Upside-Down, Mirror Looking and Water Reflection Magic Squares: Orders 3 to 6, **Zenodo**,
January 07, 2025, pp. 1-93, *https://doi.org/10.5281/zenodo.14607070*

Upside-Down, Mirror Looking and Water Reflection Magic Squares: Orders 3 to 6

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For complete work acess the author's web-site links:
https://numbers-magic.com/?p=12610
https://numbers-magic.com/?p=14126

Abstract

There are many ways of representing magic squares with palindromic type entries or composite forms based on pair of Latin squares. Based on palindromic and composite magic squares we have written ***upside-down*** *and/or* ***mirror looking*** *magic squares. By* ***upside-down****, we understand that making* 180^o *rotation still we have a magic square. Applying the* ***upside-down*** *property the numbers 0, 1, 2, 5, 6, 8 and 9 remains the same, where 6 becomes 9 and 9 as 6. In this case, the numbers are written in* ***digital/special fonts****. The* ***mirror looking*** *property is same as* ***horizontal flip****. In this case, the numbers 0, 1, 2, 5 and 8 remains the same, where 2 becomes 5 and 5 as 2. There is one more property, known by* ***vertical flip****. For simplicity, let's call it as* ***water reflection****. In this case, the numbers 0, 1, 2, 3, 5 and 8 remains the same, where 2 becomes 5 and 5 as 2. Thus the numbers 0, 1, 2 5 and 8 are available in all the three property. The numbers those satisfy all the three properties, we call them as* ***universal****. The same is with magic squares. The magic squares formed by the numbers 0, 1, 2, 5 and 8 are known by* ***universal*** *magic squares. Finally, in case of* ***upside-down****, the number 6 becomes 9 and 9 as 6. In case of* ***water reflection****, the number 3 remains the same. In this paper we worked with magic squares of orders 3 to 6, satisfying one of or all the above three properties. Total project brings work on magic squares of order 3 to 25. Some of the results of this project are revised and enlarged versions the author's previous works. For more details refer [4]-[15], [28], [36]-[43].*

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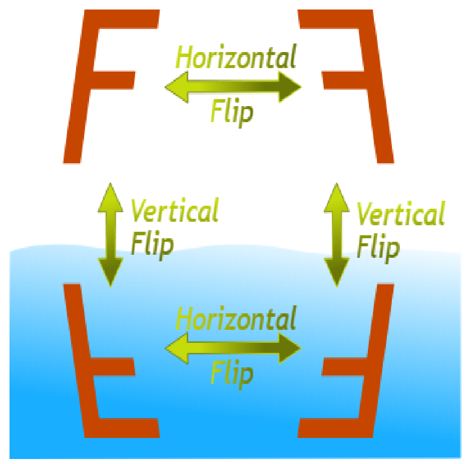
1 Introduction

Magic squares are generally constructed using sequential or consecutive numbers such as $1, 2, \ldots, n^2$, etc. Here in this work, we shall write magic squares in consecutive numbers, and then transforming it palindromic and composite forms. Based on these two representations, the work is extended to **upside-down**, **mirror looking** and **water reflection** forms by using some extra fonts such as **digital/special fonts**. The composite forms are based on Latin squares decompositions. Let's see the above discussion with examples.

Let's consider the following image:



Source: https://www.mathsisfun.com/definitions/vertical-flip.html

From the above image we understand that **horizontal flip** is same as **mirror looking image** and the **vertical flip** is same as **water reflection image**. The same terms are given in **paint brush of microsoft**. Let's see how it works on numbers.

Let's consider following 9 digits written in digital form:

0 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9

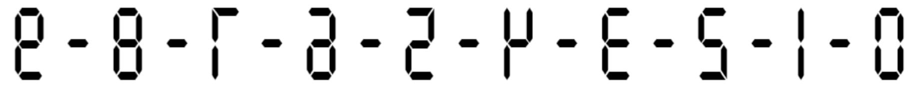
• 180^o Rotation

6 - 8 - L - 9 - 5 - h - E - 2 - 1 - 0

In this case, the readable numbers are 0, 1, 2, 5, 6, 8 and 9 where the number 6 becomes 9 and 9 as 6. Thus the **survival numbers** after 180^o **rotation** are 0, 1, 2, 5, 6, 8 and 9. Sometimes we call them as **upside-down** numbers.

• Mirror Looking

It is same as **horizontal flip** as describe above. In this case, we have



In this case, the readable numbers are 0, 1, 2, 5 and 8, where 2 becomes 5 and 5 as 2. Thus the **survival numbers** after **horizontal flip** are 0, 1, 2, 5 and 8. Sometimes we call them as **mirror-looking** numbers.

• Water Reflection

It is same as **vertical flip** as describe above. In this case, we have

In this case, the readable numbers are 0, 1, 2, 3, 5 and 8, where 2 becomes 5 and 5 as 2. Thus the **survival numbers** after **vertical flip** are 0, 1, 2, 3, 5 and 8. For the first time we call these numbers as **water reflection** or **water reflexive** numbers.

We observe that the numbers 0, 1, 2, 5 and 8 are in all the three situations, i.e., these are **upside-down**, **mirror-looking** and **water reflexive**. We call them as **universal numbers** provided they are written in **digital form**. There left only one number 3. It is only **water reflexive**. While the numbers 6 and 9 are only **upside-down**.

There is a lot of work by author on **upside-down** and **mirror-looking** numbers. This work is concentrated only towards magic squares having **water reflexive** numbers, i.e., 0, 1, 2, 3, 5 and 8. The numbers 0, 1, 2, 5 and 8 are already studied previously. This work brings magic squares of order 3 to 10 specially in number 3 along with 0, 1, 2, 5 and 8.

It is revised and enlarged version of author's one of the first work on **magic squares** started in 2010 specially the one given in [14]. For more details refer [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 27]. This work is divided in different parts for the orders 3 to 25. For more details refer author's work [36]-[43]. This part is for the orders 3 to 6 is revised and enlarged version.

This work includes three kind os magic squares, i.e., having the properties such as, **upside-down**, **mirror looking** and/or **water reflexive** magic squares.

2 Magic Squares of Order 3

Let's consider a classical Lo-Shu magic square of order 3:

Example 1. *A magic square of order 3 is given by*

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

2.1 Composite Magic Squares

Eliminating the third value in Grid 2, and then splitting in two Latin squares, we get

Grid 1. ?? *Let's consider following composite form:*

b	c	a
a	b	c
c	a	b
	A	

a	c	b
c	b	a
b	a	c
	B	

ba	cc	ab
ac	bb	ca
cb	aa	bc
	AB	

The grid AB can be written as

$$AB := 10 \times A + B$$

In particular for $a = 1$, $b = 2$ and $c = 3$, we get

2	3	1
1	2	3
3	1	2
	A	

1	3	2
3	2	1
2	1	3
	B	

21	33	12
13	22	31
32	11	23
	AB	

Applying $3 \times (A-1) + B$ over the elements of A and B given above, we get a magic square of order 3 given in Example 9. Below are some examples **composite upside-down** and **mirror looking semi-magic** squares.

2.2 Palindromic Representations

Let's consider three letters a, b and c, where $a, b, c \in \{1,2,3,4,5,6,7,8,9\}$. We can make exactly 9 palindromes of 3-digits using only these three letters:

Table 2.1. *The 9 palindromes with letters a, b and c are given by*

1	2	3	4	5	6	7	8	9
aaa	aba	aca	bab	bbb	bcb	cac	cbc	ccc

Grid 2. *Using three letters a, b and c, we have only 9 palindromes of 3-digits. This allows us to write following* ***palindromic grid****:*

bab	ccc	aba
aca	bbb	cac
cbc	aaa	bcb

where $a, b, c \in \{1,2,3,4,5,6,7,8,9\}$

Some particular examples of Grid 2 are as follows:

2.3 3-Digits Entries

Example 2. *Let's consider the following three magic squares:*

			178
61	99	16	176
19	66	91	176
96	11	69	176
176	176	176	196

			76
20	55	02	77
05	22	50	77
52	00	25	77
77	77	77	67

			165
58	*22*	*85*	*165*
82	*55*	*28*	*165*
25	*88*	*52*	*165*
165	*165*	*165*	*165*

*We observe that only the last one is magic square, while the first two are semi-magic squares. Writing the above magic squares without sums and in **digital fonts**, we have*

61	99	16
19	66	91
96	11	69

20	55	02
05	22	50
52	00	25

58	22	85
82	55	28
25	88	52

*Let's consider the **upside-down** and **mirror looking** versions of above three magic squares.*

• 180^o Rotation

69	11	96
16	99	61
91	66	19

52	00	25
05	22	50
20	55	02

25	88	52
82	55	28
58	22	85

Let's check the sums of above three magic squares.

			286
69	*11*	*96*	*176*
16	*99*	*61*	*176*
91	*66*	*19*	*176*
176	*176*	*176*	*187*

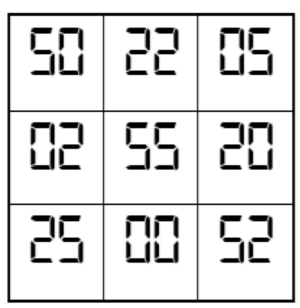
			67
52	00	25	77
05	22	50	77
20	55	02	77
77	77	77	76

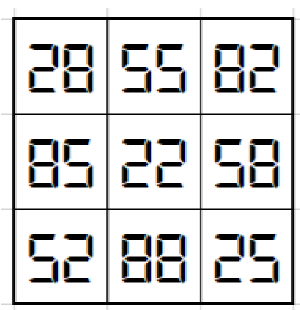
			165
25	88	52	165
82	55	28	165
58	22	85	165
165	165	165	165

The first one is a magic square while last two are semi-magic squares. The first magic square formed by digits 1, 6 and 9 is only ***upside-down****, while other two still have* ***mirror looking*** *versions. Let's see below.*

• Mirror Looking

Let's consider the last two images, where we can check about ***mirror looking****.*





			85
50	22	05	77
02	55	20	77
25	00	52	77
77	77	77	157

			156
28	55	82	165
85	22	58	165
52	88	25	165
165	165	165	75

In this case, both are semi-magic squares. Thus first semi-magic square is ***upside-down****. The last two squares* ***universal****, where the second one is all time semi-magic, and the third one is magic square for the* ***upside-down*** *situation and semi-magic in* ***mirror looking*** *position.*

2.3.1 Palindromic Representations

Example 3. *For $a = 1$, $b = 2$ and $c = 3$ in Grid 2, we have 3-digits* ***palindromic*** *magic square of order 3:*

			666
213	333	121	666
131	222	313	666
323	111	232	666
666	666	666	666

Example 4. *For $a = 2$, $b = 5$ and $c = 8$ in Grid 2, we have 3-digits* ***palindromic*** *magic square of order 3:*

			1665
525	888	252	1665
282	555	828	1665
858	222	585	1665
1665	1665	1665	1665

In **digital fonts** the above magic square is given by

525	888	252
282	555	828
858	222	585

Let's see below the **upside-down** and **mirror looking** versions of above magic square.

