Bordism Field Theories II

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Abstract

In this paper, I expand upon bordism field theories and their connections with other theories.

1 Introduction

1.1 Overview

In the first of these papers [1], we considered a functor

$$\mathcal{Z} : \texttt{p-Bord}_k \longrightarrow \texttt{TV} \tag{1.1}$$

known as the *TQFT* functor of *field functor*. We focused on some tools developed by Kock and others. In this paper, we will consider more formally the action $\mathcal{Z}_*(x)$ of an object $x \in \mathscr{O}_{\Sigma_{0\longrightarrow 1}}$; that is to say, we will consider a Σ model built on the Frobenius diagrams we encountered in [1]; i.e., diagrams of the form

$$\Sigma_0 \longrightarrow (A \times A) \otimes (A \times A) \longrightarrow \Sigma_1$$

We interpret the $\otimes$ operator here as an invariant of the topological realization of the underlying Frobenius algebra; for physical purposes, we interpret this as a quasi-particle. The gist of this construction is that the cobordism category shares some similarities with the notion of a *prequantum line bundle*, and passing to the category of topological vector spaces is essentially the application of an effective time evolution operator, i.e., $U(x)$ for an observable x in quantum mechanics.

Let us index the various states of x by affixing a subscript; i.e., x_a, x_b, etcetera will denote different states of x corresponding to different expectation values. Suppose x_i denotes a particle found in its initial state;

$$x_i = |x_t\rangle_{t=0}$$

and we will let x_f denote the "final" state of the particle (after performing a new measurement); i.e.,

$$x_f = \langle x_t|_{t=\varepsilon}$$

Then, we identify the time evolution operator $U(x)$ with the Ricci iteration

$$Ric_{g_{i+\varepsilon}}(x) \quad i \sim 0 \quad \varepsilon \sim 1$$

1.1.1 TQFT Perspective on Time Evolution

Suppose we have some expectation value:

$$\mathcal{E}(x) = (x_f|x_i + \lambda) \tag{1.2}$$

where

$$\lambda = \frac{1}{q}|y - x_i|^{d/2} \tag{1.3}$$

What is happening in the topological theory is that there is a bundle $\mathcal{P} : \Sigma_0 \longrightarrow \Sigma_1$, where each of the Σ_i corresponds to a different realization of a ground field k. The bundle $\mathcal{P}$ consists of p many fibers; i.e., for every surjection $\Sigma_0 \longrightarrow d_q$, p counts the number of homeomorphisms. When we look view the original manifold as the present state of a string's worldsheet, then p is the multiplicity of possible future states and d is simply the dimension of the fiber. Hollywood and Khoze [3] give us some remarkable insight here: namely, we can think of the bundle as a branching of the TQFT functor, or in other words a fractionation of an instanton with $\mathcal{N} = 4$ supersymmetry. We will identity this instanton with a topological invariant χ, which the cobordism picks up via a counting of holes; i.e., a counting of the operator $\otimes$.

Let us write

$$\lambda = \sum_{p=0}^{N} \frac{1}{y^{\sqrt{p}}} y^d \tag{1.4}$$

Then, by some simple algebra, by letting $\chi = d - \sqrt{p}$, we obtain:

$$\mathcal{E}(x) = \Big(x_f|x_i + \big((N+1)y\big)^\chi\Big) \tag{1.5}$$

Witten [4-5] tells us that we obtain, in the large N limit, an "infinite tower" of instantonic excitations. If we could understand anything about this tower, then we have a good shot of understanding the nonperturbative string theory and supergravity.

1.2 Concordance with Wolfram's Model

One of the upshots to the bordism field theory (BFT) framework is that it shows reasonable agreement with the Gorard-Wolfram model. In [6], the author described states with an entanglementlike relationship as sharing a common ancestor. Here, this is made manifest by a peculiar duality between modal possibility (i.e., multiplicity of future states) and superposition (i.e., multiplicity of expectation values). For instance, one could treat a d-fold branching of the TQFT functor as entanglement-like superposition between d-many discs D_q with a common ancestor with identical values for either y or χ, but with a different value for N.

