfrak{a} is prime. This implies the proposition. \square

Statement. (a) Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is **isolated** in Γ if, whenever $0 \leq \beta \leq \alpha$ and $\alpha \in \Delta$, we have $\beta \in \Delta$. Let A be a valuation ring of a field K , with value group Γ (Exercise 31; Section 6.2.31 p. 119). If \mathfrak{p} is a prime ideal of A , show that $v(A \setminus \mathfrak{p})$ is the set of elements ≥ 0 in an isolated subgroup Δ of Γ , and that the mapping so defined of $\text{Spec}(A)$ into the set of isolated subgroups of Γ is bijective.

(b) If \mathfrak{p} is a prime ideal of A , what are the value groups of the valuation rings $A/\mathfrak{p}, A_{\mathfrak{p}}$?

Hints. (a) For any domain A , let A^*, A^0 , and $Q(A)$ denote respectively the group of units of A , the monoid of nonzero elements of A , and the field of fractions of A . For any totally ordered multiplicative abelian group G , let G^+ denote the monoid of elements ≥ 1 .

Let A be a valuation domain. Recall that the group $\Gamma(A) := Q(A)^*/A^*$, written additively, is the group of values of A , that it is totally ordered, that we have $\Gamma(A)^+ = A^0/A^*$, and that the canonical projection $Q(A)^* \rightarrow \Gamma(A)$ is designated by v .

Let \mathfrak{p} be a prime ideal of A , set $S := A \setminus \mathfrak{p}$ and let Δ be the subgroup of $\Gamma := \Gamma(A)$ generated by S/A^* . If $\langle S \rangle$ is the subgroup of $Q(A)^*$ generated by S , then we have $\Delta = \langle S \rangle/A^*$.

Let us show that Δ is isolated. It suffices to prove that $0 < \beta < \alpha$ and $\alpha \in \Delta$ implies $\beta \in \Delta$. Say that α is represented by $s \in S$ and β by $b \in A^0$. By assumption we have $sb^{-1} = a \in A$, that is $s = ab$, which implies $b \in S$, that is $\beta \in \Delta$.

The map $\mathfrak{p} \mapsto \Delta$ is clearly injective. To show that it is surjective let Δ be an isolated subgroup of Γ , and set

$$\mathfrak{p} := \left(A \setminus v^{-1}(\Delta^+) \right) \cup \{0\} \subset A.$$

It suffices to verify that \mathfrak{p} is a prime ideal. Let $a, b \in \mathfrak{p}$ and let us check that $a + b \in \mathfrak{p}$. We can assume $a + b \neq 0$. If $a + b$ was not in \mathfrak{p} we would get $v(a + b) \geq \min(v(a), v(b))$, and a or b would not be in \mathfrak{p} by isolation of Δ . Let $p \in \mathfrak{p}$ and $a \in A$ and let us check that $ap \in \mathfrak{p}$. Again we can assume $ap \neq 0$, and we get $v(ap) = v(a) + v(p) > v(p)$, which entails $ap \in \mathfrak{p}$. Finally, if $s, t \notin \mathfrak{p}$, then $st \notin \mathfrak{p}$ because Δ is a subgroup of Γ .

(b) We claim

$$\Gamma(A_{\mathfrak{p}}) \simeq \frac{\Gamma}{\Delta} \quad (44)$$

and

$$\Gamma(A/\mathfrak{p}) \simeq \Delta \quad (45)$$

[isomorphisms of totally ordered abelian groups].

Proof of (44):

$$\Gamma(A_{\mathfrak{p}}) = Q(A)^*/(A_{\mathfrak{p}})^* = Q(A)^*/\langle S \rangle \simeq \frac{Q(A)^*/A^*}{\langle S \rangle/A^*} = \frac{\Gamma}{\Delta}.$$

Proof of (45): Set $\bar{A} := A/\mathfrak{p}$ and, for any $a \in A$, write \bar{a} for the image of a in \bar{A} . As $\Gamma(\bar{A})$ and Δ are totally ordered abelian groups, it suffices to show

$$\Gamma(\bar{A})^+ \simeq \Delta^+ \text{ [isomorphism of totally ordered monoids]}. \quad (46)$$

Proof of (46): We have $\Gamma(\overline{A})^+ = (\overline{A})^0/(\overline{A})^*$ and $\Delta^+ = S/A^*$. Define the monoid morphism $f : S \rightarrow (\overline{A})^0$ by $f(s) := \overline{s}$, and consider the diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & (\overline{A})^0 \\ p \downarrow & & \downarrow q \\ S/A^* & \xrightarrow[\overline{f}]{} & (\overline{A})^0/(\overline{A})^*, \end{array}$$

where p and q are the canonical projections. We claim that there is an induced morphism $\overline{f} : S/A^* \rightarrow (\overline{A})^0/(\overline{A})^*$, and that \overline{f} is bijective. The existence of \overline{f} and its surjectivity are easy to prove. The injectivity of \overline{f} follows from the fact that, A being local, any unit of \overline{A} is the image of a unit of A . This proves (46), and thus (45).

6.2.34 Page 72, Exercise 5.33

Statement. Let Γ be a totally ordered abelian group. We shall show how to construct a field K and a valuation v of K with Γ as value group. Let k be any field and let $A = k[\Gamma]$ be the group algebra of Γ over k . By definition, A is freely generated as a k -vector space by elements x_α ($\alpha \in \Gamma$) such that $x_\alpha x_\beta = x_{\alpha+\beta}$. Show that A is an integral domain.

If $u = \lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n}$ is any non-zero element of A , where the λ_i , are all $\neq 0$ and $\alpha_1 < \cdots < \alpha_n$, define $v_0(u)$ to be α_1 . Show that the mapping $v_0 : A - \{0\} \rightarrow \Gamma$ satisfies conditions (1) and (2) of Exercise 31 [Section 6.2.31 p. 119].

Let K be the field of fractions of A . Show that v_0 can be uniquely extended to a valuation v of K , and that the value group of v is precisely Γ .

Solution. Left to the reader.

6.2.35 Page 73, Exercise 5.34

Statement. Let A be a valuation ring and K its field of fractions. Let $f : A \rightarrow B$ be a ring homomorphism such that $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a **closed** mapping. Then if $g : B \rightarrow K$ is any A -algebra homomorphism (i.e., if $g \circ f$ is the embedding of A in K) we have $g(B) = A$.

[Let $C = g(B)$; obviously $C \supset A$. Let \mathfrak{n} be a maximal ideal of C . Since f^* is closed, $\mathfrak{m} = \mathfrak{n} \cap A$ is the maximal ideal of A , whence $A_{\mathfrak{m}} = A$. Also the local ring $C_{\mathfrak{n}}$ dominates $A_{\mathfrak{m}}$. Hence by Exercise 27 [Section 6.2.27 p. 117] we have $C_{\mathfrak{n}} = A$ and therefore $C = A$.]

Hint. In the hint of the book we must check that $C_{\mathfrak{n}}$ dominates A , that is $\mathfrak{m} \subset \mathfrak{n}_{\mathfrak{n}}$. But we have $\mathfrak{m} = A \cap \mathfrak{n} \subset \mathfrak{n}_{\mathfrak{n}}$.

6.2.36 Page 73, Exercise 5.35

Statement. From Exercises 1 and 3 [Section 6.2.1 p. 104 and Section 6.2.3 p. 104] it follows that, if $f : A \rightarrow B$ is integral and C is any A -algebra, then the mapping $(f \otimes 1)^* : \text{Spec}(B \otimes_A C) \rightarrow \text{Spec}(C)$ is a closed map.

Conversely, suppose that $f : A \rightarrow B$ has this property and that B is an integral domain. Then f is integral. [Replacing A by its image in B , reduce to the case where $A \subset B$ and f is the injection. Let K be the field of fractions of B and let A' be a valuation ring of K containing A . By (5.22) [Corollary 73 p. 103] it is enough to show that A' contains B . By hypothesis $\text{Spec}(B \otimes_A A') \rightarrow \text{Spec}(A')$ is a closed map. Apply the result of Exercise 34 [Section 6.2.35] to the homomorphism $B \otimes_A A' \rightarrow K$ defined by $b \otimes a' \mapsto ba'$. It follows that $ba' \in A'$ for all $b \in B$ and all $a' \in A'$; taking $a' = 1$, we have what we want.]

Show that the result just proved remains valid if B is a ring with only finitely many minimal prime ideals (e.g., if B is Noetherian). [Let \mathfrak{p}_i be the minimal prime ideals. Then each composite homomorphism $A \rightarrow B \rightarrow B/\mathfrak{p}_i$ is integral, hence $A \rightarrow \prod(B/\mathfrak{p}_i)$ is integral, hence $A \rightarrow B/\mathfrak{N}$ is integral (where \mathfrak{N} is the nilradical of B), hence finally $A \rightarrow B$ is integral.]

Hints. Replacing the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow i & \swarrow g \\ & K & \end{array}$$

implicit in Exercise 5.34 [Section 6.2.35 p. 121] with

$$\begin{array}{ccc} A' & \xrightarrow{\quad} & B \otimes_A A' \\ & \searrow f & \swarrow \\ & K, & \end{array}$$

we get $BA' = A'$ [as subrings of K], and thus $B \subset A'$.

In the second part of this exercise we can use the fact that the natural morphism $B/\mathfrak{N} \rightarrow \prod_i B/\mathfrak{p}_i$ is injective.

7 About Chapter 6

7.1 Comments

7.1.1 Jordan-Hölder Theorem

The following proof is taken from Jeffrey Daniel Kasik Carlson's text cited at the beginning of the present text.

Consider an A -module M of finite length. Proposition 6.7, stated and proved on p. 77 of the book, says that all composition series of M have the same length, and the book claims (p. 77) that the multiset of isomorphism classes of quotients of successive terms is the same for any choice of composition series. This claim is not proved, but the authors write that the proof is the same as for finite groups. We recall it here.

Let A be a ring. In this section “module” means “ A -module”. For any submodule N of any module M , we write $M \xrightarrow{a} N$ to indicate that a is the isomorphism class of M/N . Moreover, if α is the composition series

$$M = M_0 \xrightarrow{a_1} M_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} M_n = 0,$$

we denote by $\chi(\alpha)$ the element $a_1 + \cdots + a_n$ of the free abelian group over the set $\{a_1, \dots, a_n\}$.

Here is what we want to show:

If

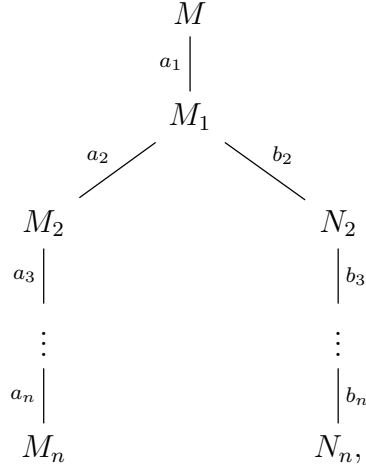
$$M = N_0 \xrightarrow{b_1} N_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} N_n = 0$$

is another composition series, denoted β , for the same module M , then we have $\chi(\alpha) = \chi(\beta)$.

The proof proceeds by induction on the length $\ell(M)$ of M . If $\ell(M) = 0$ or 1 , we are done. Assume inductively that the result holds for all modules of length less than n , and let $\ell(M) = n$. As indicated above, we suppose that M has the two composition series:

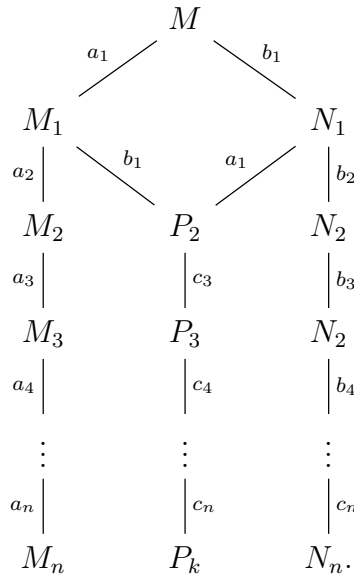
$$\begin{array}{ccc}
 & M & \\
 a_1 \swarrow & & \searrow b_1 \\
 M_1 & & N_1 \\
 a_2 \downarrow & & \downarrow b_2 \\
 M_2 & & N_2 \\
 a_3 \downarrow & & \downarrow b_3 \\
 \vdots & & \vdots \\
 a_n \downarrow & & \downarrow b_n \\
 M_n & & N_n.
 \end{array}$$

Case 1: $M_1 = N_1$. We get $a_1 = b_1$ and



and we are done because the inductive hypotheses yields $a_2 + \cdots + a_n = b_2 + \cdots + b_n$.

Case 2: $M_1 \neq N_1$. Setting $P_2 := M_1 \cap N_1$, we get



The inductive hypotheses implying

$$a_2 + \cdots + a_n = b_1 + c_3 + \cdots + c_n$$

and

$$a_1 + c_3 + \cdots + c_n = b_2 + \cdots + b_n,$$

we get

$$a_1 + a_2 + \cdots + a_n = a_1 + b_1 + c_3 + \cdots + c_n = b_1 + a_1 + c_3 + \cdots + c_n = b_1 + b_2 + \cdots + b_n.$$

This completes the proof.

7.2 Exercises

7.2.1 Page 78, Exercise 6.1

Statement. (i) Let M be a Noetherian A -module and $u : M \rightarrow M$ a module homomorphism. If u is surjective, then u is an isomorphism.

(ii) If M is Artinian and u is injective, then again u is an isomorphism. [For (i), consider the submodules $\text{Ker}(u^n)$; for (ii), the quotient modules $\text{Coker}(u^n)$.]

Solution. (i) Note that M is finitely generated by Proposition 6.2 p. 75 of the book [see statement below], and that a surjective endomorphism of a finitely generated module is bijective by Corollary 13 p. 34.

(ii) Left to the reader. [I think “for (ii), the quotient modules $\text{Coker}(u^n)$ ” should be “for (ii), the submodules $\text{Im}(u^n)$ ”.]

Here is the statement of Proposition 6.2 p. 75 of the book:

Proposition 76 (Proposition 6.2). M is a Noetherian A -module \iff every submodule of M is finitely generated.

7.2.2 Page 78, Exercise 6.2

Statement. Let M be an A -module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Hint. Use Proposition 6.2 of the book stated above.

7.2.3 Page 78, Exercise 6.3

Statement. Let M be an A -module and let N_1, N_2 be submodules of M . If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

Hint. Consider the exact sequences

$$0 \rightarrow \frac{N_1}{N_1 \cap N_2} \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow \frac{M}{N_1} \rightarrow 0$$

and

$$0 \rightarrow \frac{N_1 + N_2}{N_2} \rightarrow \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0.$$

Mild generalization: If N_1, \dots, N_k are submodules of M such that M/N_i is noetherian for all i , then $M/(N_1 \cap \dots \cap N_k)$ is noetherian.

7.2.4 Page 78, Exercise 6.4

Statement. (a) Let M be a Noetherian A -module and let \mathfrak{a} be the annihilator of M in A . Prove that A/\mathfrak{a} is a Noetherian ring.

(b) If we replace “Noetherian” by “Artinian” in this result, is it still true?

Hints. (a) Use the above generalization and Proposition 6.2 p. 75 of the book.

(b) See Example 3 p. 74 of the book.

7.2.5 Page 79, Exercise 6.5

Statement. A topological space X is said to be **Noetherian** if the open subsets of X satisfy the ascending chain condition (or, equivalently, the maximal condition). Since closed subsets are complements of open subsets, it comes to the same thing to say that the closed subsets of X satisfy the descending chain condition (or, equivalently, the minimal condition). Show that, if X is Noetherian, then every subspace of X is Noetherian, and that X is quasi-compact.

Hints. To show that every subspace of X is Noetherian one can use the maximal condition for open subsets. To show that X is quasi-compact one can use the maximal condition for the sets of finite unions of open sets occurring in a given open cover.

7.2.6 Page 79, Exercise 6.6

Statement. Prove that the following are equivalent:

- (i) X is Noetherian.
- (ii) Every open subspace of X is quasi-compact.
- (iii) Every subspace of X is quasi-compact.

Hint. To show that (ii) implies (i), note that the chain $U_1 \subset U_2 \subset \cdots$ of open subsets covers the open subset $U_1 \cup U_2 \cup \cdots$.

7.2.7 Page 79, Exercise 6.7

Statement. A Noetherian space is a finite union of irreducible closed subspaces. [Consider the set Σ of closed subsets of X which are not finite unions of irreducible closed subspaces.] Hence the set of irreducible components of a Noetherian space is finite.

See Exercises 1.19 and 1.20 p. 13 of the book.

Solution. This is a copy-and-paste of Jeffrey Daniel Kasik Carlson’s solution in

<https://spaces.ac.cn/usr/uploads/2017/07/4208763092.pdf>

Suppose, for a contradiction, that the result is false. Then there is a noetherian space X such that X is an element of the set Σ of closed subsets of X that are not unions of finitely many irreducible closed subspaces. Since Σ is nonempty and X is noetherian, Σ has a minimal element M . Since M is not a finite union of irreducible sets, it is not itself an irreducible set. Thus it is reducible, and so a union of two proper closed subspaces C and D . But C and D are both finite unions of irreducible closed sets, so M is as well, contradiction.

