Symmetries in Quantum Mechanics: from Klein-Gordon Equation to Higgs Mechanism

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Symmetry has guided the development of advances in physics, from early quantum theory to modern field theories. The challenge is to understand the relationship of quantum mechanics to relativistic invariance and ultimately to apply it to particle mass generation. By tracing the historical developments from the Klein-Gordon equation to spontaneous symmetry breaking, the unifying role of symmetry among the different formulations is demonstrated. First, the 0-spin relativistic Klein-Gordon wave equation is derived. Second, the Dirac equation is derived from it. Third, by introducing gauge principles, the Higgs mechanism is naturally arrived at, explaining massive gauge bosons in the Standard Model. The paper shows the central role of symmetry both from a historical perspective and the implication in modern frameworks.

Keywords: Symmetry, Klein-Gordon Equation, Dirac Equation, Higgs Mechanism, Gauge Theory, Supersymmetry, Spontaneous Symmetry Breaking.

I. INTRODUCTION

A. Introduction: breve overview of the paper through a concise historical perspective

Symmetry is probably one of the main mechanisms in all fields of physics, which provides a unifying point between diverse phenomena. The aim of this paper is to trace the evolution of relativistic quantum mechanics from early quantum theory, to the Klein-Gordon equation as the first relativistic wave equation for scalar fields and its derivation in the Dirac equation. Subsequently, a tour of the use of symmetry in modern developments is given, particularly culminating with the Higgs mechanism for spontaneous symmetry breaking. In this way, the emergence of quantum mechanics is connected to the framework of the Standard Model, highlighting the importance of symmetry in the description of fundamental interactions, particle mass generation, and, in general, of modern physics.

B. Historical development leading to the Klein-Gordon equation

1. *Early Quantum Mechanics*

The birth of quantum mechanics arose at the beginning of the 20th century as a break from classical physics. Several works led to the definition of the framework of quantum mechanics. Some of the most important have been the quantization of blackbody radiation by Max Planck, the photoelectric effect by Albert Einstein and the atomic model by Niels Bohr. The initial quantum theory quickly influenced the need to formulate wave mechanics that Erwin Schrödinger formulated, as well as matrix mechanics by Werner Heisenberg. Heisenberg ended up composing the first definition of modern quantum mechanics.

2. *Relativistic Quantum Equation*

It was found that the work formulated in the Schrödinger equation contained the lack of describing the dynamics of quantum particles at speeds close to that of light. Schrödinger showed that the energy levels of the hydrogen atom as well as other quantum systems can be described, but these do not comply with Lorentz invariance. Therefore, the Schrödinger equation is non-relativistic because in special relativity the equations remain invariant under Lorentz transformations.

3. *Derivation of the Klein-Gordon Equation*

The first formal description of a relativistic wave equation for the electron was proposed independently by Oskar Klein and Walter Gordon. Initially, one starts from the relativistic dispersion relation for a free particle of mass m

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^2. \tag{1}$$

In quantum theory, the substitution of energy and momentum operators is applied as follows

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = -i\hbar \nabla. \tag{2}$$

From Equation 2, the relativistic energy-momentum relation is deduced in differential equation form

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2 c^2 \nabla^2 + m^2 c^2. \tag{3}$$

Ordering the terms gives the Klein-Gordon equation in its standard form

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \Psi(\mathbf{r}, t) = 0. \tag{4}$$

Alternatively, using the four-dimensional spacetime notation

$$\left(\partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\Psi = 0. \tag{5}$$

Where $\partial_\mu = \frac{\partial}{\partial x^\mu} ex^\mu = (ct, \mathbf{r})$ with the sign metric $(+, -, -, -)$ corresponding to the convention used in the spacetime description in special relativity and field theory. It is, therefore, Minkowski space where it is modelled as a four-dimensional manifold.

4. Significance and Limitation of the Klein-Gordon Equation

The Klein-Gordon equation is the correct description of the motion of free spin-0 fields and particles. It follows Lorentz symmetry, and only the Higgs boson has been experimentally verified as a spin-0 particle [1].

Initially, it was considered to describe the electron; however, it was shown to be adequate for scalar particles such as the pion, which is a meson. In fact, the Lagrangian of the equation $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$ serves for composite systems such as the meson.

It is not the end, since for the electron it was necessary to develop the Dirac equation that incorporates the spin-$\frac{1}{2}$ of the particle [2]. In addition, difficulties were encountered in predicting probability densities, as they give negative values under the interpretation of the Schrödinger equation. A reinterpretation under the formalism of second quantization was therefore necessary in the context of Quantum Field Theory (QFT), where probabilities are understood in terms of creation and annihilation of particles.

C. Theoretical Framework: Symmetries and Conservation Laws

A system is invariant if, under a transformation, its equations of motion or action do not change. Symmetries in physics are understood as the invariance of a system under transformations or classes of transformations. The best known symmetries are translations and rotations in time or internal transformations such as global phase change. Symmetry is a unifying description of phenomena. Covariance, on the other hand, maintains the appearance of the form of the original equation once a transformation has been applied.