Denote by N_{D_q} the value N for a chosen disc D_q. Let us rewrite equation (1.5) as follows:

$$\mathcal{E}(x) = \Big(x_f|x_i + \big(\chi \overset{N_{D_q}}{\curvearrowright} y\big)\Big) \tag{1.6}$$

where the final term is given by dilaton coupling.

Then, we obtain an isomorphism:

$$\mathcal{P}_*(\Sigma_0) = \bigcup_{q=1}^{N} \mathcal{E}(\Sigma_1) \tag{1.7}$$

which generalizes to

$$\mathcal{P}_*(\Sigma_i) = \bigcup_{q=1}^{N} \mathcal{E}(\Sigma_\varepsilon) \tag{1.8}$$

1.3 Recapitulation

Recollection 1.1. *A cobordism is a d-dimensional manifold consisting of $(d-1)$-dimensional submanifolds sharing a common boundary. The boundary is called a* bordism *between the submanifolds, and the submanifolds themselves are said to be* bordant.

Recollection 1.2. *The category $\texttt{p-Bord}_k$ is a the category whose objects are $(p-1)$-dimensional submanifolds and whose morphisms are bordisms $\Sigma_{in} \to \Sigma_{out}$ with ground field k.*

Recollection 1.3. *The Frobenius algebra over k has a topological realization $|\texttt{Frob}_k|$, where for every map $A \to A \times A$, we have that A is a Frobenius algebra, and the product works like a product of Hilbert spaces, i.e., as a vector sum.*

Each A is endowed with the toplogy of a disc and each morphism is a bordism. For a cobordism whose ends are closed, we write k for the end instead of A ad endow it with the topology of the upper half-plane. For two cobordisms $\mathcal{C}_0 : A \to A \times A$ and $\mathcal{C}_1 : A \times A \to A$, we may glue them via the functor $Glu_p : A \to (A \times A) \otimes (A \times) \to A$. In this example $p = 2$, but in general, p will be the multiplicity of the glued discs.

To generalize this a bit, instead of discs, let us use $\mathcal{A}d\mathcal{S}_d$ branes of equivalent dimension. Suppose we have a $d = 5$ cobordism network with supersymmetry. Then, our target manifold will be dual to $\mathbb{R}^6/\mathbb{Z}_p$; specifically, we will have:

$$|\texttt{Frob}_k| = \mathcal{A}d\mathcal{S}_5 \times S^5/\mathbb{Z}_p \tag{1.9}$$

and our projected gauge group will become

$$U(N)^{\otimes D_p} = U(N_{D_1}) \times ... \times U(N_{D_q}) \tag{1.10}$$

which in the infra-red [3] will become, effectively:

$$\sum_{q=1}^{p} SU(N_q) \tag{1.11}$$

2 Bridging BFT and String Theory

The kinship between the BFT model and string theory runs deep in the blood. In BFT, physical processes are described via bordisms between manifolds, whereas in string theory, the worldsheet dynamics of strings are described by conformal field theories (CFTs). This section bridges these two frameworks by interpreting bordisms as worldsheet geometries and identifying concrete correlation functions as observables.

2.1 Correlators

In BFT, a bordism $\mathscr{W} : \Sigma_{in} \to \Sigma_{out}$ represents a cobordant transformation between boundary manifolds. This concept naturally aligns with the string worldsheet. The boundary $\Sigma_{in} \cup \Sigma_{out}$ represents the asymtotic states of a string process (e.g. $|k_1\rangle, |k_2\rangle, ...$), and the bordism $\mathscr{W}$ represents the worldsheet interpolating between these states, potentially with punctures encoding the insertion of operators.