Recall from Exercise 1.20iii p. 13 of the book that the irreducible components of a space X are

the maximal irreducible subsets of X , and that they are closed and cover X . Since a noetherian space X is a union of finitely many irreducible closed subspaces, it is a fortiori a union of finitely many maximal such, so it is a union of finitely many irreducible components. Let n be the least possible number needed to cover X , and let C_1, \dots, C_n be irreducible components covering X . If C is any other irreducible closed set, then $C = \bigcup_{j=1}^n (C \cap C_j)$ expresses C as a union of closed subsets; as C is irreducible, $C \subset C_j$ for some j . Thus C_1, \dots, C_n are the only irreducible components of X .

7.2.8 Page 79, Exercise 6.8

Statement. If A is a Noetherian ring then $\text{Spec}(A)$ is a Noetherian topological space. Is the converse true?

Hint. If $V(\mathfrak{a}_1) \supset V(\mathfrak{a}_2) \supset \dots$, with $r(\mathfrak{a}_i) = \mathfrak{a}_i$, is a weakly decreasing chain of closed subsets of X , then $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \dots$ is a weakly increasing chain of ideals of A .

7.2.9 Page 79, Exercise 6.9

Statement. Deduce from Exercise 6.8 that the set of minimal prime ideals in a Noetherian ring is finite.

Hint. See Exercises 1.20iv p. 13 and 6.7 p. 79 [see Section 7.2.7 p. 126] of the book.

A slightly stronger result holds: If $\text{Spec}(A)$ is noetherian, then the set of minimal prime ideals in A is finite.

7.2.10 Page 79, Exercise 6.10

Statement. If M is a Noetherian module (over an arbitrary ring A) then $\text{Supp}(M)$ is a closed Noetherian subspace of $\text{Spec}(A)$.

Solution. It suffices to note that $\text{Supp}(M) = V(\mathfrak{a})$ with $\mathfrak{a} := \text{Ann}(M)$ by Exercise 3.19v p. 46 of the book [Section 4.2.19 p. 70]; that $V(\mathfrak{a}) \simeq \text{Spec}(A/\mathfrak{a})$; that A/\mathfrak{a} is noetherian by Exercise 6.4 p. 78 of the book [cf. Section 7.2.4 p. 125 above]; and that $\text{Spec}(A/\mathfrak{a})$ is noetherian by Exercise 6.8 p. 79 of the book [cf. Section 7.2.8 p. 127 above].

7.2.11 Page 79, Exercise 6.11

Statement. Let $f : A \rightarrow B$ be a ring homomorphism and suppose that $\text{Spec}(B)$ is a Noetherian space (Exercise 5 [Exercise 7.2.5 p. 126]). Prove that $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a closed mapping if and only if f has the going-up property (Chapter 5, Exercise 10).

Solution. Since it is stated in Exercise 5.10i p. 68 of the book that f has the going-up property if f^* is closed, it suffices to prove the converse. So, assuming that f has the going-up property, let us show that f^* is closed.

Let \mathfrak{b} be a radical ideal of B , and let \mathfrak{a} be its contraction in A . It suffices to prove $f^*(V(\mathfrak{b})) = V(\mathfrak{a})$. By the previous Exercises, there are $\mathfrak{q}_1, \dots, \mathfrak{q}_n \in V(\mathfrak{b})$ such that the set of minimal elements of $V(\mathfrak{b})$

is $\{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ and $V(\mathfrak{b}) = \bigcup_{i=1}^n V(\mathfrak{q}_i)$. Let \mathfrak{p}_i be the contraction of \mathfrak{q}_i in A . It suffices to prove:

$$V(\mathfrak{a}) = \bigcup_{i=1}^n V(\mathfrak{p}_i) \quad (47)$$

and

$$f^*(V(\mathfrak{q}_i)) = V(\mathfrak{p}_i) \text{ for all } i. \quad (48)$$

Condition (48) follows from the going-up property. To prove (47), let \mathfrak{p} be in $V(\mathfrak{a})$. It is enough to check that \mathfrak{p} is in $V(\mathfrak{p}_i)$ for some i . We have

$$\mathfrak{p} \supset \mathfrak{a} = \mathfrak{b}^c = \left(\bigcap \mathfrak{q}_i \right)^c = \bigcap \mathfrak{p}_i,$$

and Proposition 4 p. 14 implies $\mathfrak{p} \supset \mathfrak{p}_i$ for some i , as was to be shown.

7.2.12 Page 79, Exercise 6.12

Statement. Let A be a ring such that $\text{Spec}(A)$ is a Noetherian space. Show that the set of prime ideals of A satisfies the ascending chain condition. Is the converse true?

Hint. To show that the converse is not true, let K be a field and $A := K^{\mathbb{N}}$ the ring of K -valued functions on \mathbb{N} . As A is absolutely flat, its prime ideals are maximal. Hence it suffices to show that there is an ascending chain $\mathfrak{a}_1 < \mathfrak{a}_2 < \dots$ of radical ideals of A . We can set

$$\mathfrak{a}_n := \{f \in A \mid f(k) = 0 \text{ for all } k > n\}.$$

8 About Chapter 7

8.1 Comments

8.1.1 Page 80, a remark

Here is an example of a descending chain $A_0 \supset A_1 \supset \cdots$ such that each A_n is noetherian but the intersection is not.

Consider the submonoid M_n of \mathbb{N}^2 defined by

$$M_n := \{a \in \mathbb{N}^2 \mid a_2 \geq 1 \vee a_1 \geq n\} \supset M_{n+1},$$

note that we have $M_n \supset M_{n+1}$, and that

$$M := \bigcap_n M_n = \{a \in \mathbb{N}^2 \mid a_2 \geq 1\}$$

is **not** finitely generated. Observe that the finite set

$$G_n := \{(n, 0), (n+1, 0), \dots, (2n-1, 0), (0, 1), (1, 1), (2, 1), \dots, (n-1, 1)\} \subset M_n$$

generates M_n .

Here is a picture for $n = 3$: The black dots are the points of G_3 , the white dots are the points on $M_3 \setminus G_3$, the crosses are the points in $\mathbb{N}^2 \setminus M_3$.

2	○	○	○	○	○	○	○
1	●	●	●	○	○	○	○
0	×	×	×	●	●	●	○
	0	1	2	3	4	5	6.

Let K be a field and x and y indeterminates, and set $A_n := K[(x^i y^j)_{(i,j) \in G_n}]$, that is, A_n is the sub- K -algebra of $K[x, y]$ generated by $\{x^i y^j \mid (i, j) \in G_n\}$. Then the A_n satisfy the conditions stated at the beginning of this section.

8.1.2 Page 81, Hilbert Basis Theorem

Theorem 77. *If M is a noetherian A -module and t is an indeterminate, then $M[t]$ is a noetherian $A[t]$ -module. In particular, if A is a noetherian ring, then so is $A[t]$.*

Proof. Assume by contradiction some sub- $A[t]$ -module of $A[t]$ is **not** finitely generated, and let f_1, f_2, \dots be a sequence in $M[t]$ such that the sequence $N_i := A[t]f_1 + \cdots + A[t]f_i$ of sub- $A[t]$ -modules of $A[t]$ increases, and $\deg f_i$ is minimum for this condition. Let $x_i \in M$ be the leading coefficient of f_i ; let n be such that $Ax_1 + \cdots + Ax_n$ is the sub- A -module of M generated by the x_i ; let $a_1, \dots, a_n \in A$ satisfy $x_{n+1} = a_1 x_1 + \cdots + a_n x_n$; set $d(i) := \deg f_{n+1} - \deg f_i$,

$$g := \sum_{i=1}^n a_i t^{d(i)} f_i \in N_n \subset A[t];$$

and observe that $\deg(f_{n+1} - g)$ is less than $\deg f_{n+1}$, which is impossible. □

8.1.3 Page 82, Proof of Proposition 7.8

Kevin Buzzard writes in this MathOverflow answer:

The following slip on p. 82 was found by Kenny Lau when he was formalising Proposition 7.8 in Lean: In the line "Substituting (1) and making repeated use of (2) shows that each element of C is..."

there's an implicit induction proof, but the base case where the element is 1 is not dealt with. This can be fixed in a number of ways, e.g. by adding a new condition

$$(0) \quad 1 = \sum_i b_i y_i$$

and using the b_i as further generators of B_0 .

Another way of fixing this would be to take 1 as one of the y_j .

8.1.4 Page 83, Lemma 7.12

The statement of Lemma 7.12, which is: "In a Noetherian ring every irreducible ideal is primary", should be replaced by "In a Noetherian ring every **proper** irreducible ideal is primary".

In the proof, the equalities $xy = 0$ and $\text{Ann}(x^n) = \text{Ann}(x^{n+1})$ imply $(x^n) \cap (y) = (0)$. Indeed, if a is in $(x^n) \cap (y)$ we have $a = bx^n = cy$ for some b and c in A , and thus $bx^{n+1} = cyx = 0$. Now $bx^{n+1} = 0$ implies $bx^n = 0$, that is $a = 0$.

Note that primary ideals in noetherian rings can be reducible: the ideal (x^2, xy, y^2) of $K[x, y]$ [where K is a field, x and y are indeterminates] is (x, y) -primary but reducible because $(x, y^2) \cap (y, x^2) = (x^2, xy, y^2)$.

Here is a related result:

The following conditions on a ring A are equivalent:

- (a) *the Krull dimension of A is at most zero,*
- (b) *A/\mathfrak{N} is absolutely flat, where \mathfrak{N} is the nilradical of A ,*
- (c) *for each a in A the descending chain $(a) \supset (a^2) \supset \dots$ stabilizes.*

Proof. (a) \implies (b): We can assume $\mathfrak{N} = (0)$. Let a be in A . It suffices to show $(a) = (a^2)$. Let \mathfrak{p} be a prime ideal of A . Then the nilradical of $A_{\mathfrak{p}}$ is (0) and $\mathfrak{p}_{\mathfrak{p}}$ is the only prime ideal of $A_{\mathfrak{p}}$. This implies that $A_{\mathfrak{p}}$ is a field, and we get successively the equalities

$$(a)_{\mathfrak{p}} = (a^2)_{\mathfrak{p}}, \quad (a)_{\mathfrak{p}}/(a^2)_{\mathfrak{p}} = (0), \quad ((a)/(a^2))_{\mathfrak{p}} = (0).$$

As \mathfrak{p} is an arbitrary prime ideal of A , this forces $(a) = (a^2)$.

(b) \implies (c): With obvious notation we have $\bar{a} = \bar{a}^2 \bar{b}$ for some b in A , that is $(a - a^2 b)^n = 0$ for some $n \geq 1$. This is easily seen to imply $a^{n+1} \in (a^n)$ and thus $(a^{n+1}) = (a^n)$.

(c) \implies (a): Let \mathfrak{p} be a prime ideal of A and let a be in $A \setminus \mathfrak{p}$. We have $a^n(1 - ab) = 0$ for some b in A and some n in \mathbb{N} . In particular $a^n(1 - ab) \in \mathfrak{p}$, and thus $1 - ab \in \mathfrak{p}$. This implies that A/\mathfrak{p} is a field, and therefore that \mathfrak{p} is maximal. \square

8.1.5 Page 83, Theorem 7.13

Recall the statement:

In a Noetherian ring A every ideal has a primary decomposition.

Let E be a subset of a ring A . Recall that, if A is noetherian, we have, by Section 2.2.9 p. 21 $V(E) = \bigcup_{\mathfrak{p} \in M} V(\mathfrak{p})$, where M is the set of minimal elements of $V(E)$.

Assume now that A is **noetherian**.

Then the set M is finite. As a result, we have

The topology of $\text{Spec}(A)$ depends only on its poset structure.

More precisely:

A subset of $\text{Spec}(A)$ is closed if and only if it is a finite union of subsets of the form $V(\mathfrak{p})$ with $\mathfrak{p} \in \text{Spec}(A)$.

Equivalently:

The closed subsets of $\text{Spec}(A)$ are the closures of the finite subsets.

8.2 Exercises

8.2.1 Page 84, Exercise 7.1

Statement. Let A be a non-Noetherian ring and let Σ be the set of ideals in A which are not finitely generated. Show that Σ has maximal elements and that the maximal elements of Σ are prime ideals.

Hence a ring in which every prime ideal is finitely generated is Noetherian (I. S. Cohen).

Solution. Following the hint in the book, let \mathfrak{a} be a maximal element of Σ . Suppose by contradiction that there exist $x, y \in A$ such that $x \notin \mathfrak{a}$, $y \notin \mathfrak{a}$, $xy \in \mathfrak{a}$.

There are $s_i \in A$ such that $\mathfrak{a} + (x) = (s_1, \dots, s_n)$. We have $s_i = b_i + t_i x$ with $b_i \in \mathfrak{a}$, $t_i \in A$. Putting $\mathfrak{b} = (b_1, \dots, b_n) \subset \mathfrak{a}$, we get $\mathfrak{a} + (x) = \mathfrak{b} + (x)$.

We claim $\mathfrak{a} = \mathfrak{b} + x(\mathfrak{a} : x)$.

The inclusion $\mathfrak{b} + x(\mathfrak{a} : x) \subset \mathfrak{a}$ is clear. To prove the other inclusion, let a be in \mathfrak{a} and let us show $a \in \mathfrak{b} + x(\mathfrak{a} : x)$. We have $a = \sum u_i (b_i + t_i x) = b + vx$ with $u_i \in A$, $b \in \mathfrak{b}$, $v \in A$. This yields $vx = a - b \in \mathfrak{a}$, and thus $v \in (\mathfrak{a} : x)$, proving the claim.

Note that y is in $(\mathfrak{a} : x)$ but not in \mathfrak{a} . This implies that $(\mathfrak{a} : x)$ is finitely generated, and thus, in view of the claim, that \mathfrak{a} is finitely generated, contradiction.

8.2.2 Page 84, Exercise 7.2

Statement. Let A be a Noetherian ring and let $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$. Prove that f is nilpotent if and only if each a_n is nilpotent.

Solution. In view of Exercise 1.5ii [Section 2.2.6 p. 20] it suffices to show: each a_i nilpotent $\implies f$ nilpotent. We have $a_i \in \mathfrak{N}$ for all i . Recall Corollary 7.15 p. 83 of the book: In a noetherian ring the nilradical is nilpotent. We have $f \in \mathfrak{N}[[x]]$, hence $f^n \in \mathfrak{N}^n[[x]]$ for all n . Since $\mathfrak{N}^n = (0)$ for n large enough by Corollary 7.15, we get $f^n = 0$ for n large enough.

8.2.3 Page 84, Exercise 7.3

Statement. Let \mathfrak{a} be an irreducible ideal in a ring A . Then the following are equivalent:

- (i) \mathfrak{a} is primary,
- (ii) for every multiplicatively closed subset S of A we have $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : s)$ for some $s \in S$,
- (iii) for every $x \in A$ the sequence $(\mathfrak{a} : x^n)$ is stationary.

Hints.

- (i) \implies (ii): Use Proposition 4.8 p. 53. [This implication holds even if \mathfrak{a} is reducible.]
- (ii) \implies (iii): Use Proposition 3.11ii p. 41 [Proposition 33 p. 58]. [This implication also holds even if \mathfrak{a} is reducible.] [Hint: set $S := x^{\mathbb{N}}$.]
- (iii) \implies (i): Use the proof of Lemma 7.12 p. 83.

8.2.4 Page 84, Exercise 7.4

Statement. Which of the following rings are Noetherian?

- (i) The ring of rational functions of z having no pole on the circle $|z| = 1$.
- (ii) The ring of power series in z with a positive radius of convergence.
- (iii) The ring of power series in z with an infinite radius of convergence.
- (iv) The ring of polynomials in z whose first k derivatives vanish at the origin (k being a fixed integer).
- (v) The ring of polynomials in z, w all of whose partial derivatives with respect to w vanish for $z = 0$.

In all cases the coefficients are complex numbers.

Solution. Let A be the ring in question.

- (i) Hint: we have $A = S^{-1}\mathbb{C}[z]$ for some multiplicative subset S , so A is noetherian.
- (ii) Hint: A is a principal ideal domain [the only ideals being the (z^n)].
- (iii) A is **not** noetherian. Proof: Set $f(z) := \sin(\pi z)$ and

$$f_n(z) := \frac{f(z)}{(z-1)\cdots(z-n)}$$

for $n = 1, 2, \dots$ and note that we have $(f_1) < (f_2) < \cdots \subset A$ [details left to the reader].

(iv) We prove a slightly more general result: If B is a noetherian ring and X is an indeterminate, then the ring $A := B + X^{k+1}B[X]$ is noetherian. Indeed, we have

$$A = B[X^{k+1}] + X^{k+2}B[X^{k+1}] + \cdots + X^{2k+2}B[X^{k+1}],$$

so A is a noetherian module over the subring $B[X^{k+1}]$, *a fortiori* a noetherian ring.