A group formalizes the idea of a group of transformations with associative binary operation and the existence of neutral and inverse elements. Consider the rotation $90° + 180° = 270°$ on a plane $\mathbb{R}^2$. In this rotation, there is the null rotation that acts as identity, each rotation has its inverse, the composition is associative and the set of transformations is called closure.

In relativistic physics, Minkowski spacetime symmetry is based on the invariance of the object [3], in this case, the metric $\eta_{\mu\nu}$ under the group symmetry called the Poincaré group.

1. Lie Groups and Lie Algebra

A Lie group $(G, \circ)$ is a continuous group and hence a differentiable manifold, where the group operation and the inverse operation are smooth. The properties are

- Closure: $\forall\, g,\, h \in G$, the product $g \circ h \in G$.
- Existence of identity element: $\exists\, e \in G : e \circ g = g \circ e = g$.
- Existence of inverse: $\forall g \in G, \quad \exists\, g^{-1}$ with $g \circ g^{-1} = g^{-1} \circ g = e$.
- Associativity: $(g \circ h) \circ k = g \circ (h \circ k) \quad \forall g,\, h\, k \in G$.

A Lie algebra $\mathfrak{g}$ associated with G is a vector space provided with a Lie bracket $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies

- Bilinearity: $[aX + bY,\, Z] = a[X,\, Z] + b[Y,\, Z]$, $[Z,\, aX + bY] = a[Z,\, X] + b[Z,\, Y]$, with scalars a, b.
- Anticommutativity: $[X,\, Y] = -[Y,\, X]$.
- Jacobi identity: $[X,\, [Y,\, Z]] + [Y,\, [Z,\, X]] + [Z,\, [X,\, Y]] = 0$.

There is a one-to-one correspondence between Lie algebras and connected and simply connected Lie groups, i.e., to each Lie algebra $\mathfrak{g}$ there corresponds a simply connected Lie group called a distinguished covering group —except coverings. $SO(2)$ is associated with the circle S^1, but is isomorphic to $U(1)$. In $SO(3)$ it is associated with the sphere S^2 of rotations, but its double covering group is $SU(2)$, which topologically corresponds to S^3.

1. $U(1)$ Described by complex 1×1 unitary matrices. Phase symmetry in quantum electrodynamics. 2-D rotations.
2. $SU(2)$ Set of 2×2 unitary matrices with $\det(SU(2)) = 1$. Symmetry in the weak interaction or quantum spin. Every Hermitian matrix traceless 2×2 can be expanded into linear combinations of Pauli matrices. 3-D rotations.
3. $SU(3)$ Complex 3×3 complex unitary matrices with $\det(SU(2)) = 1$. Symmetry in the strong interaction (quantum chromodynamics).
4. $SO(3,1)$ and $ISO(3,1)$ are the Lorentz group and the Poincaré group, which describe the relativistic structure of the field theory. The Poincaré group arises from the sum of translations and Lorentz transformations in special relativity.

The relationship between the composition of exponentials in the group and the bracket in the algebra is formalized by the Backer-Campbell-Hausdorff (BCH) formula [4]

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\cdots},$$

where X and Y belongs to $\mathfrak{g}$. The BCH performs an expansion in terms of commutators and nested commutators that make explicit the non-commutativity of the Lie operation. Thus, it describes how the product of exponentials in the group translates into a combination of elements of the algebra with higher order corrections to the sum $X+Y$.

A representation G on a vector space V assigns to each element $g \in G$ a linear operator $R(g)$ on V. In terms of algebra, each generator $T^a \in \mathfrak{g}$ is associated with a matrix $\rho(T^a)$. Furthermore, a Casimir operator is an element of the universal envelope of $\mathfrak{g}$ which commutes with all generators. It typically classifies irreducible representations, such as the total spin operator in $SU(2)$. Cartan decompositions and Cartan algebras are essential to classify representations of semisimple groups, such as $SU(3)$. In particle physics it allows to identify spins, charges and families of particles as irreducible representations under the gauge group of the theory and/or under the Poincaré group.

2. *Poincaré Group*

The Poincaré Group is denoted by $\mathcal{P} = \mathrm{ISO}(3,\,1) = SO(3,\,1) \ltimes \mathbb{R}^{3,\,1}$, where $SO(3,\,1)$ is a 4×4 matrix that left invariant the $\eta_{\mu\nu}$, and $\mathbb{R}^{3,\,1}$ represents the translations in space-time. $SO(3,\,1)$ have spatial rotation with three J^i generators and boosts with three generators K^i. Fulfilling the Lorentz algebra. $\mathbb{R}^{3,\,1}$ introduce four translations generators P^μ. Semidirect composition $\ltimes$ reflects that translations and Lorentz transformations do not commute in general. The Poincaré algebra denoted by $\mathfrak{p}$ is characterized by

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= i\big(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho}\big), \\ [P^\mu, P^\nu] &= 0, \\ [M^{\mu\nu}, P^\rho] &= i\big(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu\big), \end{aligned} \tag{6}$$

where $M^{\mu\nu}$ generate rotations and boosts, and P^μ generate translations. In relativistic theory, mass and spin arise from the classification of elementary particles which are identified by irreducible representations of $\mathfrak{p}$, described by two Casimir invariants. First, $P^\mu P_\mu = m^2$. Second, the Pauli-Lubanski four-vector W^μ and the spin s in the invariant $W^\mu W_\mu = -m^2\, s(s+1)$.