- **180^o Rotation**

585	222	858
828	555	282
252	888	525

Let's check the sum of above magic square:

			1665
585	222	858	*1665*
828	555	282	*1665*
252	888	525	*1665*
1665	*1665*	*1665*	*1665*

In this case, the magic sum is same as of original magic square.

- **Mirror Looking**

252	888	525
585	222	858
828	555	282

Let's check the sum of above magic square:

			1575
252	888	525	*1665*
585	222	858	*1665*
828	555	282	*1665*
1665	*1665*	*1665*	*756*

In this case it becomes **semi-magic** square.

Example 5. *For $a = 1$, $b = 6$ and $c = 9$ in Grid 2, we have 3-digits **palindromic semi-magic** square of order 3:*

			1796
616	*999*	*161*	*1665*
191	*666*	*919*	*1665*
969	*111*	*696*	*1665*
1665	*1665*	*1665*	*1978*

In **digital fonts** the above magic square is given by

616	999	161
191	666	919
969	111	696

Let's see below the **upside-down** version of above magic square.

• 180^o Rotation

969	111	696
616	999	161
191	666	919

Let's check the sum of above magic square:

			1886
969	111	696	*1776*
616	999	161	*1776*
191	666	919	*1776*
1776	*1776*	*1776*	*2887*

In this case also the magic square is **semi-magic**.

Note 2.1. *We observe that the Examples 10 and 12 are magic squares, while Examples 5 is* ***semi-magic****. The reason is on the choices of a, b and c. If we choose them with the property that $b = \frac{a+c}{2}$, then we always get a magic square, otherwise it becomes* ***semi-magic****. This happens in two ways, one when we have consecutive numbers such as, {1,2,3}, {6,7,8}, etc. Second when there is uniform difference between the numbers, for example, {2,5,8}, {1,5,9}, etc.*

2.4 2-Digits Entries

In this subsection, we shall work with 2-digits entries, i.e., only with numbers: (1,8), (2,5) and (6,9).

2.4.1 The Numbers 1 and 8

Example 6. *Let's consider a=11, b=18 and c=81 in the Grid* **??**, *we get the following the* ***semi-magic*** *square of order 3 for the digits 1 and 8 is given by*

			11054
1811	8181	1118	11110
1181	1818	8111	11110
8118	1111	1881	11110
11110	11110	11110	5510

Let's write the above magic square in ***spcial fonts****:*

1811	8181	1118
1181	1818	8111
8118	1111	1881

• 180^o **Rotation**

1881	1111	8118
1118	8181	1811
8111	1818	1181

Let's check the sum of row and columns:

			24410
1881	*1111*	*8118*	*11110*
1118	*8181*	*1811*	*11110*
8111	*1818*	*1181*	*11110*
11110	*11110*	*11110*	*11243*

• Mirror Looking

8111	1818	1181
1118	8181	1811
1881	1111	8118

Let's check the sum of row and columns:

			11243
8111	*1818*	*1181*	*11110*
1118	*8181*	*1811*	*11110*
1881	*1111*	*8118*	*11110*
11110	*11110*	*11110*	*24410*

In this case it is ***universal semi-magic*** *square with* ***magic sum:*** $S_{3\times 3} := 11110$

2.4.2 The Numbers 2 and 5

Example 7. *Let's consider a=22, b=25 and c=52 in the Grid* **??***, we get the following the* ***semi-magic*** *square of order 3 for the digits 2 and 5 is given by*

			9975
2522	*5252*	*2225*	*9999*
2252	*2525*	*5222*	*9999*
5225	*2222*	*2552*	*9999*
9999	*9999*	*9999*	*7599*

*Let's write the above **semi-magic** square in **digital fonts***

2522	5252	2225
2252	2525	5222
5225	2222	2552

• 180^o Rotation

2552	2222	5225
2225	5252	2522
5222	2525	2252

Let's check the sum of row and columns:

			15699
2552	2222	5225	9999
2225	5252	2522	9999
5222	2525	2252	9999
9999	9999	9999	10056

• Mirror Looking

2555	5252	5525
5552	2525	5255
5225	5555	2552

Let's check the sum of row and columns:

			13275
2555	5252	5525	13332
5552	2525	5255	13332
5225	5555	2552	13332
13332	13332	13332	7632

*In this case, it is **universal semi-magic** square with different **semi-magic** sums, i.e., in case of original and upside-down the **semi-magic** sums are same, i.e.,* $S_{3\times3}(2,5) := 9999$. *In case of **mirror looking**, the **semi-magic** sum is different,* $S_{3\times3}(2,5) := 13332$.

2.4.3 The Numbers 6 and 9

Example 8. *Let's consider a=66, b=69 and c=96 in the Grid **??**, we get the following the **semi-magic** square of order 3 for the digits 6 and 9 is given by*

6966	9999	6669
6699	6969	9966
9969	6666	6999

*In this case it is **upside-down semi-magic** square with **magic sum:*** $S_{3\times3}(6,9) := 23634$

Remark 2.1. *The Examples 6 and 8 are **universal semi-magic** squares with digits (1,8) and (2,5). The Example **??** one with the digits (6,9) is **upside-down**.*

3 Water Reflection Magic Squares of Order 3

3.1 3-Digits Magic Squares

Let's consider the following 4 magic squares having the digits 0, 1, 2, 5 and 8 with 3.

Example 9. *A magic square of order 3 with digits 0, 1 and 3 is given by*

			43
10	33	01	44
03	11	30	44
31	00	13	44
44	44	44	34

Example 10. *A magic square of order 3 with digits 0, 3 and 8 is given by*

			119
30	88	03	121
08	33	80	121
83	00	38	121
121	121	121	101

Example 11. *A magic square of order 3 with digits 1, 3 and 8 is given by*

			129
31	88	13	132
18	33	81	132
83	11	38	132
132	132	132	102

Example 12. *A magic square of order 3 with digits 2, 3 and 5 is given by*

			109
32	55	23	110
25	33	52	110
53	22	35	110
110	110	110	100

Thus, we observe that all the above four magic squares are **semi-magic**. Let's write the above **semi-magic** squares in digital/special fonts:

10	33	01
03	11	30
31	00	13

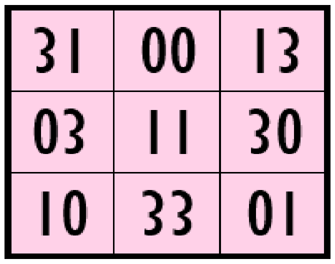
30	88	03
08	33	80
83	00	38

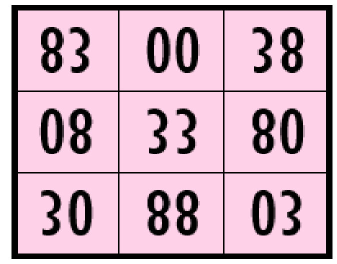
31	88	13
18	33	81
83	11	38

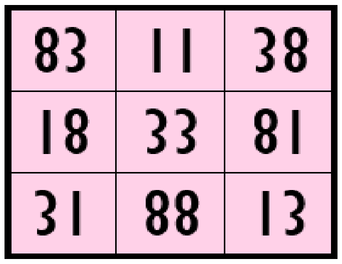
32	55	23
25	33	52
53	22	35

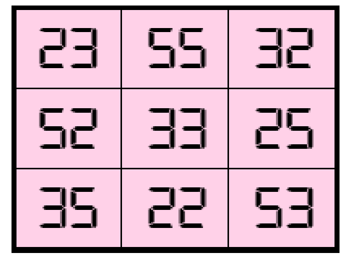
3.1.1 Water Reflection

Water reflection of above magic squares is given by









Let's verify the magic sum of above four magics squares:

Example 13. *A magic square of order 3 with digits 0, 1 and 3 is given by*

			34
31	*00*	*13*	44
03	*11*	*30*	44
10	*33*	*01*	44
44	44	44	*43*

Example 14. *A magic square of order 3 with digits 0, 3 and 8 is given by*

			101
83	*00*	*38*	121
08	*33*	*80*	121
30	*88*	*03*	121
121	121	121	*119*

Example 15. *A magic square of order 3 with digits 1, 3 and 8 is given by*

			102
83	*11*	*38*	132
18	*33*	*81*	132
31	*88*	*13*	132
132	132	132	*129*

Example 16. *A magic square of order 3 with digits 2, 3 and 5 is given by*

			100
23	*55*	*32*	*110*
52	*33*	*25*	*110*
35	*22*	*53*	*110*
110	*110*	*110*	*109*

Thus, the above four **semi-magic** squares remains the same even in **water reflection**

3.2 2-Digits Magic Squares

Let's consider following three magic squares of order 3 with 2 digits.

Example 17. *A magic square of order 3 with digits 0 and 3 is given by*

			6690
3003	*3333*	*0330*	*6666*
0333	*3030*	*3303*	*6666*
3330	*0303*	*3033*	*6666*
6666	*6666*	*6666*	*9066*

Example 18. *A magic square of order 3 with digits 1 and 3 is given by*

			7793
3113	*3333*	*1331*	*7777*
1333	*3131*	*3313*	*7777*
3331	*1313*	*3133*	*7777*
7777	*7777*	*7777*	*9377*

Example 19. *A magic square of order 3 with digits 3 and 8 is given by*

			15514
3833	*8383*	*3338*	*15554*
3383	*3838*	*8333*	*15554*
8338	*3333*	*3883*	*15554*
15554	*15554*	*15554*	*11554*

The above three magic squares are **semi-magic**. Let's write the above **semi-magic** squares in digital/special fonts:

3003	3333	0330
0333	3030	3303
3330	0303	3033

3113	3333	1331
1333	3131	3313
3331	1313	3133

3833	8383	3338
3383	3838	8333
8338	3333	3883

3.2.1 Water Reflection

Let's write consider the above **semi-magic** squares as **water reflection**

3330	0303	3033
0333	3030	3303
3003	3333	0330

3331	1313	3133
1333	3131	3313
3113	3333	1331

8338	3333	3883
3383	3838	8333
3833	8383	3338

Let's verify the sum of above magic squares

Example 20. *A magic square of order 3 with digits 0 and 3 is given by*

			9066
3330	*0303*	*3033*	*6666*
0333	*3030*	*3303*	*6666*
3003	*3333*	*0330*	*6666*
6666	*6666*	*6666*	*6690*

Example 21. *A magic square of order 3 with digits 1 and 3 is given by*

			9377
3331	*1313*	*3133*	*7777*
1333	*3131*	*3313*	*7777*
3113	*3333*	*1331*	*7777*
7777	*7777*	*7777*	*7793*

Example 22. *A magic square of order 3 with digits 3 and 8 is given by*

			11554
8338	*3333*	*3883*	*15554*
3383	*3838*	*8333*	*15554*
3833	*8383*	*3338*	*15554*
15554	*15554*	*15554*	*15514*

Thus, the above three **semi-magic** squares remains the same even in **water reflection**.