(v) We prove a slightly more general result. Let B be a ring, let X and Y be indeterminates, and set $A := B[X] + XB[X, Y]$. Then A is a subring of $B[X, Y]$. Claim 1: A is **not** noetherian. [The case in the book is $B := \mathbb{C}$.] Set

$$C := A/(X^2) = B[x, Y] = B[x] + xB[x, Y] = B + xB + xB[Y] = B[x][Y],$$

where x is the image of X ; in particular $x^2 = 0$. It suffices to prove: Claim 2: C is **not** noetherian. It suffices to show that we have $xy^{n+1} \notin (xy, xy^2, \dots, xy^n)$ in C . Otherwise we would get

$$xy^{n+1} = b_1xy + b_2xy^2 + \cdots + b_nxy^n$$

with $b_i \in B$, contradiction.

8.2.5 Page 84, Exercise 7.5

Statement. Let A be a Noetherian ring, B a finitely generated A -algebra, G a finite group of A -automorphisms of B , and B^G the set of all elements of B which are left fixed by every element of G . Show that B^G is a finitely generated A -algebra.

Hint. Use Exercise 5.12 p. 68 [see Section 6.2.12 p. 108 above] and Proposition 7.8 p. 81.

8.2.6 Page 84, Exercise 7.6

Statement. If a finitely generated ring is a field K , it is a finite field.

Hint. Let A be the prime subring of K . Then $A = \mathbb{Z}$ or $A = \mathbb{F}_p$ for some prime p . Proposition 5.7 p. 61 of the book and Noether Normalization Theorem [Theorem 74 p. 110] imply that the first case is impossible and that, in the second case, K is a finite degree extension of \mathbb{F}_p .

8.2.7 Page 85, Exercise 7.7

Statement. Let X be an affine algebraic variety given by a family of equations $f_\alpha(t_1, \dots, t_n) = 0$ ($\alpha \in I$) (Chapter 1, Exercise 27 [Section 2.2.28 p. 31]). Show that there exists a finite subset I_0 of I such that X is given by the equations $f_\alpha(t_1, \dots, t_n) = 0$ for $\alpha \in I_0$.

Solution. Since $k[t_1, \dots, t_n]$ is noetherian, there is a finite subset I_0 of I such that the ideal generated by the $f_\alpha(t_1, \dots, t_n)$ with $\alpha \in I$ is also generated by the $f_\alpha(t_1, \dots, t_n)$ with $\alpha \in I_0$.

8.2.8 Page 85, Exercise 7.9

Hints. The inclusion

$$\mathfrak{a}_{\mathfrak{m}_1} \subset \left(\frac{x_0}{1}, \dots, \frac{x_t}{1} \right)$$

holds by the choice of x_{s+1}, \dots, x_t .

The inclusion

$$\mathfrak{a}_{\mathfrak{m}_{r+1}} \subset \left(\frac{x_0}{1}, \dots, \frac{x_t}{1} \right)$$

holds by the choice of x_1 [indeed, $x_1/1$ is a unit of $A_{\mathfrak{m}_{r+1}}$].

If \mathfrak{m} is a maximal ideal distinct from all the \mathfrak{m}_i , then the inclusion

$$\mathfrak{a}_{\mathfrak{m}} \subset \left(\frac{x_0}{1}, \dots, \frac{x_t}{1} \right)$$

holds by the choice of $\mathfrak{m}_1, \dots, \mathfrak{m}_{r+s}$ [indeed, $x_0/1$ is a unit of $A_{\mathfrak{m}}$].

8.2.9 Page 85, Exercise 7.10

See Theorem 77 p. 129 above.

8.2.10 Page 85, Exercise 7.11

Statement. Let A be a ring such that each local ring $A_{\mathfrak{p}}$ is Noetherian. Is A necessarily Noetherian?

Hints. See Exercises 2.28 and 3.10ii [cf. Section 4.2.10 p. 66] pages 35 and 44 of the book.

8.2.11 Page 85, Exercise 7.13

Statement. Let $f : A \rightarrow B$ be a ring homomorphism of finite type and let $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the mapping associated with f . Show that the fibers of f^* are Noetherian subspaces of B . [Typo: it should be “subspaces of $\text{Spec}(B)$ ”.]

Hint. Let \mathfrak{p} be in $\text{Spec}(A)$ and set $k := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. By Exercise 3.21iv p. 47 of the book [Section 4.2.21 p. 72 above] and Exercise 6.8 p. 79 of the book [cf. Section 7.2.8 p. 127 above], it suffices to show that $k \otimes_A B$ is noetherian. To do this, note that there are indeterminates x_1, \dots, x_n such that B is a quotient of $A[x_1, \dots, x_n]$, and conclude that $k \otimes_A B$ is a quotient of $k[x_1, \dots, x_n]$.

8.2.12 Page 85, Exercise 7.14

Statement. Let k be an algebraically closed field, let A denote the polynomial ring $k[t_1, \dots, t_n]$ and let \mathfrak{a} be an ideal in A . Let V be the variety in k^n defined by the ideal \mathfrak{a} , so that V is the set of all $x = (x_1, \dots, x_n) \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$. Let $I(V)$ be the ideal of V , i.e. the ideal of all polynomials $g \in A$ such that $g(x) = 0$ for all $x \in V$. Then $I(V) = r(\mathfrak{a})$.

Solution. This was proved in Paragraph 6.2.16 p. 112.

8.2.13 Page 86, Exercise 7.15

Statement. Let A be a Noetherian local ring, \mathfrak{m} its maximal ideal and k its residue field, and let M be a finitely generated A -module. Then the following are equivalent:

- (i) M is free;
- (ii) M is flat;
- (iii) the mapping of $\mathfrak{m} \otimes M$ into $A \otimes M$ is injective;
- (iv) $\text{Tor}_i^A(k, M) = 0$.

[To show that (iv) \implies (i), let x_1, \dots, x_n be elements of M whose images in $M/\mathfrak{m}M$ form a k -basis of this vector space. By (2.8)⁶, the x_i generate M . Let F be a free A -module with basis e_1, \dots, e_n and define $\phi : F \rightarrow M$ by $\phi(e_i) = x_i$. Let $E = \text{Ker}(\phi)$. Then the exact sequence $0 \rightarrow E \rightarrow F \rightarrow M \rightarrow 0$ gives us an exact sequence

$$0 \rightarrow k \otimes_A E \rightarrow k \otimes_A F \xrightarrow{1 \otimes \phi} k \otimes_A M \rightarrow 0.$$

Since $k \otimes F$ and $k \otimes M$ are vector spaces of the same dimension over k , it follows that $1 \otimes \phi$ is an isomorphism, hence $k \otimes E = 0$, hence $E = 0$ by Nakayama's Lemma (E is finitely generated because it is a submodule of F , and A is Noetherian).]

Solution. Omitted.

8.2.14 Page 86, Exercise 7.16

Statement. Let A be a Noetherian ring, M a finitely generated A -module. Then the following are equivalent:

- (i) M is a flat A -module;
- (ii) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module, for all prime ideals \mathfrak{p} ;
- (iii) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module, for all maximal ideals \mathfrak{m} .

In other words, flat = locally free. [Use Exercise 15.]

Solution. Omitted.

8.2.15 Page 86, Exercise 7.17

Statement. Let A be a ring and M a Noetherian A -module. Show (by imitating the proofs of (7.11) and (7.12)) that every submodule N of M has a primary decomposition (Chapter 4, Exercises 20-23).

Hint. Let M be a noetherian A -module whose zero submodule $0 \subset M$ is irreducible. Let us show that 0 is primary in M .

⁶Proposition 2.8 p. 22 of the book says "Let A be a local ring, \mathfrak{m} its maximal ideal, M a finitely generated A -module, and x_i ($1 \leq i \leq n$) elements of M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then the x_i generate M ."

Let a be an element of A which is a zero-divisor in M . It suffices to show that a is nilpotent in M .

The chain of submodules $(0 : a) \subset (0 : a^2) \subset \cdots$ stabilizes. Say $(0 : a^n) = (0 : a^{n+1})$. It suffices to show $a^n M = 0$.

Let $x \in M$ satisfy $x \neq 0$ and $ax = 0$. It is enough to prove $a^n M \cap Ax = 0$.

Let y be in $a^n M \cap Ax = 0$. We only need to show $y = 0$.

We have $y = a^n z = bx$ for some $z \in M$ and some $b \in A$, and we get

$$ay = a^{n+1} z = abx = bax = 0,$$

and thus $0 = a^n z = y$. This completes the proof.

8.2.16 Page 86, Exercise 7.18

Statement. Let A be a Noetherian ring, \mathfrak{p} a prime ideal of A , and M a finitely generated A -module. Show that the following are equivalent:

- (i) \mathfrak{p} belongs to 0 in M ;
- (ii) there exists $x \in M$ such that $\text{Ann}(x) = \mathfrak{p}$;
- (iii) there exists a submodule of M isomorphic to A/\mathfrak{p} .

Deduce that there exists a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each quotient M_i/M_{i-1} is of the form A/\mathfrak{p}_i where \mathfrak{p}_i is a prime ideal of A .

Solution. Recall Proposition 7.17 p. 83 of the book:

Proposition 78 (Proposition 7.17). *Let $\mathfrak{a} \neq (1)$ be an ideal in a Noetherian ring. Then the prime ideals which belong to \mathfrak{a} are precisely the prime ideals which occur in the set of ideals $(\mathfrak{a} : x)$ ($x \in A$).*

This proposition and its proof can be generalized to modules according to the following table:

$\mathfrak{a} \neq (1)$	$x \in A$	\mathfrak{q}_i	$\mathfrak{p}_i = r(\mathfrak{q}_i)$	\mathfrak{a}_i	$\mathfrak{p}_i^m \subset \mathfrak{q}_i$	$\mathfrak{a}_i \cap \mathfrak{p}_i^m$
$N \subsetneq M$	$x \in M$	Q_i	$\mathfrak{p}_i = r(Q_i : M)$	N_i	$\mathfrak{p}_i^m M \subset Q_i$	$N_i \cap \mathfrak{p}_i^m M$

Here is the generalized proposition:

Proposition 79. *Let A be a noetherian ring, let N be a proper submodule of an A -module M . Then the prime ideals which belong to N in M are precisely the prime ideals which occur in the set of ideals $(N : x)$ ($x \in M$).*

The generalized proposition yields the equivalence (i) \iff (ii). The equivalence (ii) \iff (iii) is clear. To prove the existence of the stated chain, the generalized proposition furnishes M_0 . The same argument with M replaced by M/M_0 gives a submodule M'_1 of M/M_0 , and we can let $M_1 \subset M$ be the inverse image of M'_1 . Continuing this way, we get the desired chain after a finite number of steps.

8.2.17 Page 86, Exercise 7.19

Statement. Let \mathfrak{a} be an ideal in a Noetherian ring A . Let

$$\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{b}_i = \bigcap_{j=1}^s \mathfrak{c}_j$$

be two minimal decompositions of \mathfrak{a} as intersections of irreducible ideals. Prove that $r = s$ and that [possibly after re-indexing the \mathfrak{c}_i] $r(\mathfrak{b}_i) = r(\mathfrak{c}_i)$ for all i . State and prove an analogous result for modules.

Solution. Our goal is to prove:

Proposition 80. (a) Let A be a ring, M an A -module and W a submodule. Let

$$W = \bigcap_{i=1}^n U_i = \bigcap_{j=1}^m V_j$$

be two minimal decompositions of W as intersections of irreducible submodules. Then $n = m$.

(b) If M is noetherian, we have [possibly after re-indexing the V_j] $r(U_i : M) = r(V_i : M)$ for all i .

Proof of Part (a) of Proposition 80.

Lemma 81. Let A be a ring, M an A -module and let $U_1, U_2, V_1, \dots, V_m, W$ be submodules of M such that U_1 is irreducible and $W = U_1 \cap U_2 = V_1 \cap \dots \cap V_m$. Then we have $W = V_i \cap U_2$ for some i .

We summarize Lemma 81 by saying that “we have replaced U_1 with V_i in the equality $W = U_1 \cap U_2$ ”.

Before proving Lemma 81, we show that it implies Part (a) of Proposition 80.

It suffices to derive a contradiction from the assumption $n < m$. Using Lemma 81 repeatedly we get $V_{i_1} \cap \dots \cap V_{i_n} = V_1 \cap \dots \cap V_m$, in contradiction with the minimality of the right side.

Proof of Lemma 81. We follow Matthew Emerton: <https://mathoverflow.net/q/12322/461>

We can assume $W = 0$. Let $\phi_i : M \rightarrow M/U_i$ ($i = 1, 2$) and $\phi : M \rightarrow M/U_1 \times M/U_2$ be the natural morphisms, note that ϕ is injective, and set $X_j := V_j \cap U_2$ for $1 \leq j \leq m$.

It suffices to show $X_j = 0$ for some j .

We have $\bigcap X_j \subset \bigcap V_j = 0$ and $\phi(X_j) = \phi_1(X_j) \times 0$. By injectivity of ϕ we also have

$$\bigcap \phi(X_j) = \phi \left(\bigcap X_j \right) = 0,$$

and thus $\bigcap \phi_1(X_j) = 0$. The zero submodule of M/U_1 being irreducible, this implies $\phi_1(X_j) = 0$, hence $X_j = 0$, for some j . This proves Lemma 81.

Part (a) of Proposition 80 has been proved, and it only remains to prove Part (b).

Proof of Part (b) of Proposition 80. Recall the setting: A is a ring, M is a noetherian A -module, W is a submodule,

$$W = \bigcap_{i=1}^n U_i = \bigcap_{j=1}^n V_j$$

are two minimal decompositions of W as intersections of irreducible submodules. We must show that we have [possibly after re-indexing the V_j] $r(U_i : M) = r(V_i : M)$ for all i .

Irreducible submodules being primary, the sets

$$\{r(U_1 : M), \dots, r(U_n : M)\} \text{ and } \{r(V_1 : M), \dots, r(V_n : M)\}$$

are equal. Denote this set by P , write $[n]$ for the set $\{1, \dots, n\}$ and define the maps f and g from $[n]$ to P by $f(i) := r(U_i : M)$ and $g(i) := r(V_i : M)$. It suffices to show that, for all $\mathfrak{p} \in P$, the fibers $f^{-1}(\mathfrak{p})$ and $g^{-1}(\mathfrak{p})$ are equipotent.

Let I be an isolated subset of P . The Second Uniqueness Theorem for modules implies

$$\bigcap_{i \in f^{-1}(I)} U_i = \bigcap_{i \in g^{-1}(I)} V_i.$$

Then Part (a) of Proposition 80 entails that $f^{-1}(I)$ and $g^{-1}(I)$ are equipotent. In particular, if $\mathfrak{p} \in P$ is minimal, $f^{-1}(\mathfrak{p})$ and $g^{-1}(\mathfrak{p})$ have same cardinality, and an obvious induction completes the proof.

8.2.18 Page 87, Exercise 7.20

Statement. Let X be a topological space and let \mathcal{F} be the smallest collection of subsets of X which contains all open subsets of X and is closed with respect to the formation of finite intersections and complements.

- (i) Show that a subset E of X belongs to \mathcal{F} if and only if E is a finite union of sets of the form $U \cap C$, where U is open and C is closed.
- (ii) Suppose that X is irreducible [see Section 2.2.20 p. 26 above] and let $E \in \mathcal{F}$. Show that E is dense in X [i.e., that $\overline{E} = X$] if and only if E contains a non-empty open set in X .

Solution.

- (i) Let \mathcal{F}' be the set of those subsets E of X such that E is a finite union of sets of the form $U \cap C$, where U is open and C is closed. It suffices to check that, if two sets are in \mathcal{F}' , then so are their respective complements and their intersection. This is straightforward.
- (ii) If E contains a non-empty open set U , then $X = \overline{E} \cup U^*$, where U^* is the complement of U , and the irreducibility of X implies $X = \overline{E}$. Conversely, if $E = (U_1 \cap C_1) \cup \dots \cup (U_n \cap C_n)$ [obvious notation] is dense in X , then one of the $U_i \cap C_i$ is already dense in X . This implies $C_i = X$ and thus $U_i \subset E$.

8.2.19 Page 87, Exercise 7.21

Statement. Let X be a Noetherian topological space [Chapter 6, Exercise 5 — see Section 8.2.18 p. 138 above] and let $E \subset X$. Show that $E \in \mathcal{F}$ if and only if Condition (\star) below holds:

(\star) for each irreducible closed set $X_0 \subset X$, either $\overline{E \cap X_0} \neq X_0$ or else $E \cap X_0$ contains a non-empty open subset of X_0 .

Solution. Let us denote by $\mathcal{F}(X)$ the set designated by \mathcal{F} in Exercise 7.20 [see Section 8.2.18 p. 138 above].