Performing a path integral over a bordism $\mathscr{W}$ in BFT yields the correlation function:

$$\langle \mathscr{O}_1 ... \mathscr{O}_n \rangle_{\mathscr{W}} = \int_{Bord(\mathscr{W})} e^{-S} \prod_{j=1}^{n} \mathscr{O}_j \tag{2.1}$$

where $\mathscr{O}_i$ are local observables associated with points on the bordism, and S is the action functional on $\mathscr{W}$. There is in fact an isomorphism

$$\mathscr{W} \simeq \mathcal{M}_{g,2}$$

such that integrating over a standard cobordism can be interpreted as integrating over the moduli space of genus g surfaces with two punctures. Here, $g \propto \chi$ is simply the level of the Glu functor used to sew the cobordism to its mirror image.

Example 2.1. *A disk-like bordism interpolates between a single boundary and a puncture, analogous to a tree-level scattering amplitude. The boundary states correspond to external string states.*

Example 2.2. *A cylinder bordism represents a one-loop process in string theory, capturing the propagation of a closed string between two boundaries.*

E.g.

$$\mathscr{Z}_{cylinder} = \int_0^\infty d\tau \; Tr(q^{L_0 - c/24}),$$

where $q = e^{2\pi i \tau}$ and L_0 is the Virasoro generator.

Example 2.3. *Consider the two-point function of scalar observables on a genus-g bordism:*

$$\langle \mathscr{O}_1(x_1) \mathscr{O}_2(x_2) \rangle_{\mathscr{W}} = \int_{\mathcal{M}_{g,2}} e^{-S} \mathscr{O}_1(x_1) \mathscr{O}_2(x_2).$$

In string theory, this corresponds to a two-point function of vertex operators:

$$\langle V_1 V_2 \rangle_g = \int_{\mathcal{M}_{g,2}} \langle V_1 | b_k \rangle \langle V_2 | b_k \rangle e^{-S_{CFT}},$$

where $|b_k\rangle$ are intermediate states in the Hilbert space of the CFT.

2.2 Bubbles

Suppose we are given a cobordism whose Frobenius representation is of the form $A \times (A \times A) \otimes (A \times A) \times A$; in other words, this is a cobordism which starts with a single disc, branches into a pair of pants, and is sewn together at the feet with another pair of pants. What are we to make of the space between the legs? That is to say, what physical environment must exist for the *Glu* functor to make any sense?

One possible interpretation is as a physical hole. Witten [7] came across such a hole in the early 1980s when he was studying Kaluza Klein compactifications of $\mathbb{M}^4 \times S^1$, where $\mathbb{M}$ is good ol' Minkowski space. The vacuum instabilities led him to consider absolute spacetime holes in which nothing - no color or charge - exists or could possibly exist. These became known as "end-of-the-world" branes (ETW) branes, or alternatively, "bubbles of nothing." Alarmingly, the expansion of these bubbles seemed unmarred by any other physical process outside of collsion with another bubble. The nucleation rates of these bubbles have been estimated to be [8]:

$$\mathfrak{N}_{ETW} = \exp\Big(- (2\pi^3 R_{KK}^3)\Big)$$

A more recent, and optimist development, within the string landscape, has been to look at the converse - so-called "bubbles of something" (BoS). If we replace equations (75) and (76) from [8] with the appropriate variables, we have that the creation rate of a bubble is:

$$B_{creation} = -B_{decay} \tag{2.2}$$

where

$$B_{decay} = \xi \ell_{\mathscr{W}} \left(\sqrt{\frac{T_{\mathscr{W}}}{T_{\mathscr{W}} - \frac{\xi}{p\pi}}} - 1 \right) \tag{2.3}$$

where

$$\xi = p\pi^{p/2} M_P^{p/2} \tag{2.4}$$

$$T_{\mathscr{W}} = T_p^{p/2} \ell_{\mathscr{W}} \tag{2.5}$$

and

$$\ell_{\mathscr{W}} = \ell_{\mathcal{A}p\mathcal{S}}^{p/2} \tag{2.6}$$

and T_4 is the string tension. Of course, we need to introduce loop-level correction terms to account for the fractionation of the instanton. The rewritten equation reads:

$$qB_{creation}^{New} = \chi \overset{\mathcal{N}_{D_q}}{\curvearrowright} \mathcal{Z}_*^{1/d}(B_{creation}) \tag{2.7}$$

which gives us

$$B_{creation}^{New} = \frac{\Big((N+1) B_{creation} \Big)^{\chi}}{q} \tag{2.8}$$