4 Magic Squares of Order 4

This section brings examples magic squares of order 4. Some of these examples are **universal magic squares** (upside-down and mirror looking), while some of them are only **upside-down**. Let consider a classical **Khajuraho magic** square of order 4.

Example 23. *The famous* ***Khajuraho magic*** *square of order 4 is given by*

		34	*34*	*34*	*34*
	7	*12*	*1*	*14*	*34*
34	*2*	*13*	*8*	*11*	*34*
34	*16*	*3*	*10*	*5*	*34*
34	*9*	*6*	*15*	*4*	*34*
	34	*34*	*34*	*34*	*34*

This is considered as one of the most **perfect magic square** of order 4 studied around 10^{th} century. Some times it is called as **dense** or **fully distributed** magic square. Dense in the sense that there are so many blocks of order 2×2 has the same sum as of magic square. It is found in India in *Khajuraho in the Parshvanath Jain temple*. This magic square is not only a normal magic square, also a **pandiagonal**. It has lot of other properties. For details see Taneja [8].

4.1 Composite Magic Squares

Grid 3. *Eliminating the third value in Grid 4, and then splitting in two Latin squares, we get*

b	c	a	d
a	d	b	c
d	a	c	b
c	b	d	a
	A		

c	d	a	b
b	a	d	c
d	c	b	a
a	b	c	d
	B		

bc	cd	aa	db
ab	da	bd	cc
dd	ac	cb	ba
ca	bb	dc	ad
	AB		

The grid AB can be written as

$$AB := 10 \times A + B$$

Example 24. *In particular for $a = 1$, $b = 2$, $c = 3$ and $d = 4$, we get*

2	3	1	4
1	4	2	3
4	1	3	2
3	2	4	1
	A		

3	4	1	2
2	1	4	3
4	3	2	1
1	2	3	4
	B		

23	34	11	42
12	41	24	33
44	13	32	21
31	22	43	14
	AB		

4.2 Palindromic Representations

Let's consider four letters a, b, c and d, where $a, b, c, d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We can make exactly 16 palindromes of 3-digits with these four letters:

Table 4.1. *The palindromes are as follows:*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
aaa	aba	aca	ada	bab	bbb	bcb	bdb	cac	cbc	ccc	cdc	dad	dbd	dcd	ddd

Replacing the above values with their respective palindromes in Example 35, we get the following grid of order 4:

Grid 4. *Using three letters a, b, c and d, we have only 16 palindromes of 3-digits. This allows to write as the following* ***palindromic grid****:*

bcb	cdc	aaa	dbd
aba	dad	bdb	ccc
ddd	aca	cbc	bab
cac	bbb	dcd	ada

where $a, b, c, d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

For all $a,b,c,d \in \{1,2,3,4,5,6,7,8,9\}$, the grid given in Grid 4 represents a palindromic magic square of order 4. If it exists, then its sum is given by

$$S_{4\times4}(a,b,c,d) := (a+b+c+d) \times 111.$$

Depending upon the choices of a,b,c and d, it may be magic or semi-magic square.

4.2.1 7-digit Palindromic Magic Squares with 2 Letters

We have only 16 choices of 7-digits palindromes for the two letters, a and b. This allows us to write following grid of order 4:

Grid 5. *We have following palindromic grid of order 4 with 7-digits and 2 letters a and b:*

$abbabba$	$babbbab$	$aaaaaaa$	$bbababb$
$aaabaaa$	$bbaaabb$	$abbbbba$	$bababab$
$bbbbbbb$	$aababaa$	$baabaab$	$abaaaba$
$baaaaab$	$abababa$	$bbbabbb$	$aabbbaa$

For all where $a,b \in \{1,2,3,4,5,6,7,8,9\}$, above grid represent a **palindromic** magic square of order 4. If it exists, then it magic sum is given by

$$S_{4\times4}(a,b) := 2 \times (aaaaaaa + bbbbbbb) = (a+b) \times 2222222.$$

4.3 4-Digits Entries

Example 25. *The following two examples are **upside-down** and **mirror looking** magic squares just with four numbers with sums $S_{4\times4}(1,6,8,9) := (1+6+8+9)\times11 = 264$ and $S_{4\times4}(0,1,2,5) := (0+1+2+5)\times11 = 88$ respectively:*

				264
61	86	99	18	264
19	98	81	66	264
88	69	16	91	264
96	11	68	89	264
264	264	264	264	264

				88
10	21	55	02	88
05	52	20	11	88
22	15	01	50	88
51	00	12	25	88
88	88	88	88	88

*In **digital fonts** the above two magic square are given by*

61	86	99	18
19	98	81	66
88	69	16	91
96	11	68	89

10	21	55	02
05	52	20	11
22	15	01	50
51	00	12	25

*The **upside-down** and **mirror looking** versions of above magic squares are given by*

• 180^o Rotation

68	89	11	96
16	91	69	88
99	18	86	61
81	66	98	19

52	21	00	15
05	10	51	22
11	02	25	50
20	55	12	01

Let's check the sums of above magic squares.

				264
68	89	11	96	264
16	91	69	88	264
99	18	86	61	264
81	66	98	19	264
264	264	264	264	264

				88
52	*21*	*00*	*15*	*88*
05	*10*	*51*	*22*	*88*
11	*02*	*25*	*50*	*88*
20	*55*	*12*	*01*	*88*
88	*88*	*88*	*88*	*88*

- ## Mirror Looking

50	22	15	01
11	05	52	20
02	10	21	55
25	51	00	12

Let's check the sums of above magic squares.

				88
50	*22*	*15*	*01*	*88*
11	*05*	*52*	*20*	*88*
02	*10*	*21*	*55*	*88*
25	*51*	*00*	*12*	*88*
88	*88*	*88*	*88*	*88*

Note 4.1. *The first example is* ***upside-down****, while the second example is* ***upside-down*** *and* ***mirror looking****. Moreover, second example is well-known author's stamp published at Macau - China - 2015:*

Example 26. *For $a=1$, $b=2$, $c=3$ and $d=4$ in Grid 4, the 3-digits **palindromic** magic square of order 4 with magic sum $S_{4\times4}(1,2,3,4) := (1+2+3+4)\times 111 = 1110$ is given by*

		1110	1110	1110	1110
	232	343	111	424	1110
1110	121	414	242	333	1110
1110	444	131	323	212	1110
1110	313	222	434	141	1110
	1110	1110	1110	1110	1110

Example 27. *For $a=1$, $b=2$, $c=5$ and $d=8$ in Grid 4, the 3-digits **palindromic** magic square of order 4 with magic sum $S_{4\times4}(1,2,5,8) := (1+2+5+8)\times 111 = 1776$ is given by.*

				1776
252	585	111	828	1776
121	818	282	555	1776
888	151	525	212	1776
515	222	858	181	1776
1776	1776	1776	1776	1776

In **digital fonts** the above magic square is given by

252	585	111	828
121	818	282	555
888	151	525	212
515	222	858	181

Let’s see below the **upside-down** and **mirror looking** versions of above magic square:

• 180^o Rotation

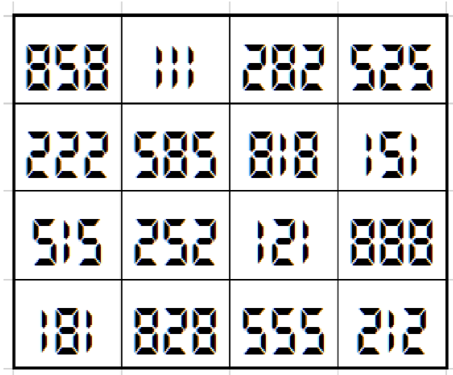
181	858	222	515
212	525	151	888
555	282	818	121
828	111	585	252

Let’s check the sum of above magic square:

				1776
181	858	222	515	*1776*
212	525	151	888	*1776*
555	282	818	121	*1776*
828	111	585	252	*1776*
1776	*1776*	*1776*	*1776*	*1776*

In this case, it remains a magic square.

• Mirror Looking



Let's check the sum of above magic square:

				1776
858	111	282	525	*1776*
222	585	818	151	*1776*
515	252	121	888	*1776*
181	828	555	212	*1776*
1776	*1776*	*1776*	*1776*	*1776*

In this case, it remains a magic square. Thus we have a magic square of order 4, where **upside-down** and **mirror looking** versions are also magic squares.

Example 28. *For $a = 1$, $b = 6$, $c = 8$ and $d = 9$ in Grid 4, the 3-digits **palindromic** magic square of order 4 with magic sum $S_{4\times4}(1, 6, 8, 9) := (1 + 6 + 8 + 9) \times 111 = 2664$ is given by.*

				2664
161	*686*	*999*	*818*	*2664*
919	*898*	*181*	*666*	*2664*
888	*969*	*616*	*191*	*2664*
696	*111*	*868*	*989*	*2664*
2664	*2664*	*2664*	*2664*	*2664*

In **digital fonts** the above magic square is given by

161	686	999	818
919	898	181	666
888	969	616	191
696	111	868	989

Let's see below the **upside-down** version of above magic square:

• 180^o Rotation

686	898	111	969
161	919	696	888
999	181	868	616
818	666	989	191

Let's check the sum of above magic square

				2664
686	898	111	969	*2664*
161	919	696	888	*2664*
999	181	868	616	*2664*
818	666	989	191	*2664*
2664	*2664*	*2664*	*2664*	*2664*

Thus, in this case the original magic square and its **upside-down** version, both are magic squares with same sum, i.e., $S_{4\times 4} := 2664$.

Note 4.2. *We observe that the Example 36 is a **pandiagonal** magic square, while the Examples 37 and 38 are just magic squares. It depends on the choices of letters a, b, c and d. If the choices are consecutive numbers such as, {1,2,3, 4}, {6,7,8, 9}, etc. or with equal differences, such as, {1, 3,5,7}, {3, 5,7,9}, etc. then we always have a **pandiagonal** magic square.*

Note 4.3. *In case of magic square of order 3, the mirror looking magic square (Note* **??***) turns as **semi-magic**. In case of order 4, the Example 37 , it remains a magic square even in mirror looking.*

4.4 2-Digits Entries

In this subsection, we shall work with 2-digits entries, i.e., only with numbers: (1,8), (2,5) and (6,9).