3. *Noether's Theorem*

The Principle of Least Action postulates that the path followed by a physical system makes the action S stationary. In a system of particles with generalized coordinates $q_i(t)$, the action is defined as

$$S[q(t)] = \int_{t1}^{t2} \mathcal{L}(q_i,\, \dot{q}_i,\, t)\, dt \tag{7}$$

In field theory it is generalized to

$$S[\Phi] = \int d^4\, x \mathscr{L}(\Phi,\, \partial_\mu \Phi,\, x^\mu), \tag{8}$$

where $\mathscr{L}$ is the Lagrangian density and Φ represents one of several fields, e.g. Φ^i for multiplets. The variational condition $\delta S = 0$ leads to the Euler-Lagrange equations, which describe the dynamics.

For one-particle mechanics, consider a variation as $q(t) \to q(t) + \epsilon(t)$ with $\epsilon(t_1) = \epsilon(t_2) = 0$. Expand as

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} \right) dt. \tag{9}$$

Integrating by parts, the term with $\delta\dot{q}$

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}\, dt = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q\, dt. \tag{10}$$

Since $\delta q(t_1) = \delta q(t_2) = 0$, the boundary term vanishes. Then,

$$\delta S = \int_{t_1}^{t_2} dt\, \delta q \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right). \tag{11}$$

By imposing $\delta S = 0$ for arbitrary variations δq, it is obtained the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0. \tag{12}$$

Similarly, for a field $\Phi(\mathbf{x}, t)$, the action is written

$$S[\Phi] = \int d^4x\, \mathcal{L}(\Phi, \partial_\mu \Phi). \tag{13}$$

The variation $\Phi \to \Phi + \delta\Phi$ gives

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi}\delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\delta(\partial_\mu \Phi)\right]. \tag{14}$$

Integrate the second term by parts to obtain

$$\delta S = \int d^4x\, \delta\Phi \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\right]\right). \tag{15}$$

The condition $\delta S = 0$ for any $\delta\Phi$ implies the Euler-Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) = 0. \tag{16}$$

Noether's Theorem ensures that every continuous symmetry of the action $S[\phi]$ is related to a conservation law. Let be an infinitesimal transformation of the field, $\Phi \to \Phi + \delta\Phi$, which leaves the action $\delta S = 0$ invariant. Then it is shown that there exists a Noether current j^μ such that $\partial_\mu j^\mu = 0$, whose associated charges $Q = \int d^3x\, j^0$ are constant over time. Noether's Theorem is important in the construction of field theories and in the classifications of dynamical invariants, among other current applications. For example, a translation in time implies conservation of energy (Hamiltonian), a translation in space implies conservation of linear momentum, rotations imply conservation of angular momentum, and global phase transformations $U(1)$ imply conservation of charge (e.g. leptonic, baryonic, etc.).

D. Development of Klein-Gordon Equation and Symmetry

Consider the application of the above formalism to the formal derivation of the Klein-Gordon equation introduced into the Equations 4, 5.

1. *Action Principle for a Free Scalar Field*

Consider a scalar field $\Phi(x)$ (with $x^\mu = (t, \mathbf{x})$) which is transformed according to the representation $(0,0)$ of the Lorentz group—true scalar. The minimum Lagrangian density that respects Lorentz invariance and describes a degree of freedom with mass m is postulated as

$$\mathscr{L} = \tfrac{1}{2}\Big[\partial_\mu \Phi\, \partial^\mu \Phi - m^2\, \Phi^2\Big]. \tag{17}$$

Here, $\partial_\mu \Phi\, \partial^\mu \Phi$ is the contraction $\eta^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi$. The action remains

$$S[\Phi] = \int d^4x\; \tfrac{1}{2}\big[\partial_\mu \Phi\, \partial^\mu \Phi - m^2\, \Phi^2\big]. \tag{18}$$

2. *Derivation of the Klein-Gordon Equation*

Applying the Euler-Lagrange equation for fields,

$$\frac{\partial \mathscr{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \right) = 0, \tag{19}$$

it follows

1. $\frac{\partial \mathscr{L}}{\partial \Phi}$:

$$\frac{\partial}{\partial \Phi}\big[\tfrac{1}{2}(\partial_\mu \Phi\, \partial^\mu \Phi - m^2\, \Phi^2)\big] = -m^2\, \Phi. \tag{20}$$

2. $\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}$:

$$\frac{\partial}{\partial(\partial_\mu \Phi)}\big[\tfrac{1}{2}\, \partial_\nu \Phi\, \partial^\nu \Phi\big] = \partial^\mu \Phi. \tag{21}$$