To prove that $E \in \mathcal{F}(X)$ implies (\star) , note that $E \cap X_0 \in \mathcal{F}(X_0)$ if $E \in \mathcal{F}(X)$ and use Exercise 7.20.

To prove that (\star) implies $E \in \mathcal{F}(X)$, we follow the hint, that is, we assume by contradiction that (\star) holds but that E is not in $\mathcal{F} := \mathcal{F}(X)$.

Let Σ be the set of all closed subsets X' of X such that $E \cap X' \notin \mathcal{F}$.

Then Σ is nonempty because $X \in \Sigma$. Let X_0 be a minimal element of Σ . In particular

$$E \cap X_0 \notin \mathcal{F}. \quad (49)$$

The subset X_0 is irreducible, for if we had $X_0 = C \cup D$ with C, D closed and $\subsetneq X_0$, we would have $C, D \notin \Sigma$ by minimality of X_0 , and thus $E \cap C$ and $E \cap D$ would be in \mathcal{F} , which would imply

$$\mathcal{F} \ni (E \cap C) \cup (E \cap D) = E \cap (C \cup D) = E \cap X_0 \notin \mathcal{F}.$$

We claim

$$\overline{E \cap X_0} = X_0. \quad (50)$$

To prove (50), assume by contradiction that we have $\overline{E \cap X_0} \subsetneq X_0$. The minimality of X_0 implies

$$E \cap \overline{E \cap X_0} \in \mathcal{F}. \quad (51)$$

We have $E \cap X_0 \subset E \cap \overline{E \cap X_0}$ because $E \cap X_0 \subset E$ and $E \cap X_0 \subset \overline{E \cap X_0}$. This implies

$$E \cap X_0 = E \cap \overline{E \cap X_0}.$$

In view of (49) and (51), this gives the contradiction needed to prove (50).

Now (\star) implies that there is a nonempty open subset U of X_0 such that $U \subset E$.

We have $U \subsetneq X_0$ because $U = X_0$ would imply $\mathcal{F} \ni X_0 = E \cap X_0 \notin \mathcal{F}$ by (49).

The set $C := X_0 \setminus U$ is closed in X , and we have $X_0 = U \sqcup C$ [disjoint union], $U \neq \emptyset \neq C$, and thus $E \cap X_0 = (E \cap U) \sqcup (E \cap C) = U \sqcup (E \cap C)$. As $U \neq \emptyset$, we get $E \cap C \subsetneq E \cap X_0$, and thus $E \cap C \in \mathcal{F}$ by minimality of X_0 . Then the above display implies $E \cap X_0 \in \mathcal{F}$, contradicting again (49).

8.2.20 Page 87, Exercise 7.22

Statement. Let X be a Noetherian topological space and let E be a subset of X . Show that E is open in X if and only if, for each irreducible closed subset X_0 in X , either $E \cap X_0 = \emptyset$ or else $E \cap X_0$ contains a non-empty open subset of X_0 .

Solution. If E is open in X , then the indicated condition holds because $E \cap X_0 \neq \emptyset$ implies that $E \cap X_0$ is a non-empty open subset of X_0 contained in $E \cap X_0$.

Assume that E is not open in X . Set $F := X \setminus E$. Then F is not closed. Put

$$\Sigma := \{ X' \subset X \mid X' \text{ is closed, } F \cap X' \text{ is not closed} \}.$$

In particular X belongs to Σ . Let X_0 be a minimal element of Σ .

It suffices to show:

- (a) X_0 is irreducible,
- (b) $E \cap X_0 \neq \emptyset$,
- (c) $E \cap X_0$ contains no non-empty open subset of X_0 .

Proof of (a): The conditions $X_0 = Y_1 \cup Y_2$ with $Y_i \subsetneq X_0$ and Y_i closed would imply $F \cap X_0 = (F \cap Y_1) \cup (F \cap Y_2)$ with $F \cap Y_i$ closed and $F \cap X_0$ not closed, which is impossible.

Proof of (b): The equality $E \cap X_0 = \emptyset$ would imply $F \cap X_0 = X_0$ with $F \cap X_0$ not closed and X_0 closed, contradiction.

Proof of (c): Assume $E \cap X_0 \supset U \neq \emptyset$ with U open in X_0 . It suffices to derive a contradiction. Set $X_1 := X_0 \setminus U$. We clearly have $X_1 \subsetneq X_0$.

We claim: $F \cap X_1 = F \cap X_0$.

It is enough to show $F \cap X_0 \subset F \cap X_1$. Let a be in $F \cap X_0$. As a cannot be in U [because this would imply $a \in E \cap F = \emptyset$], the point a is in $X_0 \setminus U = X_1$, and the claim is proved.

The claim implies $X_1 \in \Sigma$, contradicting the minimality of X_0 . This completes the proof of (c).

8.2.21 Page 87, Exercise 7.23

Statement. Let A be a Noetherian ring, $f : A \rightarrow B$ a ring homomorphism of finite type (so that B is Noetherian). Let $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and let $f^* : Y \rightarrow X$ be the mapping associated with f . Then the image under f^* of a constructible subset E of Y is a constructible subset of X .

[By Exercise 20 [Section 8.2.18 p. 138] it is enough to take $E = U \cap C$ where U is open and C is closed in Y ; then, replacing B by a homomorphic image, we reduce to the case where E is open in Y . Since Y is Noetherian, E is quasi-compact and therefore a finite union of open sets of the form $\text{Spec}(B_g)$. Hence reduce to the case $E = Y$. To show that $f^*(Y)$ is constructible, use the criterion of Exercise 21 [Section 8.2.19 p. 138]. Let X_0 be an irreducible closed subset of X such that $f^*(Y) \cap X_0$ is dense in X_0 . We have $f^*(Y) \cap X_0 = f^*(f^{*-1}(X_0))$, and $f^{*-1}(X_0) = \text{Spec}((A/\mathfrak{p}) \otimes_A B)$, where $X_0 = \text{Spec}(A/\mathfrak{p})$. Hence reduce to the case where A is an integral domain and f is injective. If Y_1, \dots, Y_n are the irreducible components of Y , it is enough to show that some $f^*(Y_j)$ contains a non-empty open set in X . So finally we are brought down to the situation in which A, B are integral domains and f is injective (and still of finite type); now use Chapter 5, Exercise 21 [Section 6.2.21 p. 138] to complete the proof.]

Hints. The fact that “ E is quasi-compact” in the hint of the book follows from Exercise 6.6 p. 79 of the book. The phrase “where $X_0 = \text{Spec}(A/\mathfrak{p})$ ” follows from Section 2.2.20 p. 26 above.

Here are some details about various steps:

- Restriction to the case when A is a domain and f is injective: We have $X_0 = V(\mathfrak{p}) \simeq \text{Spec}(A/\mathfrak{p})$ for some $\mathfrak{p} \in X$. If \bar{f} is the natural monomorphism $A/\mathfrak{p} \rightarrow B/\mathfrak{p}^e$, then the diagram below, where the vertical arrows are the obvious homeomorphisms, commutes:

$$\begin{array}{ccc} \text{Spec}(B/\mathfrak{p}^e) & \xrightarrow{\bar{f}^*} & \text{Spec}(A/\mathfrak{p}) \\ \uparrow & & \uparrow \\ f^{*-1}(X_0) & \xrightarrow{f^*} & X_0. \end{array}$$

- Restriction to the case when A and B are domains and f is injective: As X is irreducible, at least one of the $f^*(Y_i)$ is dense. We have $Y_i \simeq \text{Spec}(B/\mathfrak{q})$ for some $\mathfrak{q} \in Y$. We can replace B with B/\mathfrak{q} , i.e. we can assume that B is a domain and $f^*(Y)$ is dense, and Exercise 1.21v p. 13 of the book [Section 2.2.22 p. 27] implies that f is injective.
- Last step: See Section 6.2.21 p. 114 above.

8.2.22 Page 87, Exercise 7.24

Statement. With the notation and hypotheses of Exercise 23 [Section 8.2.21 p. 140], f^* is an open mapping $\iff f$ has the going-down property (Chapter 5, Exercise 10 [Section 6.2.10 p. 108]). [Suppose f has the going-down property. As in Exercise 23 [Section 8.2.21 p. 140], reduce to proving that $E = f^*(Y)$ is open in X . The going-down property asserts that if $\mathfrak{p} \in E$ and $\mathfrak{p}' \subset \mathfrak{p}$ then $\mathfrak{p}' \in E$: in other words, that if X_0 is an irreducible closed subset of X and X_0 meets E , then $E \cap X_0$ is dense in X_0 . By Exercises 20 and 22 [Section 8.2.18 p. 138 and Section 8.2.20 p. 139], E is open in X .]

Solution. Let us spell out the last two sentences of the hint.

Let \mathfrak{p}_0 be in X , and set $X_0 := V(\mathfrak{p}_0)$. Assuming that f has the going-down property, we want to show that $E := f^*(Y)$ is open in X . Suppose $E \cap X_0 \neq \emptyset$, and thus

$$\mathfrak{p}_0 \in E \tag{52}$$

[by the going-down property]. As E is constructible by Exercise 7.23, it suffices, in view of Exercise 7.22, to show that $E \cap X_0$ contains a nonempty open subset of X_0 , and Exercise 7.20ii [see Section 8.2.18 p. 138 above] tells us that it is even enough to check the inclusion

$$X_0 \subset \overline{E \cap X_0}. \tag{53}$$

Setting $\mathfrak{a} := \bigcap_{\mathfrak{p} \in E \cap X_0} \mathfrak{p}$, we get $\mathfrak{a} \subset \mathfrak{p}_0$ because $\mathfrak{p}_0 \in E \cap X_0$ by (52), that is

$$X_0 = V(\mathfrak{p}_0) \subset V(\mathfrak{a}) = \overline{E \cap X_0},$$

proving (53).

8.2.23 Page 87, Exercise 7.25

Statement. Let A be Noetherian, $f : A \rightarrow B$ of finite type and flat (i.e., B is flat as an A -module). Then $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is an open mapping. [Exercise 24 and Chapter 5, Exercise 11; Section 8.2.22 p. 141 and Section 6.2.11 p. 108.]

Solution. Omitted. [The hint in the book is fairly complete.]

8.2.24 Page 88, Exercise 7.26

Statement. Let A be a Noetherian ring and let $F(A)$ denote the set of all isomorphism classes of finitely generated A -modules. Let C be the free abelian group generated by $F(A)$. With each short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of finitely generated A -modules we associate the element $(M') - (M) + (M'')$ of C , where (M) is the isomorphism class of M , etc. Let D be the subgroup of C generated by these elements, for all short exact sequences. The quotient group C/D is called the **Grothendieck group** of A , and is denoted by $K(A)$. If M is a finitely generated A -module, let $\gamma(M)$, or $\gamma_A(M)$, denote the image of (M) in $K(A)$.

(i) Show that $K(A)$ has the following universal property: for each additive function, λ on the class of finitely generated A -modules, with values in an abelian group G , there exists a unique homomorphism $\lambda_0 : K(A) \rightarrow G$ such that $\lambda(M) = \lambda_0(\gamma(M))$ for all M .

(ii) Show that $K(A)$ is generated by the elements $\gamma(A/\mathfrak{p})$, where \mathfrak{p} is a prime ideal of A . [Use Exercise 18, Section 8.2.16 p. 136.]

(iii) If A is a field, or more generally if A is a principal ideal domain, then $K(A) \simeq \mathbb{Z}$.

(iv) Let $f : A \rightarrow B$ be a **finite** ring homomorphism. Show that restriction of scalars gives rise to a homomorphism $f_! : K(B) \rightarrow K(A)$ such that $f_!(\gamma_B(N)) = \gamma_A(N)$ for a B -module N . If $g : B \rightarrow C$ is another finite ring homomorphism, show that $(g \circ f)_! = f_! \circ g_!$.

Hints. (i) Left to the reader.

(ii) In the notation of Exercise 18 Section 8.2.16 we have

$$\begin{aligned} \gamma(M) &= \gamma(A/\mathfrak{p}_r) + \gamma(M_{r-1}) \\ &= \gamma(A/\mathfrak{p}_r) + \gamma(A/\mathfrak{p}_{r-1}) + \gamma(M_{r-1}) \\ &\vdots \\ &= \gamma(A/\mathfrak{p}_r) + \gamma(A/\mathfrak{p}_{r-1}) + \cdots + \gamma(A/\mathfrak{p}_1). \end{aligned}$$

(iii) If \mathfrak{p} is a nonzero prime ideal, we have $0 = \gamma(A) - \gamma(A) + \gamma(A/\mathfrak{p}) = \gamma(A/\mathfrak{p})$ in $K(A)$. This shows that $K(A)$ is generated by $\gamma(A)$. Using again the notation of Exercise 18 Section 8.2.16, we have a surjective **rank morphism** $C \rightarrow \mathbb{Z}$ which induces by (i) a surjective morphism $K(A) \rightarrow \mathbb{Z}$. This implies $K(A) \simeq \mathbb{Z}$.

(iv) Left to the reader.

Statement. Let A be a Noetherian ring and let $F_1(A)$ be the set of all isomorphism classes of finitely generated **flat** A -modules. Repeating the construction of Exercise 26 we obtain a group $K_1(A)$. Let $\gamma_1(M)$ denote the image of (M) in $K_1(A)$.

- (i) Show that tensor product of modules over A induces a commutative ring structure on $K_1(A)$, such that $\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M \otimes N)$. The identity element of this ring is $\gamma_1(A)$.
- (ii) Show that tensor product induces a $K_1(A)$ -module structure on the group $K(A)$, such that $\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes N)$.
- (iii) If A is a (Noetherian) local ring, then $K_1(A) \simeq \mathbb{Z}$.
- (iv) Let $f : A \rightarrow B$ be a ring homomorphism, B being Noetherian. Show that extension of scalars gives rise to a ring homomorphism $f^! : K_1(A) \rightarrow K_1(B)$ such that $f^!(\gamma_1(M)) = \gamma_1(B \otimes_A M)$. [If M is flat and finitely generated over A , then $B \otimes_A M$ is flat and finitely generated over B .] If $g : B \rightarrow C$ is another ring homomorphism (with C Noetherian), then $(f \circ g)^! = f^! \circ g^!$.
- (v) If $f : A \rightarrow B$ is a finite ring homomorphism then

$$f_!(f^!(x)y) = xf_!(y)$$

for $x \in K_1(A)$, $y \in K(B)$. In other words, regarding $K(B)$ as a $K_1(A)$ -module by restriction of scalars, the homomorphism $f^!$ is a $K_1(A)$ -module homomorphism.

Remark. Since $F_1(A)$ is a subset of $F(A)$ we have a group homomorphism $\varepsilon : K_1(A) \rightarrow K(A)$, given by $\varepsilon(\gamma_1(M)) = \gamma(M)$. If the ring A is finite-dimensional and **regular**, i.e., if all its local rings $A_{\mathfrak{p}}$ are regular (Chapter 11) it can be shown that ε is an isomorphism.

Hints. (i) Hint: use Exercise 2.8i [Section 3.2.8 p. 38].

(ii) Left to the reader.

(iii) Hint: use Exercise 7.16 [Section 8.2.14 p. 135].

(iv) Left to the reader.

(v) Setting $x = \gamma_{1A}(M)$, $y = \gamma_B(N)$ we get

$$\begin{aligned} f_!(f^!(x)y) &= f_!(f^!(\gamma_{1A}(M))\gamma_B(N)) = f_!(\gamma_{1B}(B \otimes_A M)\gamma_B(N)) = f_!(\gamma_B(N \otimes_B B \otimes_A M)) \\ &= f_!(\gamma_B(M \otimes_A N)) = \gamma_A(M \otimes_A N) \end{aligned}$$

and

$$xf_!(y) = \gamma_{1A}(M)f_!(\gamma_B(N)) = \gamma_{1A}(M)\gamma_A(N) = \gamma_A(M \otimes_A N).$$

9 About Chapter 8

9.1 Comments

9.1.1 Theorem 8.5 p. 90

Recall Theorem 8.5:

Theorem 82 (Theorem 8.5 p. 90 of the book). *A ring A is Artin $\iff A$ is Noetherian and $\dim A = 0$.*

Theorem 8.5 implies that an Artin ring A is a finite length A -module. In particular Propositions 6.7 and 6.8 p. 77 of the book apply to chains of ideals of A .

9.1.2 Proposition 8.6 p. 90

I suggest the following restatement of Proposition 8.6:

Let A be a noetherian local ring, \mathfrak{m} its maximal ideal. Then exactly one of the following two statements is true:

- (i) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all n and A is **not** Artin,
- (ii) $\mathfrak{m}^n = 0$ for some n , in which case A is an Artin local ring.

In particular

$$A \text{ is Artin} \iff \mathfrak{m} \text{ is nilpotent.} \quad (54)$$

By Proposition 8.6 p. 90 of the book [Section 9.1.2 p. 144] and Proposition 48 p. 87, we have:
Every proper ideal of an Artin local ring is \mathfrak{m} -primary (where \mathfrak{m} is the maximal ideal).

This fact is implicitly used in the proof of Theorem 8.7.