4.4.1 The Numbers 1 and 8

Example 29. *Let's consider* $a = 11, b = 18, c = 81$ *and* $d = 88$ *in 5, we get the following the pandiagonal square of order 4 for the digits 1 and 8:*

		19998	*19998*	*19998*	*19998*
	1881	*8188*	*1111*	*8818*	*19998*
19998	*1118*	*8811*	*1888*	*8181*	*19998*
19998	*8888*	*1181*	*8118*	*1811*	*19998*
19998	*8111*	*1818*	*8881*	*1188*	*19998*
	19998	*19998*	*19998*	*19998*	*19998*

Let's write the above magic square in ***special fonts****:*

1881	8188	1111	8818
1118	8811	1888	8181
8888	1181	8118	1811
8111	1818	8881	1188

• 180^o Rotation

8811	1888	8181	1118
1181	8118	1811	8888
1818	8881	1188	8111
8188	1111	8818	1881

Let's check the sum of row and columns:

		19998	*19998*	*19998*	*19998*
	8811	*1888*	*8181*	*1118*	*19998*
19998	*1181*	*8118*	*1811*	*8888*	*19998*
19998	*1818*	*8881*	*1188*	*8111*	*19998*
19998	*8188*	*1111*	*8818*	*1881*	*19998*
	19998	*19998*	*19998*	*19998*	*19998*

• Mirror Looking

8188	1111	8818	1881
1818	8881	1188	8111
1181	8118	1811	8888
8811	1888	8181	1118

Let's check the sum of row and columns:

		19998	*19998*	*19998*	*19998*
	8188	*1111*	*8818*	*1881*	*19998*
19998	*1818*	*8881*	*1188*	*8111*	*19998*
19998	*1181*	*8118*	*1811*	*8888*	*19998*
19998	*8811*	*1888*	*8181*	*1118*	*19998*
	19998	*19998*	*19998*	*19998*	*19998*

Thus, we have a **universal pandiagonal** magic square of order 4 with magic sum, $S_{4\times4} := 19998$.

4.4.2 The Numbers 2 and 5

Example 30. *Let's consider* $a = 22, b = 25, c = 52$ *and* $d = 55$ *in 5, we get the following the pandiagonal square of order 4 for the digits 2 and 5:*

		15554	15554	15554	15554
	2552	5255	2222	5525	15554
15554	2225	5522	2555	5252	15554
15554	5555	2252	5225	2522	15554
15554	5222	2525	5552	2255	15554
	15554	15554	15554	15554	15554

Let's write the above magic square in ***special fonts****:*

2552	5255	2222	5525
2225	5522	2555	5252
5555	2252	5225	2522
5222	2525	5552	2255

• 180^o Rotation

5522	2555	5252	2225
2252	5225	2522	5555
2525	5552	2255	5222
5255	2222	5525	2552

Let's check the sum of row and columns:

		15554	15554	15554	15554
	5522	2555	5252	2225	15554
15554	2252	5225	2522	5555	15554
15554	2525	5552	2255	5222	15554
15554	5255	2222	5525	2552	15554
	15554	15554	15554	15554	15554

• Mirror Looking

2522	5555	2252	5225
5252	2225	5522	2555
5525	2552	5255	2222
2255	5222	2525	5552

Let's check the sum of row and columns:

		15554	*15554*	*15554*	*15554*
	2522	*5555*	*2252*	*5225*	*15554*
15554	*5252*	*2225*	*5522*	*2555*	*15554*
15554	*5525*	*2552*	*5255*	*2222*	*15554*
15554	*2255*	*5222*	*2525*	*5552*	*15554*
	15554	*15554*	*15554*	*15554*	*15554*

Thus, we have a **universal pandiagonal** magic square of order 4 with magic sum, $S_{4\times4} := 15554$.

4.4.3 The Numbers 6 and 9

Example 31. *Let's consider* $a = 66, b = 69, c = 96$ *and* $d = 99$ *in 5, we get the following the pandiagonal square of order 4 for the digits 6 and 9:*

		33330	*33330*	*33330*	*33330*
	6996	*9699*	*6666*	*9969*	*33330*
33330	*6669*	*9966*	*6999*	*9696*	*33330*
33330	*9999*	*6696*	*9669*	*6966*	*33330*
33330	*9666*	*6969*	*9996*	*6699*	*33330*
	33330	*33330*	*33330*	*33330*	*33330*

Let's write the above magic square in ***special fonts****:*

6996	9699	6666	9969
6669	9966	6999	9696
9999	6696	9669	6966
9666	6969	9996	6699

• 180^o Rotation

6699	9666	6969	9996
9969	6996	9699	6666
9696	6669	9966	6999
6966	9999	6696	9669

Let's check the sum of row and columns:

		33330	33330	33330	33330
	6699	9666	6969	9996	33330
33330	9969	6996	9699	6666	33330
33330	9696	6669	9966	6999	33330
33330	6966	9999	6696	9669	33330
	33330	33330	33330	33330	33330

Thus, we have an **upside-down** pandiagonal magic square of order 4 with 2-digits 6 and 9. The magic sum is $S_{4\times4} := 33330$.

Example 32. *For* $a = 2$ *and* $b = 5$ *in 5 we have 7-digits palindromic* ***pandiagonal upside-down*** *and* ***mirror looking*** *magic square of order 4 with sum* $S_{4\times4}(2,5) := (2+5) \times 2222222 = 15555554$*:*

		15555554	15555554	15555554	15555554
	2552552	5255525	2222222	5525255	15555554
15555554	2225222	5522255	2555552	5252525	15555554
15555554	5555555	2252522	5225225	2522252	15555554
15555554	5222225	2525252	5552555	2255522	15555554
	15555554	15555554	15555554	15555554	15555554

*In this case, it is **pandiagonal** magic square. Let's write it in **digital fonts**.*

2552552	5255525	2222222	5525255
2225222	5522255	2555552	5252525
5555555	2252522	5225225	2522252
5222225	2525252	5552555	2255522

*Let's check the properties of **upside-down** and **mirror looking***

• 180^o Rotation and Mirror Looking

2255522	5552555	2525252	5222225
2522252	5225225	2252522	5555555
5252525	2555552	5522255	2225222
5525255	2222222	5255525	2552552

2252522	5555555	2522252	5225225
2525252	5222225	2255522	5552555
5255525	2552552	5525255	2222222
5522255	2225222	5252525	2555552

The magic sum in each case is given by

		15555554	*15555554*	*15555554*	*15555554*
	2255522	*5552555*	*2525252*	*5222225*	*15555554*
15555554	*2522252*	*5225225*	*2252522*	*5555555*	*15555554*
15555554	*5252525*	*2555552*	*5522255*	*2225222*	*15555554*
15555554	*5525255*	*2222222*	*5255525*	*2552552*	*15555554*
	15555554	*15555554*	*15555554*	*15555554*	*15555554*

		15555554	*15555554*	*15555554*	*15555554*
	2252522	*5555555*	*2522252*	*5225225*	*15555554*
15555554	*2525252*	*5222225*	*2255522*	*5552555*	*15555554*
15555554	*5255525*	*2552552*	*5525255*	*2222222*	*15555554*
15555554	*5522255*	*2225222*	*5252525*	*2555552*	*15555554*
	15555554	*15555554*	*15555554*	*15555554*	*15555554*

*In both the case, the magic sum is same as of original magic sum. It is a **universal** magic square. It is also **pandiagonal**.*

Example 33. *For* $a = 1$ *and* $b = 8$ *in* 5 *we have 7-digits palindromic* ***pandiagonal upside-down*** *and* ***mirror looking*** *magic square of order 4 with sum* $S_{4\times4}(1,8) := (1+8) \times 2222222 = 19999998$*:*

		19999998	19999998	19999998	19999998
	1881881	8188818	1111111	8818188	19999998
19999998	1118111	8811188	1888881	8181818	19999998
19999998	8888888	1181811	8118118	1811181	19999998
19999998	8111118	1818181	8881888	1188811	19999998
	19999998	19999998	19999998	19999998	19999998

In terms of ***digital fonts*** *the above magic square is given by*

1881881	8188818	1111111	8818188
1118111	8811188	1888881	8181818
8888888	1181811	8118118	1811181
8111118	1818181	8881888	1188811

• 180^o Rotation and Mirror Looking

1188811	8881888	1818181	8111118
1811181	8118118	1181811	8888888
8181818	1888881	8811188	1118111
8818188	1111111	8188818	1881881

8818188	1111111	8188818	1881881
8181818	1888881	8811188	1118111
1811181	8118118	1181811	8888888
1188811	8881888	1818181	8111118

Let's check the magic sums of these magic squares.

		19999998	19999998	19999998	19999998
	1188811	8881888	1818181	8111118	19999998
19999998	1811181	8118118	1181811	8888888	19999998
19999998	8181818	1888881	8811188	1118111	19999998
19999998	8818188	1111111	8188818	1881881	19999998
	19999998	19999998	19999998	19999998	19999998

		19999998	19999998	19999998	19999998
	8818188	1111111	8188818	1881881	19999998
19999998	8181818	1888881	8811188	1118111	19999998
19999998	1811181	8118118	1181811	8888888	19999998
19999998	1188811	8881888	1818181	8111118	19999998
	19999998	19999998	19999998	19999998	19999998

In both the case, the magic sum is same as of original magic sum. It is a ***universal*** *magic square. It is also* ***pandiagonal****.*

Example 34. *For* $a = 6$ *and* $b = 9$ *in 5 we have 7-digits palindromic* ***pandiagonal upside-down*** *and* ***mirror looking*** *magic square of order 4 with sum* $S_{4\times4}(6,9) := (6+9) \times 2222222 = 33333330$*:*

		33333330	33333330	33333330	33333330
	6996996	9699969	6666666	9969699	33333330
33333330	6669666	9966699	6999996	9696969	33333330
33333330	9999999	6696966	9669669	6966696	33333330
33333330	9666669	6969696	9996999	6699966	33333330
	33333330	33333330	33333330	33333330	33333330

The above magic square without magic sums is given by

6996996	9699969	6666666	9969699
6669666	9966699	6999996	9696969
9999999	6696966	9669669	6966696
9666669	6969696	9996999	6699966

• 180^o Rotation

9966699	6669666	9696969	6999996
9699969	6996996	9969699	6666666
6969696	9666669	6699966	9996999
6696966	9999999	6966696	9669669

Let's check sums of the above magic square.