3. The d'Alembert operator[1]:

$$\partial_\mu(\partial^\mu \Phi) \equiv \Box\, \Phi = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi. \tag{22}$$

Combining the results

$$-m^2\, \Phi - \Box\, \Phi = 0 \quad \Longrightarrow \quad (\Box + m^2)\, \Phi = 0, \tag{23}$$

The Klein-Gordon equation is obtained

$$\left(\tfrac{1}{c^2}\, \tfrac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \Phi(\mathbf{x}, t) = 0. \tag{24}$$

Note that $\mathscr{L}$ is a Lorentz-low scalar $(0,0)$ representation, which guarantees the covariant form of the equation of motion. Translation invariance gives rise to conservation of the energy-momentum tensor. Finally, considering that Φ is complex, the eventual global phase invariance $U(1)$ implies Noether charge conservation. The Klein-Gordon equation arises as a fundamental description of free and relativistic spin-0 fields.

E. Beyond Klein-Gordon Equation

1. *Deduction of the Dirac Equation from the Klein-Gordon Equation*

a. Motivation.

1. The $\frac{1}{2}$-spin cannot be described by the Klein-Gordon equation.
2. Probability densities can become negative if the Schrödinger interpretation is applied to the Klein-Gordon equation.
3. The Dirac equation is linear in the ∂_μ derivatives, solving previous problems and predicting antiparticles.

The Klein-Gordon Equation 24, in natural units ($c = 1$) for simplicity, can be written as

$$(\Box + m^2)\, \Phi(x) = (\partial_\mu \partial^\mu + m^2)\, \Phi(x) = 0. \tag{25}$$

This equation involves a quadratic relationship in derivative operators. Let a linear operator $\mathcal{D}$ such that

[1] It is the fundamental relativistic operator in flat Minkowski spacetime. The name comes from the non-relativistic limit of the classical d'Alembert equation.

$$\mathcal{D}\,\Psi(x) = 0 \quad \Longrightarrow \quad \mathcal{D}^2\,\Psi(x) \equiv (\Box{+}m^2)\,\Psi(x) = 0. \quad (26)$$

If $\mathcal{D}$ is linear in ∂_μ, it can be expressed as $\mathcal{D} = i\,\Gamma^\mu\,\partial_\mu - m$, where Γ^μ are objects —matrices— to be determined that satisfy a specific condition to reproduce $\Box + m^2$.

b. Condition on Γ^μ To ensure $\mathcal{D}^2 = \Box + m^2$, expand

$$\mathcal{D}^2 = (i\,\Gamma^\mu\,\partial_\mu - m)(i\,\Gamma^\nu\,\partial_\nu - m). \quad (27)$$

1. Linear term in derivatives

$$i\,\Gamma^\mu\,\partial_\mu(i\,\Gamma^\nu\,\partial_\nu) = -\Gamma^\mu\,\Gamma^\nu\,(\partial_\mu\partial_\nu). \quad (28)$$

2. Since $\partial_\mu\partial_\nu$ is commutative, i.e. $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, while $\Gamma^\mu\Gamma^\nu$ need not commute, the symmetric part must match $\eta^{\mu\nu}\,\partial_\mu\partial_\nu = \Box$. This requires $\{\Gamma^\mu, \Gamma^\nu\} = 2\,\eta^{\mu\nu}$.

3. The cross terms involving m and ∂_μ cancel, also the $m^2 = (-m)(-m)$ ensures consistency.

Thus, the necessary condition for $\mathcal{D}^2\Psi = (\Box + m^2)\Psi$ is

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\,\eta^{\mu\nu}. \quad (29)$$

2. Gamma Matrices

The solution to the Equation 29 is provided by the so-called gamma matrices γ^μ, which are 4×4 matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}$. Note $\Gamma^\mu = \gamma^\mu$; consequently, the Dirac operator becomes

$$\mathcal{D} = i\,\gamma^\mu\,\partial_\mu - m. \quad (30)$$

$\Psi(x)$ must be a spinor with four components to ensure the action of γ^μ is well-defined, which is known as Dirac spinors.

3. Lorentz-Invariant Lagrangian

The linearized operator, Equation 30, allows the construction of the Dirac Lagrangian. The conjugate Dirac spinor $\bar{\Psi} = \Psi^\dagger\gamma^0$ facilitates Lorentz-invariant combinations

$$\mathscr{L}_{\text{Dirac}} = \bar{\Psi}\big(i\,\gamma^\mu\,\partial_\mu - m\big)\,\Psi. \quad (31)$$

Note that the kinetic term, $\bar{\Psi}i\gamma^\mu\partial_\mu\Psi$, is first order in ∂_μ and Lorentz scalar under spinor transformations. Moreover, the mass term $\bar{\Psi}\Psi$ is a Lorentz scalar.