9.1.3 Theorem 8.7 p. 90

Recall the statement of the theorem:

Theorem 83 (Theorem 8.7 p. 90 of the book, structure theorem for Artin rings). *An Artin ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings.*

It seems to me that the second part of the proof of Theorem 8.7 can be simplified. We must check the essential uniqueness of the decomposition of an Artin ring A as a finite product of Artin local rings A_i . But, looking at minimal idempotents, one sees that, if a ring can be decomposed as a product of finitely many local rings, such a decomposition is essentially unique. More precisely, the kernels of the morphisms from the ring to the various factors are exactly those ideals which are maximal among the proper ideals generated by an idempotent.

9.1.4 Page 91, Proposition 8.8

Let A be an Artin local ring with maximal ideal \mathfrak{m} and consider the condition

(*) there is an x in \mathfrak{m} such that every ideal of A is of the form (x^r) .

The proof of Proposition 8.8 shows that (*) is equivalent to any of the conditions (i), (ii) or (iii).

9.2 Exercises

9.2.1 Page 91, Exercise 8.1

Statement. Let $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = (0)$ be a minimal primary decomposition of the zero ideal in a Noetherian ring, and let \mathfrak{q}_i be \mathfrak{p}_i -primary. Let $\mathfrak{p}_i^{(r)}$ be the r th **symbolic power** of \mathfrak{p}_i (Chapter 4, Exercise 13; Section 5.3.10 p. 88). Show that for each $i = 1, \dots, n$ there exists an integer r_i such that $\mathfrak{p}_i^{(r_i)} \subset \mathfrak{q}_i$.

Suppose \mathfrak{q}_i is an isolated primary component. Then $A_{\mathfrak{p}_i}$ is an Artin local ring, hence if \mathfrak{m}_i is its maximal ideal we have $\mathfrak{m}_i^r = (0)$ for all sufficiently large r , hence $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r .

If \mathfrak{q}_i is an embedded primary component, then $A_{\mathfrak{p}_i}$ is **not** Artinian, hence the powers \mathfrak{m}_i^r are all distinct, and so the $\mathfrak{p}_i^{(r)}$ are all distinct. Hence in the given primary decomposition we can replace \mathfrak{q}_i by any of the infinite set of \mathfrak{p}_i -primary ideals $\mathfrak{p}_i^{(r)}$ where $r \geq r_i$ and so there are infinitely many minimal primary decompositions of (0) which differ only in the \mathfrak{p}_i -component.

Solution. Reminder: Proposition 7.14 p. 83 of the book says that, in a noetherian ring, every ideal a contains a power of its radical.

To show that for each $i = 1, \dots, n$ there exists an integer r_i such that $\mathfrak{p}_i^{(r_i)} \subset \mathfrak{q}_i$, note that, by Proposition 7.14 [see reminder above] there exists an integer r_i such that $(\mathfrak{p}_i)_{\mathfrak{p}_i}^{r_i} \subset (\mathfrak{q}_i)_{\mathfrak{p}_i}$, and the inclusion $\mathfrak{p}_i^{(r_i)} \subset \mathfrak{q}_i$ follows by contraction.

In the second paragraph, we suppose in addition that \mathfrak{q}_i is an isolated primary component. By Corollary 45a p. 81 \mathfrak{q}_i is the smallest \mathfrak{p}_i -primary ideal. We have just shown that there exists an integer r such that $\mathfrak{p}_i^{(r)} \subset \mathfrak{q}_i$. By Exercise 4.13i [Section 5.3.10 p. 88] $\mathfrak{p}_i^{(r)}$ is \mathfrak{p}_i -primary. This implies $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$.

The claims in the second paragraph, follow from Proposition 8.6 p. 90 of the book [Section 9.1.2 p. 144].

9.2.2 Page 91, Exercise 8.2

Statement. Let A be a Noetherian ring. Prove that the following are equivalent:

- (i) A is Artinian;
- (ii) $\text{Spec}(A)$ is discrete and finite;
- (iii) $\text{Spec}(A)$ is discrete.

Hint. Use Exercise 1.22 [Section 2.2.23 p. 28], Theorem 82 p. 144 [recall that Theorem 82 says that a ring A is Artin $\iff A$ is Noetherian and $\dim A = 0$] and Theorem 83 p. 144 [recall that Theorem 8.7 is the structure theorem for Artin rings and that it says that an Artin ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings].

9.2.3 Page 91, Exercise 8.3

Statement. Let k be a field and A a finitely generated k -algebra. Prove that the following are equivalent:

- (i) A is Artinian;
- (ii) A is a finite k -algebra.

[To prove that (i) \implies (ii), use Theorem 83 p. 144 [recall that Theorem 8.7 is the structure theorem for Artin rings and that it says that an Artin ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings] to reduce to the case where A is an Artin local ring. By the Nullstellensatz, the residue field of A is a finite extension of k . Now use the fact that A is of finite length as an A -module. To prove (ii) \implies (i), observe that the ideals of A are k -vector subspaces and therefore satisfy d.c.c.]

Solution. Left to the reader. [Recall that Theorem 8.7 is the structure theorem for Artin rings and that it says: “An Artin ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings.”]

9.2.4 Page 92, Exercise 8.4

Statement. Let $f : A \rightarrow B$ be a ring homomorphism of finite type. Consider the following statements:

- (i) f is finite;
- (ii) the fibres of f^* are discrete subspaces of $\operatorname{Spec}(B)$;
- (iii) for each prime ideal \mathfrak{p} of A , the ring $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$ -algebra ($k(\mathfrak{p})$ is the residue field of $A_{\mathfrak{p}}$);
- (iv) the fibres of f^* are finite.

Prove that (i) \implies (ii) \iff (iii) \implies (iv). [Use Exercises 2 and 3.]

If f is integral and the fibres of f^* are finite, is f necessarily finite?

Hints. To prove (i) \implies (iii) one can use Section 4.1.12 p. 60, Claim 1 above and the following Lemma:

Lemma 84. *Let $A \rightarrow B$ be an integral ring morphism, let $S \subset A$ be a multiplicative subset, and let \mathfrak{p} be a prime ideal of A disjoint from S . Then the induced morphism $S^{-1}A/S^{-1}\mathfrak{p} \rightarrow S^{-1}B/(S^{-1}\mathfrak{p})^e$ is integral.*

Proof. By Proposition 5.6ii p. 61 of the book, $S^{-1}A \rightarrow S^{-1}B$ is integral. By Proposition 3.11iv

p. 41 of the book [Proposition 33 p. 58], $S^{-1}\mathfrak{p}$ is prime. By Theorem 5.10 p. 62 of the book, $S^{-1}\mathfrak{p}$ is contracted. Now the Lemma follows from Proposition 5.6i p. 61 of the book. \square

Hint for the last question, which is “If f is integral and the fibres of f^* are finite, is f necessarily finite?”: Consider the case when A and B are fields.

Note that $\mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{2}]$ satisfies (ii) but not (i).

9.2.5 Page 91, Exercise 8.5

Statement. In Chapter 5, Exercise 16 [Section 6.2.16 p. 110; this is the section about the Noether’s Normalization Theorem], show that X is a finite covering of L (i.e., the number of points of X lying over a given point of L is finite and bounded).

Solution. Omitted.

9.2.6 Page 92, Exercise 8.6

Statement. Let A be a Noetherian ring and \mathfrak{q} a \mathfrak{p} -primary ideal in A . Consider chains of primary ideals from \mathfrak{q} to \mathfrak{p} . Show that all such chains are of finite bounded length, and that all maximal chains have the same length.

Hints. See Section 9.1.1 p. 144 above.

Note that

the poset of \mathfrak{p} -primary ideals of A between \mathfrak{q} and \mathfrak{p}
is canonically isomorphic to

the poset of $\mathfrak{p}_{\mathfrak{p}}$ -primary ideals of $A_{\mathfrak{p}}$ containing $\mathfrak{q}_{\mathfrak{p}}$,
and that the above poset is equal to

the poset of proper ideals of $A_{\mathfrak{p}}$ containing $\mathfrak{q}_{\mathfrak{p}}$.

In particular all proper ideals of $A_{\mathfrak{p}}$ containing $\mathfrak{q}_{\mathfrak{p}}$ are $\mathfrak{p}_{\mathfrak{p}}$ -primary. [This is because all proper ideals of an Artin local ring (A, \mathfrak{m}) are \mathfrak{m} -primary.]

10 About Chapter 9

10.1 Comments

10.1.1 Page 94, Proposition 9.2

Recall Proposition 9.2:

Proposition 85 (Proposition 9.2). *Let A be a Noetherian local domain of dimension one, \mathfrak{m} its maximal ideal, $k = A/\mathfrak{m}$ its residue field. Then the following are equivalent:*

- (i) *A is a discrete valuation ring;*
- (ii) *A is integrally closed;*
- (iii) *\mathfrak{m} is a principal ideal;*
- (iv) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$;
- (v) *Every non-zero ideal is a power of \mathfrak{m} ;*
- (vi) *There exists $x \in A$ such that every non-zero ideal is of the form (x^n) , $n \neq 0$.*

In the setting of Proposition 9.2, the condition

(v') every non-zero ideal can be written in a unique way as a power of \mathfrak{m}

is equivalent to any of the conditions (i) to (vi). The same holds for

(vi') there exists $x \in A$ such that every non-zero ideal can be written in a unique way as (x^n) .

This follows from Statement (B) in the proof of Proposition 9.2.

10.1.2 Page 95

- Proof of the implication (iv) \implies (v) in Proposition 9.2 p. 94 [Proposition 85 p. 148]. It is written: “from (8.8) (applied to A/\mathfrak{m}^n) it follows that \mathfrak{a} is a power of \mathfrak{m} ”. The fact that A/\mathfrak{m}^n is Artin follows from Proposition 8.6 p. 90 of the book [see Section 9.1.2 p. 144 above].
- The proof of the equivalence (ii) \iff (iii) in Theorem 9.3 uses Statement (B) in the proof of Proposition 9.2 p. 94.
- Corollary 9.4. See Section 10.1.1 above.
- The domain $\mathbb{Z}[-5]$ is Dedekind but does not have unique factorization. The fact that $\mathbb{Z}[-5]$ does not have unique factorization follows from the fact that $2 \cdot 3$ and $(1 + \sqrt{5})(1 - \sqrt{5})$ are irreducible factorizations of 6.

10.1.3 Page 97, proof of Theorem 9.8

The last sentence of the proof of Theorem 9.8 is “Then \mathfrak{a} is invertible, hence $\mathfrak{b} = \mathfrak{a}_{\mathfrak{p}}$ is invertible by (9.7)”. I think the authors meant (9.6). Here are more details: We have $\mathfrak{b} = \mathfrak{a}_{\mathfrak{p}}$ by Proposition 3.11i

p. 41 [Proposition 33 p. 58] and Proposition 1.17.iii p. 10 [see the proof of Proposition 7. p. 80]. Moreover \mathfrak{a}_p is invertible by Proposition 9.6 [(i) \implies (ii)].

10.2 Exercises

10.2.1 Page 99, Exercise 9.1

Statement. Let A be a Dedekind domain, S a multiplicatively closed subset of A . Show that $S^{-1}A$ is either a Dedekind domain or the field of fractions of A .

Suppose that $S \neq A \setminus \{0\}$, and let H, H' be the ideal class groups of A and $S^{-1}A$ respectively. Show that extension of ideals induces a surjective homomorphism $H \rightarrow H'$.

Hint. See Section 6.1.6 p. 102 above.

10.2.2 Page 99, Exercise 9.2

Statement. Let A be a Dedekind domain. If $f = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial with coefficients in A , the **content** of f is the ideal $c(f) = (a_0, \dots, a_n)$ in A . Prove **Gauss's lemma** that $c(fg) = c(f)c(g)$.

[Localize at each maximal ideal.]

Hint. See Exercise 1.2iv p. 11 of the book [Section 2.2.2 p. 17]. It suffices to show $c(f)_m c(g)_m \subset c(fg)_m$ for all maximal ideal \mathfrak{m} .

10.2.3 Page 99, Exercise 9.3

Statement. A valuation ring (other than a field) is Noetherian if and only if it is a discrete valuation ring.

Solution. It is proved on p. 94 of the book that discrete valuation ring are noetherian. Exercise 5.28 p. 72 of the book [Section 6.2.28 p. 118] says that the ideals of a valuation ring are totally ordered. In particular finitely generated ideals are principal. Thus any noetherian valuation ring A is a principal ideal domain. Hence, if A is not a field, then it is of dimension one, and is therefore a discrete valuation ring by Proposition 9.2 p. 94 of the book.

10.2.4 Page 99, Exercise 9.4

Statement. Let A be a local domain which is not a field and in which the maximal ideal \mathfrak{m} is principal and $\bigcap_{n \geq 1} \mathfrak{m}^n = (0)$. Prove that A is a discrete valuation ring.

Hint. Let p be a generator of \mathfrak{m} .

There is a unique surjection $w : A \setminus \{0\} \rightarrow \mathbb{N}$ such that $a \in \mathfrak{m}^{w(a)} \setminus \mathfrak{m}^{w(a)+1}$ for all $a \in A \setminus \{0\}$. We have $w(ab) = w(a) + w(b)$ for all $a, b \in A \setminus \{0\}$, and $a \in A \setminus \{0\}$ is a unit if and only if $w(a) = 0$. Moreover, if $a \in A \setminus \{0\}$, then $a = p^{w(a)}u$ with u a unit.

If \mathfrak{a} is a nonzero ideal of A , and if n is the least nonnegative integer such that $p^n \in \mathfrak{a}$, then $\mathfrak{a} = (p^n)$.

This implies that A is a principal ideal domain, and thus [Proposition 9.2 p. 94 of the book] a Dedekind domain.

10.2.5 Page 99, Exercise 9.6

Statement. Let M be a finitely-generated torsion module ($T(M) = M$) over a Dedekind domain A . Prove that M is uniquely representable as a finite direct sum of modules $A/\mathfrak{p}_i^{n_i}$, where the \mathfrak{p}_i are non-zero prime ideals of A .

Solution. Let M be a finitely-generated torsion module over the Dedekind domain A . Then M has a nonzero annihilator \mathfrak{a} . Let $\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ be the prime factorization of \mathfrak{a} , and note that M is a module over the ring $A/\mathfrak{a} \simeq A/\mathfrak{p}_1^{n_1} \times \cdots \times A/\mathfrak{p}_r^{n_r}$. This yields an obvious decomposition $M = M_1 \oplus \cdots \oplus M_r$ of M , where each M_i is an $A/\mathfrak{p}_i^{n_i}$ -module.

Thus we can assume $\mathfrak{a} = \mathfrak{p}^n$ with \mathfrak{p} maximal, and it suffices to prove the claim below.

Claim 86. *There is a unique k -tuple (m_1, \dots, m_k) of integers such that $1 \leq m_1 \leq \cdots \leq m_k \leq n$ and $M \simeq A/\mathfrak{p}^{m_1} \oplus \cdots \oplus A/\mathfrak{p}^{m_k}$ [isomorphism of A -modules].*

We leave it to the reader to check that there is a unique pair (ϕ, ψ) of A -algebra morphisms

$$A/\mathfrak{p}^n \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n$$

such that

$$\phi(a + \mathfrak{p}^n) = \frac{a}{1} + \mathfrak{p}_{\mathfrak{p}}^n \quad \text{and} \quad \psi\left(\frac{a}{s} + \mathfrak{p}_{\mathfrak{p}}^n\right) = s'a + \mathfrak{p}^n$$

for all $a \in A$ and all $s, s' \in A \setminus \mathfrak{p}$ satisfying $ss' - 1 \in \mathfrak{p}^n$, and that ϕ and ψ are inverse isomorphisms.

For any A/\mathfrak{p}^n -module N write N' for the the A -module N viewed as an $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n$ -module via the formula $ax := \psi(a)x$ for all $a \in A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n$ and all $x \in N$. Note that N is finitely generated if and only if N' is.

Going back to our module M , note that, $A_{\mathfrak{p}}$ being a principal ideal domain and $\mathfrak{p}_{\mathfrak{p}}$ being maximal, there is a unique k -tuple (m_1, \dots, m_k) of integers such that $1 \leq m_1 \leq \cdots \leq m_k \leq n$ and

$$M' \simeq A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^{m_1} \oplus \cdots \oplus A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^{m_k}$$

[isomorphism of $A_{\mathfrak{p}}$ -modules]. As $(A/\mathfrak{p}^{m_i})' \simeq A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^{m_i}$, this implies Claim 86.

10.2.6 Page 99, Exercise 9.7

Statement. Let A be a Dedekind domain and $\mathfrak{a} \neq 0$ an ideal in A . Show that every ideal in A/\mathfrak{a} is principal.

Deduce that every ideal in A can be generated by at most two elements.

Hint. See the previous exercise. More precisely: it suffices to show that any nontrivial quotient of A is a principal ideal ring, and to deduce from this that, given any nonzero element $x \in \mathfrak{a}$, there is a $y \in \mathfrak{a}$ such that $\mathfrak{a} = (x, y)$.