		33333330	33333330	33333330	33333330
	9966699	6669666	9696969	6999996	33333330
33333330	9699969	6996996	9969699	6666666	33333330
33333330	6969696	9666669	6699966	9996999	33333330
33333330	6696966	9999999	6966696	9669669	33333330
	33333330	33333330	33333330	33333330	33333330

Thus the above magic square is ***upside-down****. It is also a* ***pandiagonal*** *magic square.*

Note 4.4. *The three Examples 40, 41 and 42 give us following symmetry in magic sums:*

$$\frac{S_{4\times4}(2,5)}{2+5}=\frac{S_{4\times4}(1,8)}{1+8}=\frac{S_{4\times4}(6,9)}{6+9}=2222222.$$

Moreover, all the three magic squares are ***pandiagonal****. The first two are* ***universal*** *and the third is only* ***upside-down****.*

Note 4.5. *Applying* $4\times(A-1)+B$ *over the members of* A *and* B*, we get a magic square of order 4 given in Example 35. Below are some more examples in composite form giving* ***upside-down*** *and* ***mirror looking*** *magic squares. Moreover, A and B are* ***pairwise mutually orthogonal diagonal Latin squares****.*

The three Examples 40, 41 and 42 brings **upside-down** and **mirror looking** magic squares of order 4 only with two digits with 7 digits in each cell. These three are **pandiagonal** magic squares. Below are two examples with two digits 69 na 25, where each cell contains only 4 digits.

5 Water Reflection Magic Squares of Order 4

5.1 4-Digits Magic Squares

Let's consider following 4 **magic squares** of order 4 with digits 0, 1, 2, 3, 5 and 8.

Example 35. *A magic square of order 4 with digits 0, 1, 3 and 8 is given by*

				132
13	38	00	81	132
01	80	18	33	132
88	03	31	10	132
30	11	83	08	132
132	132	132	132	132

Example 36. *A magic square of order 4 with digits 2, 3, 5 and 8 is given by*

				198
35	58	22	83	198
23	82	38	55	198
88	25	53	32	198
52	33	85	28	198
198	198	198	198	198

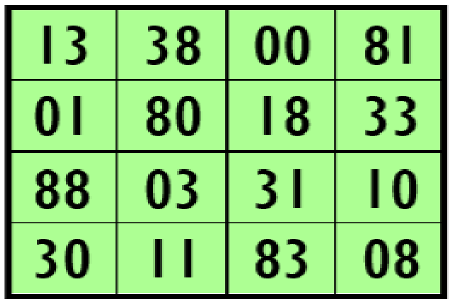
Example 37. *A magic square of order 4 with digits 1, 2, 3 and 5 is given by*

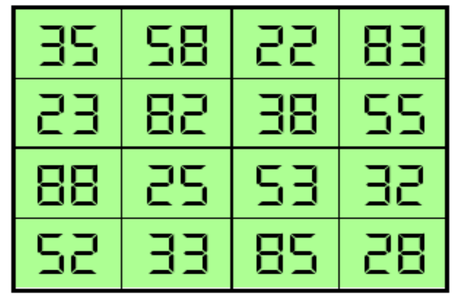
				121
23	35	11	52	121
12	51	25	33	121
55	13	32	21	121
31	22	53	15	121
121	121	121	121	121

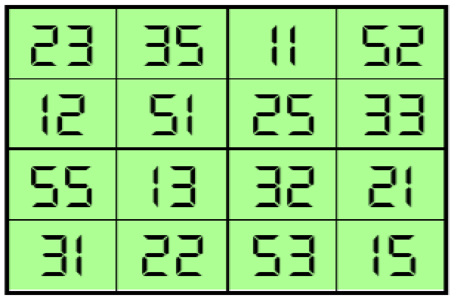
Example 38. *A magic square of order 4 with digits 0, 2, 3 and 5 is given by*

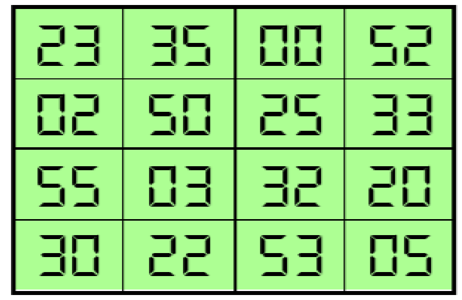
				110
23	35	00	52	110
02	50	25	33	110
55	03	32	20	110
30	22	53	05	110
110	110	110	110	110

Let’s write the above magic squares in digital/special fonts:



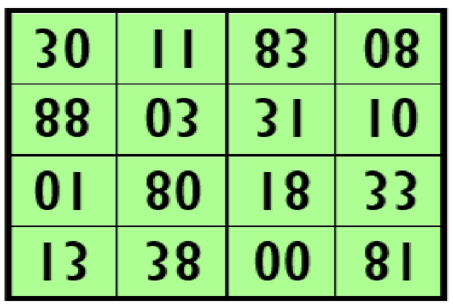


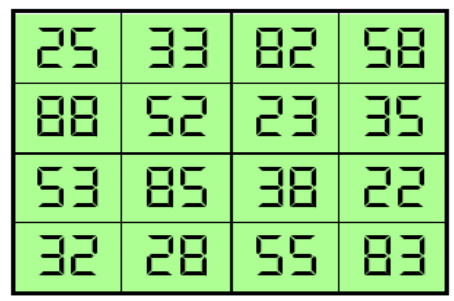


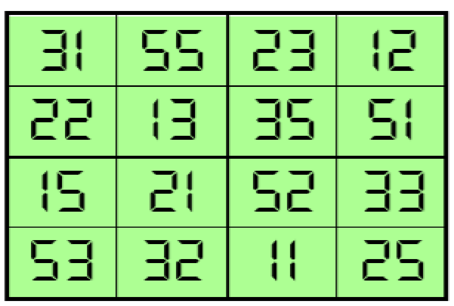


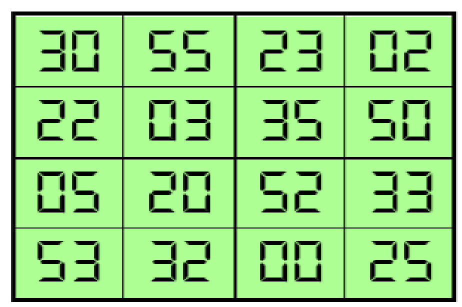
5.1.1 Water Reflection

Let's write the above magic squares with **vertical flip** i.e., **water reflection**:









Let's is check the sum of above magic squares.

Example 39. *A magic square of order 4 with digits 0, 1, 3 and 8 is given by*

				132
30	*11*	*83*	*08*	*132*
88	*03*	*31*	*10*	*132*
01	*80*	*18*	*33*	*132*
13	*38*	*00*	*81*	*132*
132	*132*	*132*	*132*	*132*

Example 40. *A magic square of order 4 with digits 2, 3, 5 and 8 is given by*

				198
25	33	82	58	198
88	52	23	35	198
53	85	38	22	198
32	28	55	83	198
198	198	198	198	198

Example 41. *A magic square of order 4 with digits 1, 2, 3 and 5 is given by*

				121
31	55	23	12	121
22	13	35	51	121
15	21	52	33	121
53	32	11	25	121
121	121	121	121	121

Example 42. *A magic square of order 4 with digits 0, 2, 3 and 5 is given by*

				110
30	55	23	02	110
22	03	35	50	110
05	20	52	33	110
53	32	00	25	110
110	110	110	110	110

Thus, the above four magic squares remains the same even with **water reflection**.

5.2 2-Digits Magic Squares

Let's consider following three magic squares of order 4 with 2 digits.

Example 43. *A* ***pandiagonal*** *magic square of order 4 with 2-digits 3 and 8 is given by*

		24442	24442	24442	24442
	3883	8388	3333	8838	24442
24442	3338	8833	3888	8383	24442
24442	8888	3383	8338	3833	24442
24442	8333	3838	8883	3388	24442
	24442	24442	24442	24442	24442

Example 44. *A **pandiagonal** magic square of order 4 with 2-digits 1 and 3 is given by*

		8888	8888	8888	8888
	1331	3133	1111	3313	8888
8888	1113	3311	1333	3131	8888
8888	3333	1131	3113	1311	8888
8888	3111	1313	3331	1133	8888
	8888	8888	8888	8888	8888

Example 45. *A **pandiagonal** magic square of order 4 with 2-digits 0 and 3 is given by*

		6666	6666	6666	6666
	0330	3033	0000	3303	6666
6666	0003	3300	0333	3030	6666
6666	3333	0030	3003	0300	6666
6666	3000	0303	3330	0033	6666
	6666	6666	6666	6666	6666

Let's write the above magic squares with **special fonts**

3883	8388	3333	8838
3338	8833	3888	8383
8888	3383	8338	3833
8333	3838	8883	3388

1331	3133	1111	3313
1113	3311	1333	3131
3333	1131	3113	1311
3111	1313	3331	1133

0330	3033	0000	3303
0003	3300	0333	3030
3333	0030	3003	0300
3000	0303	3330	0033

Let's check the sum of above three magic squares.

5.2.1 Water Reflection

Let's write the above magic squares with **water reflection**:

8333	3838	8883	3388
8888	3383	8338	3833
3338	8833	3888	8383
3883	8388	3333	8838

3111	1313	3331	1133
3333	1131	3113	1311
1113	3311	1333	3131
1331	3133	1111	3313

3000	0303	3330	0033
3333	0030	3003	0300
0003	3300	0333	3030
0330	3033	0000	3303

Let's write the sum of above magic squares:

Example 46. *A* ***pandiagonal*** *magic square of order 4 with 2-digits 3 and 8 is given by*

		24442	24442	24442	24442
	8333	3838	8883	3388	24442
24442	8888	3383	8338	3833	24442
24442	3338	8833	3888	8383	24442
24442	3883	8388	3333	8838	24442
	24442	24442	24442	24442	24442

Example 47. *A* ***pandiagonal*** *magic square of order 4 with 2-digits 1 and 3 is given by*

		8888	8888	8888	8888
	3111	1313	3331	1133	8888
8888	3333	1131	3113	1311	8888
8888	1113	3311	1333	3131	8888
8888	1331	3133	1111	3313	8888
	8888	8888	8888	8888	8888

Example 48. *A* ***pandiagonal*** *magic square of order 4 with 2-digits 0 and 3 is given by*

		6666	6666	6666	6666
	3000	0303	3330	0033	6666
6666	3333	0030	3003	0300	6666
6666	0003	3300	0333	3030	6666
6666	0330	3033	0000	3303	6666
	6666	6666	6666	6666	6666

Thus the above three magic squares are **water-reflexive**.

6 Magic Squares of Order 5

This section brings examples magic squares of order 5. Some of these examples are universal (upside-down and mirror looking) magic squares, while some of them are only upside-down.