We apply the following "reality condition" for a BoS:

$$\mathcal{Z}(bubble) = \mathrm{Tr}(\mathscr{O}) \tag{2.9}$$

so that we have

$$\frac{\langle \mathscr{O}_1 ... \mathscr{O}_n \rangle_{\mathscr{W}}}{\mathcal{Z}(bubble)} = \texttt{scalar} \tag{2.10}$$

The more traditional bubbles resemble the bubbles in the $\mathcal{N} = 4$ large N limit of our theory. That is to say, under small N, the growth of a bubble is governed by the kinematics of the surrounding branes. Specifically, the non-asymtotic dynamics of local branes control the growth of the bubble.

The dynamics of the bubble's growth are captured by the Ricci flow, where the metric $g_{\mu\nu}(\varepsilon)$ evolves in discrete time steps, with the Ricci tensor acting as the driving force behind the geometric evolution. Specifically, the change in the metric at each step is given by:

$$\frac{\partial g_{\mu\nu}(\varepsilon)}{\partial \varepsilon} = -2Ric(g_{\mu\nu}(\varepsilon)),$$

where the Ricci tensor encodes the curvature of spacetime. For small N, the interaction between the branes remains local, and their energy distribution can be expressed through the energy-momentum tensor $T_{\mu\nu}$, which sources the Ricci tensor via Einstein's field equations:

$$Ric(g_{\mu\nu}) = 8\pi G T_{\mu\nu}.$$

At each iteration step, the geometry of the surrounding spacetime is updated by applying the Ricci flow equation, with the new metric $g_{\mu\nu}(\varepsilon + \Delta\varepsilon)$ given by:

$$g_{\mu\nu}(\varepsilon + \Delta\varepsilon) = g_{\mu\nu}(\varepsilon) + \Delta\varepsilon Ric(g_{\mu\nu}(\varepsilon)).$$

This iterative process allows the growth of the bubble to be described by local curvature effects, driven by the energy-momentum tensor of the branes. As N becomes large, these local effects merge into a global description, but for small N, the evolution is controlled by the direct interaction of the branes with the surrounding geometry.

A Ricci Iteration

Mathematically, the Ricci iteration is defined as:

$$Ric_{g_{i+\varepsilon}}(x) : (X_{in} \to X_{out})|_x \tag{A.1}$$

where X_{in} and X_{out} represent the initial and final states of a system $X \ni x$, and $g_{i+\varepsilon}$ is the evolving metric under Ricci flow over a small time step ε. The iteration operates in discrete steps, with the system's stability analyzed in the limit as $\varepsilon \to 0$.

Consider a target manifold Σ with $X \sim \Sigma$. The Ricci iteration evolves a metric g according to:

$$g_{i+\varepsilon} \equiv g_i + \varepsilon Ric(g_i) \tag{A.2}$$

where $Ric(g_i)$ is the Ricci curvature of g_i. This leads to a discrete approximation of Ricci flow:

$$\frac{d}{d\varepsilon} g = -2Ric(g) \tag{A.3}$$

For instance, let X be a two dimensional Riemann surface and g its initial metric. If $g_i \to g_\infty$ where $Ric(g_\infty) = 0$, then the system is fully stable; i.e., everywhere Ricci flat, and with constant curvature. In this case, we say that the quantified function is *integrable*.

B References

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