11 About Chapter 10

11.1 Comments

11.1.1 Page 102, Completions 1

In the first paragraph after the proof of Lemma 10.1, it is written “Two Cauchy sequences are equivalent if $x_\nu - y_\nu \rightarrow 0$ in G ”. Note that a sequence may have several limits.

In the penultimate paragraph of p. 102 it is claimed that $\widehat{f} : \widehat{G} \rightarrow \widehat{H}$ is continuous, but no topologies have been defined on \widehat{G} and \widehat{H} . It is simpler to fix this problem in the setting considered in the last paragraph of p. 102. We shall use Corollary 10.4 p. 105. Note that, in this corollary, \widehat{G}_n really means $(G_n)^\wedge$, which can, and will, be viewed as a subgroup of \widehat{G} .

Then these subgroups do define a topology on \widehat{G} , and the canonical morphism $c : G \rightarrow \widehat{G}$ is continuous and its image is dense.

Moreover $\widehat{f} : \widehat{G} \rightarrow \widehat{H}$ is continuous if $f : G \rightarrow H$ is. (Here we assume that the topology of G and H are such that 0 has a countable fundamental system of neighborhoods.)

11.1.2 Page 103, Completions 2

The penultimate display is

$$\widehat{G} \simeq \varprojlim G/G_n.$$

More precisely, let $\pi_i : G \rightarrow G/G_i$ be the canonical projection; let $C \subset G^\mathbb{N}$ be the group of Cauchy sequences [this is indeed easily seen to be a subgroup of $G^\mathbb{N}$]; and note that $x \in G^\mathbb{N}$ is Cauchy if and only if for each i the sequence $j \mapsto \pi_i(x_j)$ is eventually constant, in which case we write $\pi_i(x_\infty)$ for its eventual value. Then there is a unique group morphism $\varphi : C \rightarrow \varprojlim G/G_i$ such that $\varphi(x)_i = \pi_i(x_\infty)$ for all x in C and all i in \mathbb{N} , this morphism is an epimorphism, and it induces an isomorphism $\widehat{G} \xrightarrow{\sim} \varprojlim G/G_i$.

11.1.3 Page 105

Even if it is very easy, we give additional details about the proofs of Corollary 10.4 and Proposition 10.5.

Setting $G' := G_{n_0}$ in Corollary 10.3 yields the exact sequence $0 \rightarrow \widehat{G_{n_0}} \rightarrow \widehat{G} \rightarrow (G/G_{n_0})^\wedge \rightarrow 0$. Recall that $p : G \rightarrow G/G_{n_0}$ is the natural morphism. For $n \geq n_0$ we have $pG_n = 0$. This implies $(G/G_{n_0})^\wedge \simeq G/G_{n_0}$ (canonical isomorphism), and thus $\widehat{G}/\widehat{G_{n_0}} \simeq G/G_{n_0}$, that is, $\widehat{G}/\widehat{G_n} \simeq G/G_n$ for all n . This entails

$$\widehat{\widehat{G}} \simeq \widehat{G}.$$

11.1.4 Page 109, Proposition 10.15

Proposition 10.15 (iii) says

$$\mathfrak{a}^n/\mathfrak{a}^{n+1} \simeq \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}. \quad (55)$$

Moreover (55) is derived from

$$A/\mathfrak{a}^n \simeq \hat{A}^n/\hat{\mathfrak{a}}^n, \quad (56)$$

but in the sequel (56) is needed at various places, and it is justified by stating that it follows from (55). For instance in the proof of Proposition 10.16 it is written

“By (10.15) iii) we have $\hat{A}/\hat{\mathfrak{m}} \simeq A/\mathfrak{m}$ ”.

In fact, I think it is better to prove $\hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+k} \simeq \mathfrak{a}^n/\mathfrak{a}^{n+k}$ directly by noting that we have

$$\hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+k} \simeq (\mathfrak{a}^n)^\wedge/(\mathfrak{a}^{n+k})^\wedge \simeq (\mathfrak{a}^n/\mathfrak{a}^{n+k})^\wedge \simeq \mathfrak{a}^n/\mathfrak{a}^{n+k},$$

the first isomorphism following from Proposition 10.15 (ii), the second from Corollary 10.3 p. 104, and the third being obvious.

Note that (i), (ii) and the proof of (ii) imply $(\mathfrak{a}^n)^\wedge = \hat{A} \mathfrak{a}^n = (\hat{A} \mathfrak{a})^n = \hat{\mathfrak{a}}^n \simeq \hat{A} \otimes_A \mathfrak{a}^n$. In particular, the equality $(\mathfrak{a}^n)^\wedge = \hat{A} \mathfrak{a}^n$ shows that

the \mathfrak{a} -topology and the $\hat{\mathfrak{a}}$ -topology of \hat{A} coincide.

The \mathfrak{a} -topology is finer than the $\hat{\mathfrak{a}}$ -topology even if A is not noetherian.

Note also

Proposition 87. *Let A be a noetherian ring, \mathfrak{a} an ideal of A and M a finitely generated A -module, and regard $(\mathfrak{a}M)^\wedge$ as a sub- A -module of \hat{M} . Then the sub- A -modules $(\mathfrak{a}M)^\wedge$, $\hat{\mathfrak{a}}\hat{M}$ and $\mathfrak{a}\hat{M}$ of \hat{M} coincide and are in fact sub- \hat{A} -modules of \hat{M} . Moreover they are isomorphic to $\hat{A} \otimes_A \mathfrak{a}M$.*

The proof is the same as that of Proposition 10.15i in the book.

11.1.5 Page 110, Corollary 10.19

Statement in the book:

Let A be a Noetherian ring, \mathfrak{a} an ideal of A contained in the Jacobson radical and let M be a finitely-generated A -module. Then the \mathfrak{a} -topology of M is Hausdorff, i.e. $\bigcap \mathfrak{a}^n M = 0$.

Here is a slightly stronger statement:

Let A be a Noetherian ring and \mathfrak{a} an ideal of A . Then the \mathfrak{a} -topology of M is Hausdorff for all finitely-generated A -module M , i.e. $\bigcap \mathfrak{a}^n M = 0$, if and only if \mathfrak{a} is contained in the Jacobson radical.

Let us prove that the \mathfrak{a} -topology is not necessarily Hausdorff if \mathfrak{a} is not contained in the Jacobson radical. Indeed, if \mathfrak{m} is a maximal ideal not containing \mathfrak{a} , then the \mathfrak{a} -topology of A/\mathfrak{m} is the coarse topology.

11.1.6 Page 111, Corollaries 10.20 and 10.21

About the proof of Corollary 10.20: The fact that “an \mathfrak{m} -primary ideal of A is just any ideal contained between \mathfrak{m} and some power \mathfrak{m}^n ” has already been stated as Corollary 7.16 p. 83.

About Corollary 10.21: Let a be in A . We must show:

$a \in \mathfrak{q}$ for all \mathfrak{p} -primary ideal $\mathfrak{q} \iff$ there is an s in $A \setminus \mathfrak{p}$ such that $as = 0$.

Proof. If s in $A \setminus \mathfrak{p}$ satisfies $as = 0$, and if \mathfrak{q} is a \mathfrak{p} -primary ideal, then a is in \mathfrak{q} .

If $as \neq 0$ for all $s \in A \setminus \mathfrak{p}$, then $\frac{a}{1} \neq 0$, and Corollary 10.20 implies that there is a \mathfrak{p} -primary ideal \mathfrak{q} such that $\frac{a}{1} \notin \mathfrak{q}_{\mathfrak{p}}$, and thus $a \notin \mathfrak{q}$. \square

11.2 Exercises

11.2.1 Page 113, Exercise 10.1

Statement. Let $\alpha_n : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ be the injection of abelian groups given by $\alpha_n(1) = p^{n-1}$, and let $\alpha : \rightarrow B$ be the direct sum of all the α_n (where A is a countable direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, and B is the direct sum of the $\mathbb{Z}/p^n\mathbb{Z}$). Show that the p -adic completion of A is just A but that the completion of A for the topology induced from the p -adic topology on B is the direct **product** of the $\mathbb{Z}/p\mathbb{Z}$. Deduce that p -adic completion is **not** a right-exact functor on the category of all \mathbb{Z} -modules.

Hint. For any abelian group G write \widehat{G} for the p -adic completion of G .

We claim that

$$G \mapsto \widehat{G} \text{ is neither left exact nor right exact.} \quad (57)$$

Set $C_j := \mathbb{Z}/(p^j)$ for $j \geq 0$ and $A := \bigoplus_{j \geq 1} C_1$, $B := \bigoplus_{j \geq 1} C_j$. The exact sequences

$$0 \rightarrow C_1 \xrightarrow{\alpha_j} C_j \xrightarrow{\pi_j} C_{j-1} \rightarrow 0,$$

where π_j is the multiplication by p [we use the isomorphism $pC_j \simeq C_{j-1}$ for $j \geq 1$], induce an exact sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\pi} B \rightarrow 0. \quad (58)$$

We claim

$$\widehat{A} \xrightarrow{\widehat{\alpha}} \widehat{B} \xrightarrow{\widehat{\pi}} \widehat{B} \text{ is not exact.} \quad (59)$$

This will imply (57). As \widehat{A} is isomorphic to A [details left to the reader], we can rewrite (59) as

$$A \xrightarrow{\widehat{\alpha}} \widehat{B} \xrightarrow{\widehat{\pi}} \widehat{B} \text{ is not exact.} \quad (60)$$

Write \overline{A} for the completion of A with respect to the filtration induced by the p -adic filtration of B . We leave it to the reader to check that applying Corollary 10.3 p. 104 of the book to (58) yields the exact sequence $0 \rightarrow \overline{A} \xrightarrow{\alpha'} \widehat{B} \xrightarrow{\widehat{\pi}} \widehat{B} \rightarrow 0$. We have $\overline{A} \simeq \prod_{j \geq 1} C_1$ [details again left to the reader], and thus $A \subsetneq \overline{A}$. Setting $a \in \overline{A} \setminus A$ we get $\alpha'(a) \in \text{Ker } \widehat{\pi} \setminus \text{Im } \widehat{\alpha}$. This proves (60), (59) and (57).

11.2.2 Page 114, Exercise 10.2

Statement. In Exercise 1, let $A_n = \alpha^{-1}(p^n B)$, and consider the exact sequence

$$0 \rightarrow A_n \rightarrow A \rightarrow A/A_n \rightarrow 0.$$

Show that \varprojlim is not right exact, and compute $\varprojlim^1 A_n$.

Solution. Omitted.

11.2.3 Page 114, Exercise 10.3

Statement. (a) Let A be a Noetherian ring, \mathfrak{a} an ideal and M a finitely-generated A -module. Using Krull's Theorem and Exercise 14 of Chapter 3, prove that

$$\bigcap_{n>0} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \text{Ker}(M \rightarrow M_{\mathfrak{m}}),$$

where \mathfrak{m} runs over all maximal ideals containing \mathfrak{a} .

(b) Deduce that

$$\widehat{M} = 0 \iff \text{Supp}(M) \cap V(\mathfrak{a}) = \emptyset \quad [\text{in } \text{Spec}(A)].$$

Solution. (a) By Krull's Theorem [Theorem 10.17 p. 110 of the book], we have

$$E := \text{Ker}(M \rightarrow \widehat{M}) = \bigcap \mathfrak{a}^n M = \bigcup_{a \in \mathfrak{a}} \text{Ann}_M(1 + a).$$

Set $F := \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \text{Ker}(M \rightarrow M_{\mathfrak{m}})$. We must show $E = F$. The inclusion $E \subset F$ is easy [indeed we have $\text{Ann}_M(1 - a) \subset \mathfrak{a}^n M$ for $a \in \mathfrak{a}$ and $n \in \mathbb{N}$]. To prove $F \subset E$, first note that we have $F_{\mathfrak{m}} = 0$ if $\mathfrak{m} \supset \mathfrak{a}$. By Exercise 3.14 p. 45 of the book [Section 4.2.14 p. 69], this implies $F = \mathfrak{a}F$, hence $F = \mathfrak{a}^n F \subset \mathfrak{a}^n M$ for all n , hence $F \subset E$.

(b) Set $\mathfrak{b} := \text{Ann}(M)$. We have

$$\widehat{M} = 0 \iff M = \mathfrak{a}M \iff (\exists a \in \mathfrak{a}) 1 - a \in \mathfrak{b} \iff \mathfrak{a} + \mathfrak{b} = (1) \iff \text{Supp}(M) \cap V(\mathfrak{a}) = \emptyset,$$

the successive equivalences being justified as follows:

- first equivalence: obvious,
- second equivalence: Corollary 2.5 p. 21 of the book [Corollary 10 p. 10],
- third equivalence: obvious,
- fourth equivalence: Exercise 3.19v p. 46 of the book [Section 4.2.19 p. 70].

Details about the fourth equivalence: By Exercise 3.19v p. 46 of the book we have $\text{Supp}(M) = V(\mathfrak{b})$, hence $\text{Supp}(M) \cap V(\mathfrak{a}) = V(\mathfrak{a} + \mathfrak{b})$, hence $\text{Supp}(M) \cap V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} + \mathfrak{b} = (1)$.

11.2.4 Page 114, Exercise 10.4

Statement. Let A be a Noetherian ring, \mathfrak{a} an ideal in A , and \widehat{A} the \mathfrak{a} -adic completion. For any $x \in A$, let \widehat{x} be the image of x in \widehat{A} .

(a) Show that

$$x \text{ not a zero-divisor in } A \implies \widehat{x} \text{ not a zero-divisor in } \widehat{A}.$$

(b) Does this imply that

$$A \text{ is an integral domain} \implies \widehat{A} \text{ is an integral domain?}$$

Answer to (b): No. Take $\mathfrak{a} := (1)$.

11.2.5 Page 114, Exercise 10.5

Statement. Let A be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in A . If M is any A -module, let $M^{\mathfrak{a}}, M^{\mathfrak{b}}$ denote its \mathfrak{a} -adic and \mathfrak{b} -adic completions respectively. If M is finitely generated, prove that $(M^{\mathfrak{a}})^{\mathfrak{b}} \simeq M^{\mathfrak{a}+\mathfrak{b}}$.

Hint. In view of the isomorphism $\widehat{A} \otimes_A M \simeq \widehat{M}$, it suffices to show

$$(A^{\mathfrak{a}})^{\mathfrak{b}} \simeq A^{\mathfrak{a}+\mathfrak{b}}. \quad (61)$$

Using Proposition 10.2. p. 104 and Proposition 87 p. 153, and writing L_i for \lim_i , we have

$$\begin{aligned} (A^{\mathfrak{a}})^{\mathfrak{b}} &\simeq L_j \left(\frac{L_i \frac{A}{\mathfrak{a}^i}}{\mathfrak{b}^j L_i \frac{A}{\mathfrak{a}^i}} \right) \simeq L_j \left(\frac{L_i \frac{A}{\mathfrak{a}^i}}{L_i \mathfrak{b}^j \frac{A}{\mathfrak{a}^i}} \right) \simeq L_j L_i \left(\frac{\frac{A}{\mathfrak{a}^i}}{\mathfrak{b}^j \frac{A}{\mathfrak{a}^i}} \right) \\ &\simeq L_j L_i \left(\frac{A}{\mathfrak{a}^i + \mathfrak{b}^j} \right) \simeq L_n \left(\frac{A}{(\mathfrak{a} + \mathfrak{b})^n} \right) \simeq A^{\mathfrak{a}+\mathfrak{b}}. \end{aligned}$$

11.2.6 Page 114, Exercise 10.6

Statement. Let A be a Noetherian ring and \mathfrak{a} an ideal in A . Prove that \mathfrak{a} is contained in the Jacobson radical of A if and only if every maximal ideal of A is closed for the \mathfrak{a} -topology. (A Noetherian topological ring in which the topology is defined by an ideal contained in the Jacobson radical is called a **Zariski ring**. Examples are local rings and [by Proposition 10.15iv p. 109 of the book] \mathfrak{a} -adic completions.)

Hint. Let A be a ring, \mathfrak{a} an ideal, and equip A with the \mathfrak{a} -adic topology.

Then any ideal containing \mathfrak{a} is open and closed [because such an ideal is a union of \mathfrak{a} -cosets].

Moreover any maximal ideal \mathfrak{m} not containing \mathfrak{a} is dense.

To prove this, let us show that any nonempty open subset $U \subset A$ meets \mathfrak{m} . Set $K := A/\mathfrak{m}$ and let $\pi : A \rightarrow K$ be the canonical projection. It suffices to check that $0 \in \pi(U)$. But the \mathfrak{a} -adic topology of K being the codiscrete topology, $\pi(U)$ is the unique nonempty subset of K , that is K itself.

11.2.7 Page 114, Exercise 10.7

Statement. Let A be a Noetherian ring, \mathfrak{a} an ideal of A , and \widehat{A} the \mathfrak{a} -adic completion. Prove that \widehat{A} is faithfully flat over A (Chapter 3, Exercise 16; Section 4.2.16 p. 69) if and only if A is a Zariski ring (for the \mathfrak{a} -topology).