Example 49. *Let's consider a* ***pandiagonal*** *magic square of order 5 given by*

		65	65	65	65	65
	1	9	12	20	23	65
65	17	25	3	6	14	65
65	8	11	19	22	5	65
65	24	2	10	13	16	65
65	15	18	21	4	7	65
	65	65	65	65	65	65

6.1 Palindromic Representations

Let's consider five letters a, b, c, d and e, where $a, b, c, d, e \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We can make exactly 25 palindromes of 3-digits with these five letters:

Table 6.1. *The palindromes are as follows:*

1	2	3	4	5	6	7	8	9	10	11	12	13
aaa	aba	aca	ada	aea	bab	bbb	bcb	bdb	beb	cac	cbc	ccc
14	15	16	17	18	19	20	21	22	23	24	25	
cdc	cec	dad	dbd	dcd	ddd	ded	eae	ebe	ece	ede	eee	

Replacing the above values with their respective palindromes in a magic square of order 5 given in Example 76 , we get a **palindromic grid** given below:

Grid 6. *Using five letters a, b, c, d and e, we have only 25 palindromes of 3-digits. This allows us to write as the following grid:*

aaa	bbb	ccc	ddd	eee
dcd	ede	aea	bab	cbc
beb	cac	dbd	ece	ada
ebe	aca	bdb	cec	dad
cdc	ded	eae	aba	bcb

where $a, b, c, d, e \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

For all $a, b, c, d, e \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the grid given in 6 represents a palindromic magic square of order 5. If it exists, then its magic sum is given by

$$S_{5\times 5}(a, b, c, d, e) := (a + b + c + d + e) \times 111.$$

Let's consider some examples of Grid 6

Example 50. *For $a = 1, b = 2, c = 3, d = 4$ and $e = 5$ in Grid 6, the 3-digits **palindromic** magic square of order 5 with magic sum $S_{5\times 5}(1, 2, 3, 4, 5) := (1 + 2 + 3 + 4 + 5) \times 111 = 1665$ is given by.*

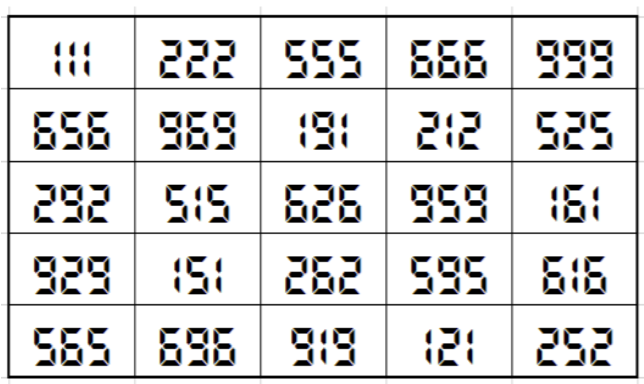
		1665	1665	1665	1665	1665
	111	222	333	444	555	1665
1665	434	545	151	212	323	1665
1665	252	313	424	535	141	1665
1665	525	131	242	353	414	1665
1665	343	454	515	121	232	1665
	1665	1665	1665	1665	1665	1665

The above Example 76 is with consecutive numbers 1 to 5 that gives us a **pandiagonal** magic square. Below is another example, with non consecutive numbers.

Example 51. *For* $a = 1, b = 2, c = 5, d = 6$ *and* $e = 9$ *in Grid 6, the 3-digits **palindromic** magic square of order 5 with magic sum* $S_{5\times5}(1, 2, 5, 6, 9) := (1 + 2 + 5 + 6 + 9) \times 111 = 2553$ *is given by.*

		2553	2553	2553	2553	2553
	111	222	555	666	999	2553
2553	656	969	191	212	525	2553
2553	292	515	626	959	161	2553
2553	929	151	262	595	616	2553
2553	565	696	919	121	252	2553
	2553	2553	2553	2553	2553	2553

In digital fonts the above magic square is given as



• 180^o Rotation

252	121	616	969	595
919	565	292	151	626
191	656	929	515	262
525	212	161	696	959
666	999	555	222	111

Let's check the sum of above magic square:

		2553	*2553*	*2553*	*2553*	*2553*
	252	121	616	969	595	*2553*
2553	919	565	292	151	626	*2553*
2553	191	656	929	515	262	*2553*
2553	525	212	161	696	959	*2553*
2553	666	999	555	222	111	*2553*
	2553	*2553*	*2553*	*2553*	*2553*	*2553*

Thus the above magic square of order 5 given in digital fonts is only **upside-down**. The next examples with digits 0, 1, 2, 5 and 8 is **universal magic square**

Example 52. *For $a = 0, b = 1, c = 2, d = 5$ and $e = 8$ in Grid 6, the 3-digits **palindromic** magic square of order 5 with magic sum $S_{5\times5}(0, 1, 2, 5, 8) := (0 + 1 + 2 + 5 + 8) \times 111 = 1776$ is given by.*

		1776	*1776*	*1776*	*1776*	*1776*
	000	*111*	*222*	*555*	*888*	*1776*
1776	*525*	*858*	*080*	*101*	*212*	*1776*
1776	*181*	*202*	*515*	*828*	*050*	*1776*
1776	*818*	*020*	*151*	*282*	*505*	*1776*
1776	*252*	*585*	*808*	*010*	*121*	*1776*
	1776	*1776*	*1776*	*1776*	*1776*	*1776*

In terms of **digital fonts** the above magic square is given by

000	111	222	555	888
525	858	080	101	212
181	202	515	828	050
818	020	151	282	505
252	585	808	010	121

• 180^o Rotation

121	010	808	585	252
505	282	151	020	818
050	828	515	202	181
212	101	080	858	525
888	555	222	111	000

Let's check the sums of above magic square:

		1776	*1776*	*1776*	*1776*	*1776*
	121	010	808	585	252	*1776*
1776	505	282	151	020	818	*1776*
1776	050	828	515	202	181	*1776*
1776	212	101	080	858	525	*1776*
1776	888	555	222	111	000	*1776*
	1776	*1776*	*1776*	*1776*	*1776*	*1776*

• Mirror Looking

888	222	555	111	000
515	101	080	828	252
020	858	212	505	181
202	585	121	050	818
151	010	808	282	525

Let's check the sums of above magic square:

		1776	*1776*	*1776*	*1776*	*1776*
	888	222	555	111	000	*1776*
1776	515	101	080	828	252	*1776*
1776	020	858	212	505	181	*1776*
1776	202	585	121	050	818	*1776*
1776	151	010	808	285	525	*1776*
	1776	*1776*	*1776*	*1776*	*1776*	*1776*

Thus the above magic square of order 5 given in **digital fonts** is **upside-down** and **mirror looking**. Moreover, it is also **pandiagonal**.

6.2 Composite Magic Squares

Grid 7. *Eliminating the third value in Grid 6, and then splitting in two Latin squares, we get*

a	*b*	*c*	*d*	*e*
d	*e*	*a*	*b*	*c*
b	*c*	*d*	*e*	*a*
e	*a*	*b*	*c*	*d*
c	*d*	*e*	*a*	*b*
		A		

a	*b*	*c*	*d*	*e*
c	*d*	*e*	*a*	*b*
e	*a*	*b*	*c*	*d*
b	*c*	*d*	*e*	*a*
d	*e*	*a*	*b*	*c*
		B		

aa	*bb*	*cc*	*dd*	*ee*
dc	*ed*	*ae*	*ba*	*cb*
be	*ca*	*db*	*ec*	*ad*
eb	*ac*	*bd*	*ce*	*da*
cd	*de*	*ea*	*ab*	*bc*
		AB		

The grid AB can be written as

$$AB := 10 \times A + B$$

As a particular case, we have following example.

Example 53. *In particular for $a = 1, b = 2, c = 3, d = 4$ and $e = 5$, we get*

1	*2*	*3*	*4*	*5*
4	*5*	*1*	*2*	*3*
2	*3*	*4*	*5*	*1*
5	*1*	*2*	*3*	*4*
3	*4*	*5*	*1*	*2*
		A		

1	*2*	*3*	*4*	*5*
3	*4*	*5*	*1*	*2*
5	*1*	*2*	*3*	*4*
2	*3*	*4*	*5*	*1*
4	*5*	*1*	*2*	*3*
		B		

11	*22*	*33*	*44*	*55*
43	*54*	*15*	*21*	*32*
25	*31*	*42*	*53*	*14*
52	*13*	*24*	*35*	*41*
34	*45*	*51*	*12*	*23*
		AB		

Note 6.1. *Applying $5 \times (A-1) + B$ over the elements of A and B given above, we get a magic square of order 5 given in Example 76. Also, A and B are* ***pairwise mutually orthogonal diagonal Latin squares****. Below are some examples of composite* ***upside-down*** *and* ***mirror looking*** *magic squares. The above Grid 7 is written for single letters a, b, c, d and e. We can choose double digits numbers to write composite examples.*

Example 54. *Let's consider $a = 0, b = 1, c = 2, d = 5$ and $e = 8$ in Grid 7 , we get following* ***pandiagonal upside-down*** *magic square:*

		176	*176*	*176*	*176*	*176*
	00	*11*	*22*	*55*	*88*	*176*
176	*52*	*85*	*08*	*10*	*21*	*176*
176	*18*	*20*	*51*	*82*	*05*	*176*
176	*81*	*02*	*15*	*28*	*50*	*176*
176	*25*	*58*	*80*	*01*	*12*	*176*
	176	*176*	*176*	*176*	*176*	*176*

In **digital fonts**, the above magic square is given as

00	11	22	55	88
52	85	08	10	21
18	20	51	82	05
81	02	15	28	50
25	58	80	01	12

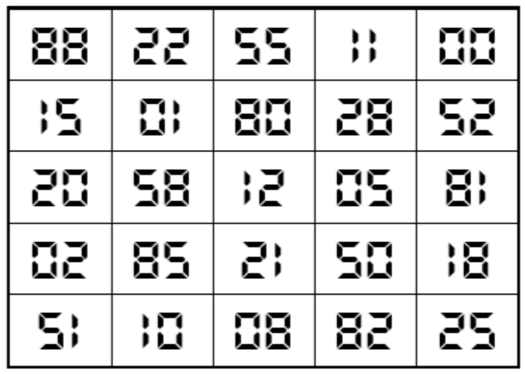
• 180^o Rotation

21	10	08	85	52
05	82	51	20	18
50	28	15	02	81
12	01	80	58	25
88	55	22	11	00

Let's check the sums of above magic square:

		176	*176*	*176*	*176*	*176*
	21	10	08	85	52	*176*
176	05	82	51	20	18	*176*
176	50	28	15	02	81	*176*
176	12	01	80	58	25	*176*
176	88	55	22	11	00	*176*
	176	*176*	*176*	*176*	*176*	*176*

- **Mirror Looking**



88	22	55	11	00
15	01	80	28	52
20	58	12	05	81
02	85	21	50	18
51	10	08	82	25

Let's check the sums of above magic square:

		176	*176*	*176*	*176*	*176*
	88	22	55	11	00	*176*
176	15	01	80	28	52	*176*
176	20	58	12	05	81	*176*
176	02	85	21	50	18	*176*
176	51	10	08	82	25	*176*
	176	*176*	*176*	*176*	*176*	*176*

Thus the above magic square of order 5 written in **digital fonts** is **upside-down** and **mirror looking**. Moreover, it is **pandiagonal** magic square.