4. Euler-Lagrange Equations for Spinor Fields

Varying the action

$$S[\Psi, \bar{\Psi}] = \int d^4x\;\mathscr{L}_{\text{Dirac}}, \quad (32)$$

regarding $\bar{\Psi}$ (keeping $\delta\Psi = 0$) yields

$$\frac{\partial\mathscr{L}}{\partial\bar{\Psi}} - \partial_\mu\Big(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\Psi})}\Big) = 0. \quad (33)$$

The resulting equation is the so-called Dirac equation

$$(i\,\gamma^\mu\,\partial_\mu - m)\,\Psi = 0. \quad (34)$$

In the Klein-Gordon Equation 24, the second-order in derivatives, $(\Box + m^2)\,\Phi = 0$, describes scalar fields of spin 0, while in the Dirac Equation 34, using the first-order in derivatives and squaring the Dirac operator, Equation 30, it is demonstrated that the Dirac equation is linear extension adapted to $\frac{1}{2}$-spin.

5. Lorentz Covariance, Gauge Fields and Larger Models

1. Both equations (Eq. 24, 34) retain their form in any inertial reference frame. Since Lagrangians are constructed as scalars or quasi-scalars under spinor transformations as opposed to Lorentz transformations, this guarantees relativistic invariance of the equations of motion[2].

2. The fields of 1-spin or gauge fields describe fundamental interactions by means of vector fields called gauge bosons[3], which transform according to the Lorentz group representation $(\frac{1}{2}, \frac{1}{2})$. They are usually denoted as $A_\mu(x)$ (electrodynamics), $W^a_\mu(x)$ (SU(2) for the weak interaction) or $G^a_\mu(x)$ (SU(3) for the strong interaction), and couple to Dirac fermions via the *covariant derivative*

$$\partial_\mu \;\longrightarrow\; D_\mu \;=\; \partial_\mu \;-\; i\,g\,A^a_\mu\,T^a, \quad (35)$$

where T^a are the gauge symmetry generators and g the coupling constant.[4] Each type of gauge field corresponds to a local Lie group whose algebra dictates the interactions.

[2] The Lorentz covariance requirement implies that the indices μ, ν rise and fall with the Minkowski metric $\eta_{\mu\nu}$.

[3] U(1), SU(2), SU(3)

[4] See, for example, the construction of quantum electrodynamics (QED) with local $U(1)$ symmetry, or the electroweak theory with $SU(2)_L \times U(1)_Y$ symmetry.

3. Starting from gauge theories, there is an extended model called the Standard Model of particle physics. It combines Dirac fields (fermions) with quarks and leptons each with spin-$\frac{1}{2}$, gauge fields (bosons) with photon (group $U(1)$), bosons $W^{\pm}, Z^0$ (group $SU(2)\times U(1)$) and gluons (group $SU(3)$), and Higgs field which is a complex scalar that spontaneously breaks electroweak symmetry (SSB) and produces the generation of masses for bosons $W^{\pm}, Z^0$ and fermions. For a representation of the SSB, refer to the Figure 1 in the Appendix A.

   The Standard Model of physics is a unifying theory ($SU(2)$ and $U(1)$) that generalizes the formalism of the Klein-Gordon equation, i.e. scalar fields, the Dirac equation, i.e. spinor fields and gauge fields i.e. 1-spin. All this, supported by the principle of local gauge invariance and Lorentz invariance.

F. Symmetry Breaking and the Higgs Mechanism

1. *Spontaneous Symmetry Breaking and the Higgs Mechanism*

From this point forward, a derivation is made from global symmetry to local symmetry —gauge.

- Global $U(1)$: If a Dirac field ψ undergoes the global transformation, $\psi \longrightarrow e^{i\alpha}\,\psi, \quad \alpha = \text{constant}$, the free Lagrangian $\mathcal{L} = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - m\,\bar{\psi}\psi$ is invariant, and Noether's theorem guarantees charge conservation.

- Local $U(1)$: To allow the phase α to depend on position $\alpha(x)$, one must introduce a gauge field A_μ and the covariant derivative $D_\mu = \partial_\mu - i\,g\,A_\mu$. The Lagrangian (quantum electrodynamics, QED) then becomes

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\left(i\,\gamma^\mu D_\mu - m\right)\psi - \tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{36}$$

  This is the simplest Abelian example of local gauge symmetry.

One is interested in extending SU(2) to define the gauge boson structure. One starts with the locally SU(2)-invariant Lagrangian. To describe two fermionic components —e.g., a doublet $\psi = (\psi_1,\,\psi_2)^T$—, one introduces the non-abelian gauge symmetry SU(2). The generators of the algebra are $T_i = \frac{1}{2}\sigma_i$, where the σ_i are the Pauli matrices. This requires

1. Three gauge fields $\mathcal{W}_\mu^i$ $(i = 1, 2, 3)$ combined into a matrix $\mathcal{W}_\mu = \mathcal{W}_\mu^i\,T_i = \mathcal{W}_\mu^i\,\frac{\sigma_i}{2}$.