[Since \widehat{A} is flat over A , it is enough to show that

$$M \rightarrow \widehat{M} \text{ injective for all finitely generated } M \iff A \text{ is Zariski;}$$

now use Corollary 10.19 p. 110 (see Section 11.1.5 p. 153) and Exercise 6.]

Hint. Let \mathfrak{m} be a maximal ideal of A . Proposition 10.15i p. 109 of the book implies that the extension \mathfrak{m}^e of \mathfrak{m} in \widehat{A} is $\widehat{\mathfrak{m}}$, and we get

$$\mathfrak{a} \subset \mathfrak{m} \implies 0 \neq (A/\mathfrak{m})^\wedge \simeq \widehat{A}/\widehat{\mathfrak{m}} \implies \mathfrak{m}^e \neq (1),$$

$$\mathfrak{a} \not\subset \mathfrak{m} \implies 0 = (A/\mathfrak{m})^\wedge \simeq \widehat{A}/\widehat{\mathfrak{m}} \implies \mathfrak{m}^e = (1).$$

11.2.8 Page 115, Exercise 10.8

Statement. Let A be the local ring of the origin in \mathbb{C}^n (i.e., the ring of all rational functions $f/g \in \mathbb{C}(z_1, \dots, z_n)$ with $g(0) \neq 0$), let B be the ring of power series in z_1, \dots, z_n which converge in some neighborhood of the origin, and let C be the ring of formal power series in z_1, \dots, z_n , so that $A \subset B \subset C$. Show that B is a local ring and that its completion for the maximal ideal topology is C . Assuming that B is Noetherian, prove that B is A -flat. [Use Chapter 3, Exercise 17 (Section 4.2.17 p. 70, and Exercise 7 above.)]

Hint. The proof that B is a local ring and that its completion for the maximal ideal topology is C is left to the reader. Exercise 3.17 says that if $A \subset B \subset C$ are rings, and if C is flat over A and faithfully flat over B , then B is flat over A . Exercise 7 above implies that C is faithfully flat over B . Note that A is also a local ring and that its completion for the maximal ideal topology is also C . Then Proposition 10.14 p. 109 of the book [see below] entails that C is flat over A .

Here is Proposition 10.14 of the book:

Proposition 88 (Proposition 10.14 p. 109 of the book). *If A is a Noetherian ring, \mathfrak{a} an ideal, \widehat{A} the \mathfrak{a} -adic completion of A , then \widehat{A} is a flat A -algebra.*

11.2.9 Page 115, Exercise 10.9, Hensel's Lemma

To solve Exercise 10.9 we need only assume that A is complete with respect to some ideal \mathfrak{m} ; we do not necessarily need \mathfrak{m} maximal or A local. To emphasize the fact that \mathfrak{m} is not necessarily maximal, we denote this ideal by \mathfrak{a} .

We shall prove:

Theorem 89 (Hensel's Lemma). *Let \mathfrak{a} be an ideal of a ring A such that A is complete for the \mathfrak{a} -adic topology, set $B := A/\mathfrak{a}$ and let $\pi : A[x] \rightarrow B[x]$ be the natural morphism. Let $F \in A[x]$ be*

monic of degree n , set $f := \pi(F)$, and assume that $g, h \in B[x]$ are coprime monic polynomials of degrees r and $n - r$ satisfying $f = gh$. Then there are monic lifts $G, H \in A[x]$ of g, h such that $F = GH$. [A **lift** of a polynomial $u \in B[x]$ is an element of $\pi^{-1}(u) \subset A[x]$.]

The proof below is stolen from Thomas J. Haines, Lectures on Commutative Algebra
<http://www.math.umd.edu/~tjh/CommAlg.pdf>

Proof. We first define two sequences G_1, G_2, \dots and H_1, H_2, \dots of monic lifts in $A[x]$ of g and h as follows.

The polynomials G_1 and H_1 are arbitrary monic lifts of g and h . We have in particular $F - G_1 H_1 \in \mathfrak{a}[x]$.

Assume that $G_k, H_k \in A[x]$ have already been constructed and satisfy

$$F - G_k H_k \in \mathfrak{a}^k[x].$$

Let i be a nonnegative integer less than n . There are $u, v \in B[x]$ such that $x^i = ug + vh$. Let $q, c_i \in B[x]$ satisfy $u = qh + c_i$ and

$$\deg c_i < \deg h = n - r.$$

Setting $d_i := q + v$ we get

$$x^i = c_i g + d_i h, \tag{62}$$

as well as $\deg d_i h = \deg(x^i - c_i g) < n$, and thus

$$\deg d_i < r.$$

Let $C_i, D_i \in A[x]$ be lifts of the same degree of c_i and d_i . We have

$$F - G_k H_k = \sum_{i=0}^{n-1} \alpha_i x^i$$

with $\alpha_i \in \mathfrak{a}^k$. Then

$$G_{k+1} := G_k + \sum_{i=0}^{n-1} \alpha_i D_i \text{ and } H_{k+1} := H_k + \sum_{i=0}^{n-1} \alpha_i C_i$$

are monic lifts of $g, h \in B[x]$, and we get

$$\begin{aligned} F - G_{k+1} H_{k+1} &= \sum_{i=0}^{n-1} \alpha_i x^i - \sum_{i=0}^{n-1} \alpha_i (C_i G_k + D_i H_k) - \sum_{i,j=0}^{n-1} \alpha_i \alpha_j C_i D_j \\ &= \sum_{i=0}^{n-1} \alpha_i x^i - \sum_{i=0}^{n-1} \alpha_i (x^i + E_i) - \sum_{i,j=0}^{n-1} \alpha_i \alpha_j C_i D_j \end{aligned}$$

for some $E_i \in \mathfrak{a}[x]$ — the last equality following from (62). This shows that $F - G_{k+1} H_{k+1}$ is in $\mathfrak{a}^{k+1}[x]$.

Write $G_{k,i}$ for the coefficient of x^i in G_k and define $H_{k,i}$ similarly. It is easy to see that the sequences $G_{1,i}, G_{2,i}, \dots$ and $H_{1,i}, H_{2,i}, \dots$ in A are Cauchy, and thus convergent. Let $G_{\infty,i}$ and $H_{\infty,i}$ be their limits. It is not hard to check that the polynomials $G := \sum G_{\infty,i} x^i$ and $H := \sum H_{\infty,i} x^i$ do the job. \square

11.2.10 Page 115, Exercise 10.10

Statement. (i) With the notation of Exercise 9, deduce from Hensel's lemma that if $\bar{f}(x)$ has a simple root $\alpha \in A/\mathfrak{m}$, then $f(x)$ has a simple root $a \in A$ such that $a \equiv \alpha \pmod{\mathfrak{m}}$.

(ii) Show that 2 is a square in the ring of 7-adic integers.

(iii) Let $f(x, y) \in k[x, y]$, where k is a field, and assume that $f(0, y)$ has $y = a_0$ as a simple root. Prove that there exists a formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f(x, y(x)) = 0$.

(This gives the “analytic branch” of the curve $f = 0$ through the point $(0, a_0)$.)

Hint. Part (iii) is a particular case of Part (i):

A	\mathfrak{m}	$\bar{f}(x)$	α	$f(x)$	a	$\bar{f}(\alpha) = 0$	$f(a) = 0$
$k[[x]]$	(x)	$f(0, y)$	a_0	$f(x, y)$	$y(x)$	$f(0, a_0) = 0$	$f(x, y(x)) = 0$

[The condition that the roots α, a, a_0 and $y(x)$ are simple is implicit.]

11.2.11 Page 115, Exercise 10.11

Statement: Show that the converse of (10.26) is false, even if we assume that A is local and that \hat{A} is a finitely-generated A -module.

Recall (10.26): If A is a Noetherian ring, \mathfrak{a} an ideal of A , then the \mathfrak{a} -completion \hat{A} of A is Noetherian.

Hint. Let A be the ring of germs at 0 of C^∞ functions from \mathbb{R} to \mathbb{R} , and \mathfrak{m} the ideal of those germs which vanish at 0. Then $\hat{A} \simeq \mathbb{R}[[x]]$ and $A \rightarrow \hat{A}$ is surjective. Note that A is not noetherian by Corollary 10.18 p. 110 of the book.

11.2.12 Page 115, Exercise 10.12

Statement: If A is Noetherian, then $B := A[[x_1, \dots, x_n]]$ is a faithfully flat A -algebra.

Solution. Using Proposition 10.14 p. 109 of the book [Proposition 88 p. 157] it is easy to see that B is A -flat. Let $\phi : A \rightarrow B$ be the inclusion, $\psi : B \rightarrow A$ the evaluation at 0, and \mathfrak{m} a maximal ideal of A . It suffices to show that the extension of \mathfrak{m} along ϕ is a proper ideal of B [see Exercise 3.16iii p. 45 of the book]. If this extension was the unit ideal of B , then the extension of \mathfrak{m} along $\psi \circ \phi$ would be the unit ideal of A . But this latter extension is \mathfrak{m} because $\psi \circ \phi$ is the identity of A .

12 About Chapter 11

12.1 Comments

12.1.1 Page 117, definition of $d(M)$

I think it would be more prudent to assume that $M \neq 0$, that A_0 is Artin, and that $\lambda(M)$ is the length of M .

12.1.2 Page 118, Proposition 11.3

It seems better to assume $xM \neq M$.

12.1.3 Page 118, Example following Proposition 11.3

We have $P(A, t) = \ell(1 - t)^s$ where ℓ is the length of A_0 .

12.1.4 Page 118, proof of Proposition 11.4

- The ring A/\mathfrak{q} is Artin by (54) p. 144.
- The fact below is used on line 5 of the proof of Proposition 11.4:

A noetherian module over an Artin ring has finite length.

This follows from Propositions 6.2 p. 75 and 6.8 p. 77.

12.1.5 Page 119, old d new d

On p. 119 it is claimed that the new d evaluated on A coincides with the old d evaluated on $G_{\mathfrak{m}}(A)$, the asserted equality being written $d(A) = d(G_{\mathfrak{m}}(A))$.

We denote the old d by d_o and the new d by d_ν , so that the equality to check becomes

$$d_o(A) = d_\nu(G_{\mathfrak{m}}(A)). \quad (63)$$

Note that $d_o(A)$ is defined when A is a noetherian graded ring, and $d_\nu(A)$ is defined when A is a noetherian local ring. By Proposition 10.22 (i) p. 111, $G_{\mathfrak{a}}(A)$ is a noetherian graded ring if A is a noetherian ring and \mathfrak{a} is an ideal of A .

As in (1) p. 118 of the book we set $\ell_n := \ell(A/\mathfrak{m}^n)$.

Corollary 11.5 p. 119 says that, for large n , the function $n \mapsto \ell_n$ is a polynomial whose degree is $d_\nu(A)$ by definition.

Corollary 11.2 p. 117 says that, for large n , the function $n \mapsto \ell(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ is a polynomial whose degree is $d_o(G_{\mathfrak{m}}(A)) - 1$.

Now (63) above follows from (1) p. 117 of the book.

12.1.6 Page 120, Proposition 11.9

We must assume that x is not a unit.

12.1.7 Page 120, proof of Proposition 11.10

The claim “ A is an Artin ring” is justified by (54) p. 144.

12.1.8 Page 121, Dimension Theorem

Here is an application of the Dimension Theorem:

Let K be a field, let x_1, x_2, \dots be indeterminates, and form the K -algebra $A := K[[x_1, x_2, \dots]]$.

Recall that A can be defined as the set of expressions of the form $\sum_u a_u u$, where u runs over the set monomials in x_1, x_2, \dots , and each a_u is in K , the addition and multiplication being the obvious ones.

Then A is a local domain, its maximal ideal \mathfrak{m} is defined by the condition $a_1 = 0$, and we claim

$$\boxed{A \text{ is not } \mathfrak{m}\text{-adically complete.}} \quad (64)$$

This result is due to Uriya First and to the MathOverflow user dhy. See <https://mathoverflow.net/a/308266/461>.

We equip A with the \mathfrak{m} -adic topology.

Let $v : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>1}$ be strictly increasing, assume that, for all $n \in \mathbb{Z}_{>0}$, the characteristic of K does not divide $v(n)$, and consider the sequence $(s_n)_{n \in \mathbb{Z}_{>0}}$ defined by $s_n = \sum_{i=1}^n x_i^{v(i)}$. This sequence being clearly Cauchy, it suffices to show that it diverges. To prove this we argue by contradiction and assume that (s_n) has a limit in A . It is easy to see that this limit is $\sum_{n>0} x_n^{v(n)}$, and that this element of A is in \mathfrak{m}^2 . Thus (64) will follow from

$$\sum_{n>0} x_n^{v(n)} \notin \mathfrak{m}^2. \quad (65)$$

Our proof of (65) starts with the following claim.

(\star) *Let k and r be positive integers; let \mathfrak{n} be the maximal ideal of $B := K[[x_1, \dots, x_k]]$; let $a_1, \dots, a_r, b_1, \dots, b_r$ be in \mathfrak{n} ; set $f = \sum_{i=1}^r a_i b_i$ and $D_j := \frac{\partial}{\partial x_j}$ for $1 \leq j \leq k$; and assume that the ideal $(D_1 f, \dots, D_k f) \subset B$ is \mathfrak{n} -primary. Then $k \leq 2r$.*

Proof of (\star). We have

$$D_j f = \sum_{i=1}^r \left((D_j a_i) b_i + a_i (D_j b_i) \right).$$

This implies that $(D_1 f, \dots, D_k f) \subset (a_1, \dots, a_r, b_1, \dots, b_r)$, and thus that $(a_1, \dots, a_r, b_1, \dots, b_r)$ is \mathfrak{n} -primary [see Corollary 7.16 p. 83 of the book]. By the Examples pages 118 and 121, and by Corollary 11.19 p. 122 of the book, we have $\dim B = k$. Now the Dimension Theorem entails $k \leq 2r$, as desired. \square

Proof of (65). Assume by contradiction that we have $\sum_{n>0} x_n^{v(n)} = \sum_{i=1}^r c_i d_i$ with $c_i, d_i \in \mathfrak{m}$. Let k be an integer $> 2r$. Mapping x_j to 0 for $j > k$ we get

$$f := \sum_{n=1}^k x_n^{v(n)} = \sum_{i=1}^r a_i b_i \in K[[x_1, \dots, x_k]]$$

for some $a_i, b_i \in (x_1, \dots, x_k) \subset K[[x_1, \dots, x_k]]$. As $(D_1 f, \dots, D_k f)$ is (x_1, \dots, x_k) -primary by Corollary 7.16 p. 83 of the book, this contradicts (\star) . \square

The case when K is **finite** is a Bourbaki exercise: Exercice 22c p. 288 in Exercices du §2 chap. III, **Algèbre commutative**, Bourbaki, Masson, Paris 1985.

12.1.9 Page 121, Proposition 11.13 and Corollary 11.16

The following statement is implicit in the book.

Proposition 90. *If \mathfrak{p} is a prime ideal of a noetherian ring, then we have*

$$\text{height } \mathfrak{p} = \min \{n \in \mathbb{N} \mid (\exists x_1, \dots, x_n \in \mathfrak{p}) \mathfrak{p} \text{ is a minimal prime ideal of } (x_1, \dots, x_n)\}.$$

We start with Lemma 91 below. The statement and the proof of this lemma are almost the same as those of Proposition 11.13 of the book. To make the analogy clearer we have used a notation as close as possible to that of the book; in particular we warn the reader that we have denoted by \mathfrak{m} a prime ideal which is not necessarily maximal!

Lemma 91. *Let A be a noetherian ring and \mathfrak{m} a prime ideal of height d . Then there exist d elements x_1, \dots, x_d of \mathfrak{m} such that \mathfrak{m} is a minimal prime ideal of (x_1, \dots, x_d) .*

Proof. Construct x_1, \dots, x_d inductively in such a way that every prime sub-ideal of \mathfrak{m} containing (x_1, \dots, x_d) has height $\geq i$, for each i . Suppose $i > 0$ and x_1, \dots, x_{i-1} constructed. Let \mathfrak{p}_j ($1 \leq j \leq s$) be the minimal prime ideals of (x_1, \dots, x_{i-1}) which are contained in \mathfrak{m} and have height **exactly** $i-1$. Since $i-1 < d = \text{height } \mathfrak{m}$, we have $\mathfrak{m} \neq \mathfrak{p}_j$ ($1 \leq j \leq s$), hence $\mathfrak{m} \neq \bigcup_{j=1}^s \mathfrak{p}_j$ by (1.11). Choose $x_i \in \mathfrak{m}$, $x_i \notin \bigcup \mathfrak{p}_j$, and let \mathfrak{q} be any prime sub-ideal of \mathfrak{m} containing (x_1, \dots, x_i) . Then \mathfrak{q} contains some minimal prime ideal of (x_1, \dots, x_{i-1}) contained in \mathfrak{m} . If $\mathfrak{p} = \mathfrak{p}_j$ for some j , we have $x_i \in \mathfrak{q}$, $x_i \notin \mathfrak{p}$, hence $\mathfrak{q} \not\subseteq \mathfrak{p}$ and therefore $\text{height } \mathfrak{q} \geq i$; if $\mathfrak{p} \neq \mathfrak{p}_j$ ($1 \leq j \leq s$), then $\text{height } \mathfrak{p} \geq i$, hence $\text{height } \mathfrak{q} \geq i$. Thus every prime ideal of (x_1, \dots, x_i) contained in \mathfrak{m} has height $\geq i$.