Example 55. *Let's consider* $a = 0, b = 1, c = 6, d = 8$ *and* $e = 9$ *in Grid 7 , we get following* ***pandiagonal upside-down*** *magic square:*

		264	264	264	264	264
	00	11	66	88	99	264
264	86	98	09	10	61	264
264	19	60	81	96	08	264
264	91	06	18	69	80	264
264	68	89	90	01	16	264
	264	264	264	264	264	264

Let's write the above magic square without sums:

00	11	66	88	99
86	98	09	10	61
19	60	81	96	08
91	06	18	69	80
68	89	90	01	16

- 180^o **Rotation**

91	10	06	68	89
08	69	81	90	16
80	96	18	09	61
19	01	60	86	98
66	88	99	11	00

Let's check the sum of above magic square.

		264	264	264	264	264
	91	10	06	68	89	264
264	08	69	81	90	16	264
264	80	96	18	09	61	264
264	19	01	60	86	98	264
264	66	88	99	11	00	264
	264	264	264	264	264	264

Thus the above magic square of order 5 is **upside-down** with magic sum $S_{5\times 5} := 330$. Moreover, it is also **pandiagonal**.

Example 56. *Let's consider $a = 2, b = 5, c = 6, d = 8$ and $e = 9$ in Grid 7 , we get following **pandiagonal upside-down** magic square:*

		330	330	330	330	330
	22	55	66	88	99	330
330	86	98	29	52	65	330
330	59	62	85	96	28	330
330	95	26	58	69	82	330
330	68	89	92	25	56	330
	330	330	330	330	330	330

In **digital fonts**, the above magic square can be written as

22	55	66	88	99
86	98	29	52	65
59	62	85	96	28
95	26	58	69	82
68	89	92	25	56

• 180^o Rotation

95	52	26	68	89
28	69	85	92	56
82	96	58	29	65
59	25	62	86	98
66	88	99	55	22

Let's check the sum of the above magic square:

		330	*330*	*330*	*330*	*330*
	95	52	26	68	89	*330*
330	28	69	85	92	56	*330*
330	82	96	58	29	65	*330*
330	59	25	62	86	98	*330*
330	66	88	99	55	22	*330*
	330	*330*	*330*	*330*	*330*	*330*

In **digital fonts** the above magic square of order 5 is **upside-down** with magic sum $S_{5\times 5} := 330$. Moreover, it is also **pandiagonal**.

Example 57. *Let's consider* $a = 22, b = 25, c = 52, d = 55$ *and* $e = 88$ *in Grid 7 , we get following* ***pandiagonal upside-down*** *and* ***mirror looking*** *magic square:*

		24442	*24442*	*24442*	*24442*	*24442*
	2222	*2525*	*5252*	*5555*	*8888*	*24442*
24442	*5552*	*8855*	*2288*	*2522*	*5225*	*24442*
24442	*2588*	*5222*	*5525*	*8852*	*2255*	*24442*
24442	*8825*	*2252*	*2555*	*5288*	*5522*	*24442*
24442	*5255*	*5588*	*8822*	*2225*	*2552*	*24442*
	24442	*24442*	*24442*	*24442*	*24442*	*24442*

In **digital fonts**, the above magic square can be written as

2222	2525	5252	5555	8888
5552	8855	2288	2522	5225
2588	5222	5525	8852	2255
8825	2252	2555	5288	5522
5255	5588	8822	2225	2552

• 180^o **Rotation**

2552	5222	2288	8855	5525
2255	8825	5552	2522	5288
5522	2588	5255	2225	8852
5225	2252	8822	5588	2555
8888	5555	2525	5252	2222

Let's check the sum of above magic square:

		24442	*24442*	*24442*	*24442*	*24442*
	2552	5222	2288	8855	5525	*24442*
24442	2255	8825	5552	2522	5288	*24442*
24442	5522	2588	5255	2225	8852	*24442*
24442	5225	2252	8822	5588	2555	*24442*
24442	8888	5555	2525	5252	2222	*24442*
	24442	*24442*	*24442*	*24442*	*24442*	*24442*

• **Mirror Looking**

8888	2222	5252	2525	5555
2552	5525	8855	2288	5222
2255	5288	2522	5552	8825
5522	8852	2225	5255	2588
5225	2555	5588	8822	2252

Let's check the sum of above magic square:

		24442	24442	24442	24442	24442
	8888	2222	5252	2525	5555	24442
24442	2552	5525	8855	2288	5222	24442
24442	2255	5288	2522	5552	8825	24442
24442	5522	8852	2225	5255	2588	24442
24442	5225	2555	5588	8822	2252	24442
	24442	24442	24442	24442	24442	24442

Thus, the magic square is **universal**, i.e., **upside down** and **mirror looking** with magic sum $S_{5\times5} := 2664$. Moreover, it is also **pandiagonal**.

Example 58. *Let's consider* $a = 11, b = 66, c = 69, d = 96$ *and* $e = 99$ *in Grid 7 , we get following* ***pandiagonal upside-down*** *magic square:*

		34441	34441	34441	34441	34441
	1111	6666	6969	9696	9999	34441
34441	9669	9996	1199	6611	6966	34441
34441	6699	6911	9666	9969	1196	34441
34441	9966	1169	6696	6999	9611	34441
34441	6996	9699	9911	1166	6669	34441
	34441	34441	34441	34441	34441	34441

Rewriting the above magic square without sums, we get

1111	6666	6969	9696	9999
9669	9996	1199	6611	6966
6699	6911	9666	9969	1196
9966	1169	6696	6999	9611
6996	9699	9911	1166	6669

• 180^o Rotation

6999	9911	1166	6696	9669
1196	6669	9699	6911	9966
9611	6966	9996	1169	6699
9969	1199	6611	9666	6996
6666	9696	6969	9999	1111

Let's check the sum of above magic square:

		34441	*34441*	*34441*	*34441*	*34441*
	6999	9911	1166	6696	9669	*34441*
34441	1196	6669	9699	6911	9966	*34441*
34441	9611	6966	9996	1169	6699	*34441*
34441	9969	1199	6611	9666	6996	*34441*
34441	6666	9696	6969	9999	1111	*34441*
	34441	*34441*	*34441*	*34441*	*34441*	*34441*

Thus, the above magic square is **upside down** with magic sum $S_{5\times 5} := 34441$. Moreover, it is also **pandiagonal**.

Example 59. *Let's consider* $a = 66, b = 69, c = 88, d = 96$ *and* $e = 99$ *in Grid 7 , we get following* ***pandiagonal upside-down*** *magic square:*

		42218	*42218*	*42218*	*42218*	*42218*
	6666	*6969*	*8888*	*9696*	*9999*	*42218*
42218	*9688*	*9996*	*6699*	*6966*	*8869*	*42218*
42218	*6999*	*8866*	*9669*	*9988*	*6696*	*42218*
42218	*9969*	*6688*	*6996*	*8899*	*9666*	*42218*
42218	*8896*	*9699*	*9966*	*6669*	*6988*	*42218*
	42218	*42218*	*42218*	*42218*	*42218*	*42218*

Rewriting the above magic squares without sums, we get

6666	6969	8888	9696	9999
9688	9996	6699	6966	8869
6999	8866	9669	9988	6696
9969	6688	6996	8899	9666
8896	9699	9966	6669	6988

• 180^o **Rotation**

8869	6999	9966	6696	9688
9996	6688	9669	8899	6966
9699	8866	6996	9988	6669
6988	9969	6699	9666	8896
6666	9696	8888	6969	9999

Let's check the sum of above magic square:

		42218	*42218*	*42218*	*42218*	*42218*
	8869	6999	9966	6696	9688	*42218*
42218	9996	6688	9669	8899	6966	*42218*
42218	9699	8866	6996	9988	6669	*42218*
42218	6988	9969	6699	9666	8896	*42218*
42218	6666	9696	8888	6969	9999	*42218*
	42218	*42218*	*42218*	*42218*	*42218*	*42218*

Thus the above magic square is **upside-down** and **mirror looking** with magic sum $S_{5\times5} := 42218$. Moreover, it is also **pandiagonal**.

Note 6.2. *We studied three magic squares of order 5, where there are 4 digits in each cell. All the three magic squares are* ***pandiagonal****. Only the magic square given in Example 57 is* ***upside-down*** *and* ***mirror looking****, i.e.,* ***universal****, while the other two given in Examples 58 and 59 are only* ***upside-down****.*

6.3 2-Digits Entries

The work given above is either with 5-digits or 3-digits entries. Below are few examples with 2-digits entries. The digits are (1,8), (2,5) and (6,9). In case of (1,8) and (2,5) the results are **universal** magic squares, while in case of (6,9) the results are **upside-down** magic squares.

6.3.1 The Numbers 1 and 8

Example 60. *Let's consider* $a = 181, b = 188, c = 818, d = 881$ *and* $e = 888$ *in Grid 7 , we get a* ***pandiagonal*** *magic square of order 5 for the digits 1 and 8 is given by*

		2958956	2958956	2958956	2958956	2958956
	181181	188881	818188	881888	888818	2958956
2958956	881188	888888	181818	188181	818881	2958956
2958956	188818	818181	881881	888188	181888	2958956
2958956	888881	181188	188888	818818	881181	2958956
2958956	818888	881818	888181	181881	188188	2958956
	2958956	2958956	2958956	2958956	2958956	2958956

Let's write the above magic square without sums in ***special fonts***

181181	188881	818188	881888	888818
881188	888888	181818	188181	818881
188818	818181	881881	888188	181888
888881	181188	188888	818818	881181
818888	881818	888181	181881	188188

- 180^o **Rotation**

881881	188181	181888	818188	888818
181188	818818	888881	881181	188888
888181	881888	188188	181818	818881
188818	181881	818181	888888	881188
818888	888188	881818	188881	181181

Let's check the sums of rows,columns and of principal diagonals:

		2958956	2958956	2958956	2958956	2958956
	881881	188181	181888	818188	888818	2958956
2958956	181188	818818	888881	881181	188888	2958956
2958956	888181	881888	188188	181818	818881	2958956
2958956	188818	181881	818181	888888	881188	2958956
2958956	818888	888188	881818	188881	181181	2958956
	2958956	2958956	2958956	2958956	2958956	2958956

- **Mirror Looking**

818888	888188	881818	188881	181181
188818	181881	818181	888888	881188
888181	881888	188188	181818	818881
181188	818818	888881	881181	188888
881881	188181	181888	818188	888818

Let's check the sums of rows,columns and of principal diagonals:

		2958956	2958956	2958956	2958956	2958956
	818888	888188	881818	188881	181181	2958956
2958956	188818	181881	818181	888888	881188	2958956
2958956	888181	881888	188188	181818	818881	2958956
2958956	181188	818818	888881	881181	188888	2958956
2958956	881881	188181	181888	818188	888818	2958956
	2958956	2958956	2958956	2958956	2958956	2958956

Thus, we have a **universal pandiagonal** magic square of order 5 with magic sum $S_{5\times5} := 2958956$.