2. Covariant derivative $D^\mu = \partial^\mu - i\,g\,\mathcal{W}^\mu$, where g is the coupling constant.

3. Curvature field —gauge kinetic term

$$\mathcal{W}^{\mu\nu} = \partial^\mu\mathcal{W}^\nu - \partial^\nu\mathcal{W}^\mu - i\,g\left[\mathcal{W}^\mu,\,\mathcal{W}^\nu\right].$$

   Expanding in components,

$$\mathcal{W}_i^{\mu\nu} = \partial^\mu\mathcal{W}_i^\nu - \partial^\nu\mathcal{W}_i^\mu + g\,\epsilon_{ijk}\,\mathcal{W}_j^\mu\,\mathcal{W}_k^\nu.$$

The SU(2)-invariant Lagrangian with a fermionic doublet becomes

$$\mathcal{L}_{\mathrm{SU(2)}} = i\,\bar{\psi}\,\gamma_\mu\,D^\mu\psi - \frac{1}{4}\,\mathrm{Tr}\big(\mathcal{W}_{\mu\nu}\,\mathcal{W}^{\mu\nu}\big). \tag{37}$$

There is no mass term for $\mathcal{W}_\mu^i$. A term like $m^2\mathcal{W}_\mu\mathcal{W}^\mu$ explicitly breaks SU(2) symmetry. Similarly, for fermions, introducing a mass term $m\,\bar{\psi}\psi$ requires ψ to be a proper doublet —or multiplet. This suggests that a spontaneous symmetry braking, or condensate of charge, is needed to give mass to some fields.

The Electroweak Model combines $\mathrm{SU(2)}_L$ with an additional $\mathrm{U(1)}_Y$ —hypercharge. Finally

1. Four bosons for SU(2) $(\mathcal{W}_\mu^1, \mathcal{W}_\mu^2, \mathcal{W}_\mu^3)$ plus one for $U(1)_Y$ (B_μ) compose the gauge fields.

2. The covariant derivative

$$D_\mu = \partial_\mu - i\,g\,\frac{\sigma^i}{2}\,\mathcal{W}_\mu^i - i\,g'\,\frac{Y}{2}\,B_\mu, \tag{38}$$

   where Y is the hypercharge, and g, g' are the coupling constants.

3. The gauge Lagrangian

$$\mathcal{L}_{\mathrm{SU(2)}\times U(1)} = i\,\bar{\psi}\,\gamma^\mu\,D_\mu\psi - \tfrac{1}{4}\,\mathrm{Tr}(\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}) - \tfrac{1}{4}\,B_{\mu\nu}B^{\mu\nu}. \tag{39}$$

   With $\mathcal{W}_{\mu\nu}$ and $B_{\mu\nu}$ as field tensors for SU(2) and $U(1)_Y$, respectively.

4. Breaking symmetry yields 3 massive bosons and 1 massless boson —photon.

5. From the unbroken global subgroup of SU(2), sometimes called as weak isospin, and the local remnant $U(1)$, one can derive conserved quantities via Noether's theorem. The weak isospin (T_i) is associated with the SU(2) generators, and after the breaking, the SU(2) gauge bosons reorganize into

$$W^{\pm},\; Z,\; \text{and}\; A_\mu \quad \text{(photon)}. \tag{40}$$

   Hence, the appearance of massive W and Z bosons is directly linked to the Higgs mechanism, while the photon remains massless, corresponding to the unbroken $U(1)$ of electromagnetism.

G. Symmetries as The Foundation of Modern Physics

1. *Role of the Symmetry in Unification Forces and SUSY*

a. Grand Unified Theories (GUTs). A central idea in the quest for unification is that the Standard Model

Group, $G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$, may be embedded as a subgroup of a larger gauge group $\mathcal{G}_{\mathrm{GUT}}$, in which quarks, leptons, and gauge bosons fit together in more symmetric multiplets.

A paradigmatic case is $SU(5)$, whose adjoint representation (**24**) decomposes into

$$\mathbf{24} \longrightarrow (\mathbf{8},\,\mathbf{1})_0 \,\oplus\, (\mathbf{1},\,\mathbf{3})_0 \,\oplus\, (\mathbf{1},\,\mathbf{1})_0 \,\oplus\, (\mathbf{3},\,\mathbf{2})_{\pm\frac{5}{6}} \,\oplus\, \ldots \tag{41}$$

which correspond, schematically, to the gluons (in $\mathrm{SU}(3)_C$), the weak bosons (W^i in $\mathrm{SU}(2)_L$), the hypercharge boson B_μ and other X, Y bosons that can induce processes such as proton decay.

On the one hand, the hypercharge Y and the colour generators ($T^a_{\mathrm{SU}(3)}$) are found as subalgebras of the algebra $\mathfrak{su}(5)$. This allows to explain the charge quantization naturally. Since quarks and leptons appear in unified multiplets —$\bar{\mathbf{5}}$ and $\mathbf{10}$—, guaranteeing relations between the electric charges. On the other hand, at very high energies ($\sim 10^{15} - 10^{16}\,\mathrm{GeV}$) [5], the coupling constants of the Standard Model (α_1, α_2, α_3) could be unified in a single coupling g_{GUT}. Although most GUTs predict proton decay via heavy gauge bosons (X, Y), so far, negative experimental limits impose very severe restrictions [6] on the unification scale or structure of each model.