If \mathfrak{p} is a prime ideal of (x_1, \dots, x_d) contained in \mathfrak{m} , then \mathfrak{p} has height $\geq d$, hence $\mathfrak{p} = \mathfrak{m}$ [for $\mathfrak{p} \subsetneq \mathfrak{m} \implies \text{height } \mathfrak{p} < \text{height } \mathfrak{m} = d$]. \square

Now Proposition 90 follows from Lemma 91 above and Corollary 11.16 of the book.

12.1.10 Page 122

- Proof of Corollary 11.16. The fact that $(x_1, \dots, x_r)_{\mathfrak{p}}$ is $\mathfrak{p}_{\mathfrak{p}}$ -primary follows easily from Propositions 4.8 (ii) p 53 and 4.9 p 54 [Proposition 42 p. 81].
- Proof of Proposition 11.20. The $d(?)$ are $d_o(?)$ in the notation of Section 12.1.5.

12.1.11 Page 123, proof of Theorem 11.22

I think “by (11.20)” should be “by (11.21)”.

12.1.12 Page 125, proof of (11.25)

For the last sentence of the proof, see the Examples on p. 121.

12.2 Exercises

12.2.1 Page 125, Exercise 11.1

Statement. Let $f \in k[x_1, \dots, x_n]$ be an irreducible polynomial over an algebraically closed field k . A point P on the variety $f(x) = 0$ is **non-singular** \iff not all the partial derivatives of $\partial f / \partial x_i$ vanish at P . Let $A = k[x_1, \dots, x_n] / (f)$, and let \mathfrak{m} be the maximal ideal of A corresponding to the point P . Prove that P is non-singular $\iff A_{\mathfrak{m}}$ is a regular local ring.

[By Corollary 11.18 p. 122 (see below) we have $\dim A_{\mathfrak{m}} = n - 1$. Now

$$\mathfrak{m} / \mathfrak{m}^2 \simeq (x_1, \dots, x_n) / (x_1, \dots, x_n)^2 + (f)$$

and has dimension $n - 1$ if and only if $f \notin (x_1, \dots, x_n)^2$.]

Hints. I think the assumption that f is irreducible is unnecessary, and that it suffices to suppose that f is nonzero.

Recall that Corollary 11.18 says: “Let A be a Noetherian local ring, x an element of \mathfrak{m} which is not a zero-divisor. Then $\dim A / (x) = \dim A - 1$.”

We can assume $P = 0$. We will use the following notation: X_1, \dots, X_n are indeterminates, A is defined by $A := k[X_1, \dots, X_n] / (f) = k[x_1, \dots, x_n]$ where x_i is the image of X_i , we set

$$\mathfrak{m} := (X_1, \dots, X_n), \quad \overline{\mathfrak{m}} := (x_1, \dots, x_n) \simeq \mathfrak{m} / (f).$$

We have

$$\dim A_{\overline{\mathfrak{m}}} = n - 1 \tag{66}$$

by Corollary 11.18. We also have

$$\frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}^2} = \frac{\mathfrak{m} / (f)}{(\mathfrak{m}^2 + (f)) / (f)} \simeq \frac{\mathfrak{m}}{\mathfrak{m}^2 + (f)}. \tag{67}$$

- If $f \in \mathfrak{m}^2$ we get

$$\frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}^2} \simeq \frac{\mathfrak{m}}{\mathfrak{m}^2} \simeq k^n$$

by (67), and $A_{\overline{\mathfrak{m}}}$ is singular by (66).

- If $f \notin \mathfrak{m}^2$ we have $n - 1 = \dim A_{\overline{\mathfrak{m}}} \leq \dim_k \overline{\mathfrak{m}} / \overline{\mathfrak{m}}^2 \leq n - 1$ by (66), Corollary 11.15 p. 121 of the book and (67), so that $A_{\overline{\mathfrak{m}}}$ is regular in this case. [Corollary 11.15 says: “ $\dim A \leq \dim_k(\mathfrak{m} / \mathfrak{m}^2)$.”]

We conclude that $A_{\overline{\mathfrak{m}}}$ is regular if and only if $f \notin \mathfrak{m}^2$. It remains to check that 0 is non-singular if and only if $f \notin \mathfrak{m}^2$, or, in other words, that $\frac{\partial f}{\partial X_i}(0) \neq 0$ for some i if and only if $f \notin \mathfrak{m}^2$. But this follows from that fact that $\frac{\partial f}{\partial X_i}(0)$ is the coefficient of X_i in f .

12.2.2 Page 125, Exercise 11.2

Statement. In (11.21) assume that A is complete. Prove that the homomorphism $k[[t_1, \dots, t_d]] \rightarrow A$ given by $t_i \mapsto x_i$ ($i = 1, \dots, d$) is injective and that A is a finitely-generated module over $k[[t_1, \dots, t_d]]$.

Recall (11.21): If (A, \mathfrak{m}) is a noetherian local ring and $k \subset A$ a field mapping isomorphically onto A/\mathfrak{m} , and if x_1, \dots, x_d is a system of parameters, then x_1, \dots, x_d are algebraically independent over k .

Solution. Let \mathfrak{q} be the \mathfrak{m} -primary ideal (x_1, \dots, x_d) , set

$$\mathfrak{n} := (t_1, \dots, t_d) \subset B_0 := k[t_1, \dots, t_d], \quad B := k[[t_1, \dots, t_d]],$$

and let $\phi : B_0 \rightarrow A$ be the obvious morphism mapping t_i to x_i . The \mathfrak{n} -topology, the \mathfrak{q} -topology and the \mathfrak{m} -topology coincide on A [see proof of Proposition 11.6 p. 119]. In particular ϕ extends uniquely to a continuous morphism $\psi : B \rightarrow A$. Let us show that ψ is injective. Let

$$0 \neq b := \sum c_\alpha t_1^{\alpha_1} \cdots t_d^{\alpha_d}$$

be in B and assume by contradiction $\psi(b) = 0$. Write $b = b_n + b_{n+1} + \cdots$ with b_i homogeneous of degree i and $b_n \neq 0$. Set $a_i := \phi(b_i)$. We get $a_n \in \mathfrak{q}^{n+1}$, and Proposition 11.20 p. 122 of the book implies $b_n = 0$, contradiction.

Let us show that A is a finitely generated B -module. By Proposition 10.24 p. 112 of the book, it suffices to show that $G_{\mathfrak{n}}(A) = G_{\mathfrak{q}}(A)$ is a finitely generated $G_{\mathfrak{n}}(B)$ -module, where \mathfrak{n} is the maximal ideal of B . Note that $G_{\mathfrak{n}}(B)$ is a sub- $G_{\mathfrak{n}}(B)$ -module of $G_{\mathfrak{q}}(A)$, the quotient being

$$\frac{A/\mathfrak{q}}{A/\mathfrak{m}} = \frac{A/\mathfrak{q}}{k},$$

which is a finite dimensional k -vector space, and *a fortiori* a finitely generated $G_{\mathfrak{n}}(B)$ -module.

12.2.3 Page 126, Exercise 11.3

Statement. Extend (11.25) to non-algebraically-closed fields.

Recall (11.25): For any irreducible variety V over K the local dimension of V at any point is equal to $\dim V$.

Solution. Let K be a field and A a finitely generated K -algebra. By Noether's Normalization Theorem [Theorem 74 p. 110], the Krull dimension n of A is finite, and there are n elements of A which are algebraically independent over K .

Claim: Any $n + 1$ elements of A are algebraically dependent over K .

Proof. If A is a domain, we are done because Noether's Normalization Theorem implies that n is the transcendence degree of the field of fractions of A over K . If A is not a domain, we can argue as follows.

Assume by contradiction that the elements x_1, \dots, x_{n+1} of A are algebraically independent over K . Set $B := K[x_1, \dots, x_{n+1}] \subset A$ and $S := B \setminus \{0\}$. This is a multiplicative subset of A which does not contain 0. Thus there is a prime ideal \mathfrak{p} of A which is disjoint from S , and B imbeds into the domain A/\mathfrak{p} , whose Krull dimension is at most n . This contradicts the first part of the argument.

12.2.4 Page 126, Exercise 11.4

The purpose of the exercise is to prove:

Theorem 92 (Nagata). *There are noetherian domains of infinite dimension.*

We sketch a proof [which will give explicit examples].

Let $\mathbb{N} = \bigsqcup_{i \in \mathbb{N}} N_i$ be a partition of \mathbb{N} such that each N_i is finite and nonempty, let K be a field, let A be the K -algebra $K[x_0, x_1, \dots]$, where the x_i are indeterminates, for each $i \in \mathbb{N}$ let \mathfrak{p}_i be the ideal of A generated by the x_j with $j \in N_i$, and let $S \subset A$ be the complement of the union of the \mathfrak{p}_i . Clearly the \mathfrak{p}_i are prime and S is a multiplicative subset of A . Set $B := S^{-1}A$.

Our main goal is to prove

(a) B is noetherian.

Reduction to Statements (d) and (e). By Exercise 7.9 p. 85 of the book, it suffices to show

(b) For each maximal ideal \mathfrak{m} of B , the local ring $B_{\mathfrak{m}}$ is noetherian.

(c) For each $b \neq 0$ in B , the set of maximal ideals of B which contain b is finite.

We claim

(d) If \mathfrak{a} is an ideal of A contained in the union of the \mathfrak{p}_i , then \mathfrak{a} is contained in some \mathfrak{p}_i .

Statement (d) will imply that the maximal ideals of B are the $S^{-1}\mathfrak{p}_i$, and thus, (d) will imply (c). Statement (d) will also reduce (b) to

(e) For each i the local ring $B_{S^{-1}\mathfrak{p}_i}$ is noetherian.

To summarize, it suffices to prove (d) and (e).

Proof of (d). Recall Statement (d):

(d) If \mathfrak{a} is an ideal of A contained in the union of the \mathfrak{p}_i , then \mathfrak{a} is contained in some \mathfrak{p}_i .

To prove (d) we shall implicitly use the following easy fact:

Note 93. *Let \mathfrak{b} be an ideal of A . Then \mathfrak{b} is generated by monomials if and only if it has the following property:*

A polynomial $f \in A$ is in \mathfrak{b} if and only if all the monomials occurring in f are in \mathfrak{b} .

Proof of (d). Assume by contradiction that \mathfrak{a} is contained in the union of the \mathfrak{p}_i , but is contained in no \mathfrak{p}_i . Let $0 \neq f \in \mathfrak{a}$.

There is an $n \in \mathbb{N}$ such that no monomial occurring in f is in $\mathfrak{p}_{n+1} \cup \mathfrak{p}_{n+2} \cup \dots$. In particular

$$(A) \quad f \notin \mathfrak{p}_{n+1} \cup \mathfrak{p}_{n+2} \cup \dots.$$

We claim that there is a $g \in \mathfrak{a}$ such that

$$(B) \quad g \notin \mathfrak{p}_0 \cup \dots \cup \mathfrak{p}_n,$$

$$(C) \quad g \text{ has no monomial in common with } f.$$

In view of (A) the claim will imply that $f + g$ is in \mathfrak{a} but not in any of the \mathfrak{p}_i , contradiction [this contradiction will complete the proof of (d)].

By Proposition 3 p. 14 there is an $h \in \mathfrak{a}$ such that $h \notin \mathfrak{p}_0 \cup \dots \cup \mathfrak{p}_n$. If j is in N_{n+1} , then $g := x_j h$ will satisfy (B) and (C). This proves the claim, and completes the proof of (d).

Proof of (e). It only remains to prove Statement (e), which we recall:

(e) For each i the local ring $B_{S^{-1}\mathfrak{p}_i}$ is noetherian.

We change the setting as follows [letting again K be a field]. Let x_1, \dots, x_n and y_1, y_2, \dots be indeterminates, denote by x the sequence (x_1, \dots, x_n) of indeterminates, and by y the sequence (y_1, y_2, \dots) of indeterminates. Let $K[x, y]$ be the polynomial K -algebra over all the above indeterminates. We claim

(g) The equality

$$K[x, y]_{(x_1, \dots, x_n)} = \left((K(y))[x] \right)_{(x_1, \dots, x_n)}$$

holds as an equality between subrings of $K(x, y)$. In particular this ring is noetherian.

We leave the proof of (g) to the reader. Clearly (g) implies (e).

12.2.5 Page 126, Exercise 11.6

Statement. Let A be a ring [not necessarily Noetherian]. Prove that

$$1 + \dim A \leq \dim A[x] \leq 1 + 2 \dim A.$$

Solution. We denote the Krull dimension of any ring A by $\dim A$ and the height of any prime ideal \mathfrak{p} by $h(\mathfrak{p})$. If we have a ring morphism $A \rightarrow B$ and a prime ideal \mathfrak{q}_i of B , we write \mathfrak{p}_i^c for $(\mathfrak{p}_i)^c$.

Proof of the inequality $\dim A[x] \geq 1 + \dim A$: If $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ is a chain of prime ideals in A , then

$$\mathfrak{p}_0[x] \subsetneq \dots \subsetneq \mathfrak{p}_n[x] \subsetneq \mathfrak{p}_n + (x)$$

is chain of prime ideals in $A[x]$ [see Exercise 4.7 p. 55 of the book — see Section 5.3.4 p. 83 — and Lemma 47 p. 84 above]. \square

Proof of the inequality $\dim A[x] \leq 1 + 2 \dim A$: This inequality follows immediately from Lemma 47 p. 84 above.

12.2.6 Page 126, Exercise 11.7

Statement. Let A be a Noetherian ring. Then

$$\dim A[x] = 1 + \dim A, \quad (68)$$

and hence, by induction on n ,

$$\dim A[x_1, \dots, x_n] = n + \dim A.$$

Hint. [This is the hint given in the book.] Let \mathfrak{p} be a prime ideal of height m in A . Then there exist $a_1, \dots, a_m \in \mathfrak{p}$ such that \mathfrak{p} is a minimal prime ideal belonging to the ideal $\mathfrak{a} = (a_1, \dots, a_m)$. By Exercise 7 of Chapter 4, the ideal $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$ and therefore the height $\mathfrak{p}[x]$ is $\leq m$. On the other hand, a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_m = \mathfrak{p}$$

gives rise to a chain

$$\mathfrak{p}_0[x] \subsetneq \mathfrak{p}_1[x] \subsetneq \dots \subsetneq \mathfrak{p}_m[x] = \mathfrak{p}[x],$$

hence the height of \mathfrak{p} is $\geq m$. Hence the height of $\mathfrak{p}[x]$ is equal to the height of \mathfrak{p} . Now use the argument of Exercise 6.

Solution. As in the previous section, we denote the Krull dimension of any ring A by $\dim A$ and the height of any prime ideal \mathfrak{p} by $h(\mathfrak{p})$. We follow the hint of the book, and we shall use the following obvious fact:

Note 94. If $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$ is a chain of prime ideals, then $h(\mathfrak{p}_n) \geq n + h(\mathfrak{p}_0)$.

The existence of a_1, \dots, a_m results from Lemma 91 p. 162 above. The phrase “and therefore the height $\mathfrak{p}[x]$ is $\leq m$ ” follows from Corollary 11.16 page 121 of the book. As explained in the hint, this implies

$$h(\mathfrak{p}[x]) = h(\mathfrak{p}). \quad (69)$$

In view of Section 12.2.5 above p. 166, (68) reduces to $\dim A[x] \leq 1 + \dim A$. We can assume $\dim A < \infty$. Set $n := \dim A$ and let $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_{n+2}$ be a chain of prime ideals in $A[x]$ of length $n + 2$.

It suffices to derive a contradiction from this assumption.

The above chain contracting to a chain of length at most n in A , there are indices i such that $\mathfrak{p}_i^c = \mathfrak{p}_{i+1}^c$. Let i be the largest such index. We have $\mathfrak{p}_i = \mathfrak{p}_i^c[x]$ by Lemma 47 p. 84 above, and

$$h(\mathfrak{p}_i^c) = h(\mathfrak{p}_i^c[x]) = h(\mathfrak{p}_i) \geq i, \quad (70)$$

the first equality following from (69) and the inequality following from Note 94. Then we get

$$h(\mathfrak{p}_{n+2}^c) \geq n + 1 - i + h(\mathfrak{p}_i^c) \geq n + 1 - i + i = n + 1,$$

the inequalities following respectively from Note 94 and Display (70). This contradicts the definition of n , proving (68).