In sum, the higher symmetry $\mathcal{G}_{\mathrm{GUT}}$ forms a framework with the three gauge interactions ($\mathrm{SU}(3)_C$, $\mathrm{SU}(2)_L$, $U(1)_Y$) where they are formally unified. However, there is no experimental confirmation, mainly because it would require energies higher than current colliders.

b. Supersymmetry (SUSY). Supersymmetry postulates a fermion-boson extension of the Poincaré algebra. Refer to the Appendix B for a more complete and extended view of supersymmetries.

In abstract terms, the SUSY algebra contains supercharge generators Q_α and their Hermitian conjugates $Q^\dagger_\alpha$, satisfying relations of the form

$$\begin{aligned} \{\,Q_\alpha,\, Q^\dagger_\beta\} \;&=\; 2\,(\gamma^\mu\, P_\mu)_{\alpha\beta}, \\ \{\,Q_\alpha,\, Q_\beta\} &= 0, \\ [\,Q_\alpha,\, P^\mu] &= 0, \end{aligned} \tag{42}$$

where P^μ is the generator of translations —momentum— and γ^μ the Dirac matrices —or Weyl spinors, according to the representation.

Each particle of the Standard Model (ψ, A_μ, ϕ) forms a multiplet with its corresponding superpartner (squark, slepton, etc.); e.g., a quark q is accompanied by a squark $\tilde{q}$ with 0-spin, etc. This multiplet is known as a supermultiplet. Furthermore, the theory postulates that at low energies particle-superparticle pairs with the same mass would be observed. So far, this does not match quantum mechanics or QFT. Therefore, it is proposed that SUSY breaks down on a certain scale by increasing the mass of the superpartner, which theoretically would justify the fact that it is not detected in particle accelerators such as the LHC. Among some conclusions of the use of supersymmetry is that theoretically the union of GUT + SUSY leads to the coupling constants unifying with greater accuracy; SUSY can stabilize the Higgs mass against huge radiative corrections and that the Lightest Supersummetric Particle, if considered neutral and stable, can be a candidate for dark matter, e.g. neutralino. Despite the controversies of new theories, the importance lies in the use of symmetry as the main mechanism for the extension of theories, such as the amplification of the Standard Model with SUSY. From Lorentz invariance to a superalgebra that relates bosons and fermions, thus maintaining the gauge principle. Furthermore, by locating supersymmetry, other theories such as supergravity are described. Or, as is also the case, the grand unification that integrates colour (QCD) and electroweak in a single group that allows to explain the relationships between quarks and leptons.

II. DISCUSSION

A demonstration of the importance of symmetry in modern physics has been made, starting with Klein-Gordon and extending to the current applications that arose from this process, such as the standard model with gauge symmetries and spontaneous breaking.

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Appendix A: Figures

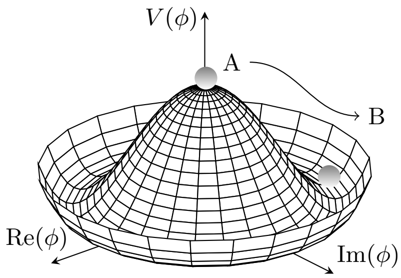

Figure 1. Higgs Potential, illustrating the spontaneous symmetry breaking.

Appendix B: Supersymmetries

1. Bosonic and Fermionic Generators from Poincaré Algebra

Supersymmetry is an extension of the Poincaré algebra $\{P_\mu, J_{\mu\nu}\}$ to include fermionic generators Q_a, $\bar{Q}_a$. The anticommutative bracket of supercharges usually gives rise to a linear combination of the four-momentum operator P_μ. Consider the simplest form in the Dirac or Majorana spinor representation, $\mathcal{N} = 1$ in 4D, of the anticommutator of the fermionic generators

$$\{\, Q_a,\ \bar{Q}_b\} \ = \ (\gamma^\mu)_{ab}\, P_\mu \ + \ (\dots), \tag{B1}$$

where P_μ is the usual bosonic translation momentum, γ^μ are Dirac matrices, and $(\dots)$ represents possible extensions such as central charges. These additional terms depend on the dimension of the spinor and the presence of extensions such as R-symmetries. In supersymmetry transformations one relates, e.g., a spin-0 Klein-Gordon field to a spin-$\frac{1}{2}$ Dirac spinor, similarly for a gauge field and a gaugino.

The Poincaré superalgebra defines

1. $\{Q_a,\, Q_b\} = 0$,
2. $\{Q_a,\, \bar{Q}_b\} \sim P_\mu$,
3. $[Q,\, P_\mu] = 0$,
4. $[Q,\, J_{\mu\nu}]$ describing the spinor transformation of Q under Lorentz.

Supercharges act as square roots of the operator P_μ.

2. Fermionic Subspace and Superfields

The fermionic coordinates θ^a —anticommutative— and the bosonic coordinates $x^\mu \in \mathbb{R}^{3,1}$ are introduced to describe supersymmetries. A superfield $\Phi(x, \theta)$ is a polynomial expansion in θ, of finite degree —for 4D, it corresponds to degree four—, whose coefficients are smooth functions of x^μ. Each order in θ corresponds to component fields of bosonic or fermionic statistics. The elementary anticommutation relations are

$$\left\{\frac{\partial}{\partial\theta^a},\, \theta^b\right\} = \delta^b_a, \quad \theta^a\,\theta^b = -\,\theta^b\,\theta^a \tag{B2}$$

The bosonic operator $\frac{\partial}{\partial x^\mu}$ and the fermionic operator $\frac{\partial}{\partial\theta^a}$ are the basis for defining supersymmetry transformations as supervectors. To represent the SUSY generators as supervectors, a concrete realization of $\{Q_a, \bar{Q}_a\}$ can be made, which is given through operators of the type

$$\begin{aligned} L_a \ &= \ (\gamma^\mu\theta)_a\, \frac{\partial}{\partial x^\mu} \ + \ (\gamma^5\epsilon)_{ab}\, \frac{\partial}{\partial\theta^b}, \\ \bar{L}_a &= (\gamma^5\epsilon)_{ab} L_b, \end{aligned} \tag{B3}$$

where γ^μ are 4D Dirac matrices, $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$, and ϵ is an appropriate antisymmetric matrix, e.g. $\epsilon = i\,\sigma^2$ for Weyl components.

Therefore, the anti-commutators:

$$\{\, L_a, \bar{L}_b\} \ = \ (\gamma^\mu)_{ab}\, \frac{\partial}{\partial x^\mu} \ \iff \ P_\mu = i\frac{\partial}{\partial x^\mu} \tag{B4}$$

3. Supersymmetric Lagrangian and Invariance

a. *Action Invariant and Total Divergence*

A Lagrangian $\mathcal{L}$ is supersymmetric if, under an infinitesimal transformation generated by δ_Q, it varies as

$$\delta_Q\, \mathcal{L} \ = \ \partial_\mu K^\mu, \tag{B5}$$

i.e. a total divergence in spacetime. Then, the functional or the action $S = \int d^4x\, \mathcal{L}$ remains invariant

$$\delta_Q S = \int d^4x\, \partial_\mu K^\mu = 0$$

The Euler-Lagrange equations in a supersymmetric model unite the equations of motion of bosonic and fermionic fields into a single supermultiplet.

b. *Relating Bosons and Fermions*

The expansion of a super field $\Phi(x,\, \theta)$ is given as

$$\Phi(x, \theta) = \phi(x) + \theta\, \psi(x) + \cdots + \theta^n\, F(x), \tag{B6}$$

where $\phi(x)$ is a bosonic field such as the Klein-Gordon scalar, $\psi(x)$ is a Dirac, Majorana or Weyl spinor and $F(x)$ auxiliary or gauge fields. The action is invariant under transformations that exchange $\phi \leftrightarrow \psi$ —boson $\leftrightarrow$ fermion.

4. Matter Field of Yang-Mills

So far, the scalar field of the Klein-Gordon equation, the electron-positron Dirac fermionic field, has been described. For a better understanding of the following sections of the appendix, the Yang-Mills matter field theory is introduced, which generalizes non-Abelian gauge interactions such as SU(2) and SU(3).

The description of fermions as quarks or nucleons under these symmetries use Dirac fields that, given a transformation, do so under non-trivial representations of the gauge groups. This allows the theory to include quarks in QCD SU(3).

5. Relating the Higgs Mechanism to Symmetry Breaking

In section I F one has seen how a doublet Higgs field H is introduced into the standard electroweak theory which, upon acquiring a non-zero vacuum expectation value $\langle \Phi_H \rangle \neq 0$, breaks the symmetry SU(2) $\times$ U(1) and gives mass to the fermions, i.e. the electron mass originates from the Yukawa interaction with the Higgs doublet H: $\mathcal{L}_{\text{Yukawa}} \sim and \bar{\ell} \, \ell \, H$, then $m_e = y \, \langle H \rangle$.

In a supersymmetric model such as the one described, the Higgs is a superfield Φ_H. The vacuum expectation value breaks the gauge symmetry generating mass for both bosons gauge and fermions. The Yukawa couplings extends to a superpotential $W(\Phi)$, where several superfields participate, and the expectation values of the Higgs superfield ϕ allow a soft breaking of supersymmetry [7].

In SUSY, quarks have bosonic superpartners called squarks, and gauge fields called gluons have gauginos —gluinos. This framework doubles the particle spectrum, pairing bosons and fermions.

6. Recapitulation of the supersymmetry relation with the symmetry descriptions of relativistic quantum mechanics

It has been shown that this framework unifies bosonic and fermionic fields by generalizing the symmetric structure used in relativistic quantum mechanics —described from the Klein-Gordon equation to the Dirac equation— and gauge field theory. It has been shown that Lagrangians invariant under boson-fermion transformations can be constructed, the equations of motion are framed in superfields, and the Higgs mechanism and Yukawa couplings are extended to a superpotential scheme.