6.3.2 The Numbers 2 and 5

Example 61. *Let's consider* $a = 222, b = 225, c = 252, d = 522$ *and* $e = 525$ *in Grid 7 , we get a* ***pandiagonal*** *magic square of order 5 for the digits 1 and 8 is given by*

		1747746	*1747746*	*1747746*	*1747746*	*1747746*
	222222	*225522*	*252225*	*522525*	*525252*	*1747746*
1747746	*522225*	*525525*	*222252*	*225222*	*252522*	*1747746*
1747746	*225252*	*252222*	*522522*	*525225*	*222525*	*1747746*
1747746	*525522*	*222225*	*225525*	*252252*	*522222*	*1747746*
1747746	*252525*	*522252*	*525222*	*222522*	*225225*	*1747746*
	1747746	*1747746*	*1747746*	*1747746*	*1747746*	*1747746*

Let's write the above magic square without sums in ***digital fonts***

222222	225522	252225	522525	525252
522225	525525	222252	225222	252522
225252	252222	522522	525225	222525
525522	222225	225525	252252	522222
252525	522252	525222	222522	225225

- 180^o **Rotation**

522522	225222	222525	252225	525252
222225	252252	525522	522222	225525
525222	522525	225225	222252	252522
225252	222522	252222	525525	522225
252525	525225	522252	225522	222222

Let's check the sums of rows,columns and of principal diagonals:

		1747746	*1747746*	*1747746*	*1747746*	*1747746*
	522522	*225222*	*222525*	*252225*	*525252*	*1747746*
1747746	*222225*	*252252*	*525522*	*522222*	*225525*	*1747746*
1747746	*525222*	*522525*	*225225*	*222252*	*252522*	*1747746*
1747746	*225252*	*222522*	*252222*	*525525*	*522225*	*1747746*
1747746	*252525*	*525225*	*522252*	*225522*	*222222*	*1747746*
	1747746	*1747746*	*1747746*	*1747746*	*1747746*	*1747746*

- ## Mirror Looking

525252	252552	255525	552255	555555
552525	555255	525555	252252	255552
252555	255252	552552	555525	525255
555552	525525	252255	255555	552252
255255	552555	555252	525552	252525

Let's check the sums of rows,columns and of principal diagonals:

		2141139	*2141139*	*2141139*	*2141139*	*2141139*
	525252	*252552*	*255525*	*552255*	*555555*	*2141139*
2141139	*552525*	*555255*	*525555*	*252252*	*255552*	*2141139*
2141139	*252555*	*255252*	*552552*	*555525*	*525255*	*2141139*
2141139	*555552*	*525525*	*252255*	*255555*	*552252*	*2141139*
2141139	*255255*	*552555*	*555252*	*525552*	*252525*	*2141139*
	2141139	*2141139*	*2141139*	*2141139*	*2141139*	*2141139*

Thus, we have a **pandiagonal universal** magic square of order 5 with 2-digits (2,5). The magic sums of original and the upside-down versions the same, but in case of **mirror looking** this sums is different.

6.3.3 The Numbers 6 and 9

Example 62. *Let's consider* $a = 666, b = 669, c = 696, d = 699$ *and* $e = 969$ *in Grid 7 , we get a* ***pandiagonal*** *magic square of order 5 for the digits 2 and 5 is given by*

		3969966	*3969966*	*3969966*	*3969966*	*3969966*
	666666	*669966*	*696669*	*966969*	*969696*	*3969966*
3969966	*966669*	*969969*	*666696*	*669666*	*696966*	*3969966*
3969966	*669696*	*696666*	*966966*	*969669*	*666969*	*3969966*
3969966	*969966*	*666669*	*669969*	*696696*	*966666*	*3969966*
3969966	*696969*	*966696*	*969666*	*666966*	*669669*	*3969966*
	3969966	*3969966*	*3969966*	*3969966*	*3969966*	*3969966*

Let's write the above magic square without sums in ***digital fonts***

666666	669966	696669	966969	969696
966669	969969	666696	669666	696966
669696	696666	966966	969669	666969
969966	666669	669969	696696	966666
696969	966696	969666	666966	669669

- 180^o **Rotation**

699699	996999	999696	969996	696969
999996	969969	696699	699999	996696
696999	699696	996996	999969	969699
996969	999699	969999	696696	699996
969696	696996	699969	996699	999999

Let's check the sums of rows,columns and of principal diagonals:

		4363359	*4363359*	*4363359*	*4363359*	*4363359*
	699699	*996999*	*999696*	*969996*	*696969*	*4363359*
4363359	*999996*	*969969*	*696699*	*699999*	*996696*	*4363359*
4363359	*696999*	*699696*	*996996*	*999969*	*969699*	*4363359*
4363359	*996969*	*999699*	*969999*	*696696*	*699996*	*4363359*
4363359	*969696*	*696996*	*699969*	*996699*	*999999*	*4363359*
	4363359	*4363359*	*4363359*	*4363359*	*4363359*	*4363359*

In this case, this magic square is **upside-down pandiagonal** of order 5 in 2-digits (6,9), but the magic sums are different, i.e., the magic sums of original magic square is different from the one obtained by rotation of 180^0.

7 Water Reflection Magic Squares of Order 5

7.1 5-Digits Magic Squares

Let's consider following three magic squares with digits 0, 1, 2, 3, 5 and 8.

Example 63. *A **pandiagonal** magic square of order 5 with digits 0, 2, 3, 5 and 8:*

		198	198	198	198	198
	00	22	33	55	88	198
198	53	85	08	20	32	198
198	28	30	52	83	05	198
198	82	03	25	38	50	198
198	35	58	80	02	23	198
	198	198	198	198	198	198

Example 64. *A **pandiagonal** magic square of order 5 with digits 1, 2, 3, 5 and 8:*

		209	209	209	209	209
	11	22	33	55	88	209
209	53	85	18	21	32	209
209	28	31	52	83	15	209
209	82	13	25	38	51	209
209	35	58	81	12	23	209
	209	209	209	209	209	209

Example 65. *A **pandiagonal** magic square of order 5 with digits 0, 1, 2, 3 and 5:*

		121	121	121	121	121
	00	11	22	33	55	121
121	32	53	05	10	21	121
121	15	20	31	52	03	121
121	51	02	13	25	30	121
121	23	35	50	01	12	121
	121	121	121	121	121	121

Let's write the above magic squares with **digital/special fonts**

00	22	33	55	88
53	85	08	20	32
28	30	52	83	05
82	03	25	38	50
35	58	80	02	23

11	22	33	55	88
53	85	18	21	32
28	31	52	83	15
82	13	25	38	51
35	58	81	12	23

00	11	22	33	55
32	53	05	10	21
15	20	31	52	03
51	02	13	25	30
23	35	50	01	12

7.1.1 Water Reflection

Using **veritical-flips** the above magic squares are given by

32	28	80	05	53
85	03	52	38	20
58	30	25	83	02
23	82	08	50	35
00	55	33	22	88

32	28	81	15	53
85	13	52	38	21
58	31	25	83	12
23	82	18	51	35
11	55	33	22	88

53	32	20	01	15
21	05	13	52	30
12	50	31	25	03
35	23	02	10	51
00	11	55	33	22

Let's check the sum of above magic squares.

Example 66. *Let's consider a* ***pandiagonal*** *magic square of order 5 with digits 0, 2, 3, 5 and 8:*

		198	198	198	198	198
	32	28	80	05	53	198
198	85	03	52	38	20	198
198	58	30	25	83	02	198
198	23	82	08	50	35	198
198	00	55	33	22	88	198
	198	198	198	198	198	198

Example 67. *Let's consider a* ***pandiagonal*** *magic square of order 5 with digits 1, 2, 3, 5 and 8:*

		209	209	209	209	209
	32	28	81	15	53	209
209	85	13	52	38	21	209
209	58	31	25	83	12	209
209	23	82	18	51	35	209
209	11	55	33	22	88	209
	209	209	209	209	209	209

Example 68. *Let's consider a **pandiagonal** magic square of order 5 with digits 0, 1, 2, 3 and 5:*

		121	121	121	121	121
	53	32	20	01	15	121
121	21	05	13	52	30	121
121	12	50	31	25	03	121
121	35	23	02	10	51	121
121	00	11	55	33	22	121
	121	121	121	121	121	121

Thus, the above three magic squares are **water-reflexive**.

7.2 3-Digits Magic Squares

Let's consider following 4 **pandiagonal** magic squares of order 5 with 3-digits.

Example 69. *Let's consider a **pandiagonal** magic square of order 5 with digits 2, 3 and 5:*

		18887	18887	18887	18887	18887
	2222	2525	3333	5252	5555	18887
18887	5233	5552	2255	2522	3325	18887
18887	2555	3322	5225	5533	2252	18887
18887	5525	2233	2552	3355	5222	18887
18887	3352	5255	5522	2225	2533	18887
	18887	18887	18887	18887	18887	18887

Example 70. *Let's consider a **pandiagonal** magic square of order 5 with digits 1, 3 and 8:*

		23331	23331	23331	23331	23331
	1111	1818	3333	8181	8888	23331
23331	8133	8881	1188	1811	3318	23331
23331	1888	3311	8118	8833	1181	23331
23331	8818	1133	1881	3388	8111	23331
23331	3381	8188	8811	1118	1833	23331
	23331	23331	23331	23331	23331	23331

Example 71. *Let's consider a **pandiagonal** magic square of order 5 with digits 0, 3 and 8:*

		21109	21109	21109	21109	21109
	0000	0808	3333	8080	8888	21109
21109	8033	8880	0088	0800	3308	21109
21109	0888	3300	8008	8833	0080	21109
21109	8808	0033	0880	3388	8000	21109
21109	3380	8088	8800	0008	0833	21109
	21109	21109	21109	21109	21109	21109

Example 72. *Let's consider a **pandiagonal** magic square of order 5 with digits 0, 1 and 3:*

		5555	5555	5555	5555	5555
	0000	0101	1010	1111	3333	5555
5555	1110	3311	0033	0100	1001	5555
5555	0133	1000	1101	3310	0011	5555
5555	3301	0010	0111	1033	1100	5555
5555	1011	1133	3300	0001	0110	5555
	5555	5555	5555	5555	5555	5555

Let's write the above magic squares in **special/digital fonts**

2222	2525	3333	5252	5555
5233	5552	2255	2522	3325
2555	3322	5225	5533	2252
5525	2233	2552	3355	5222
3352	5255	5522	2225	2533

1111	1818	3333	8181	8888
8133	8881	1188	1811	3318
1888	3311	8118	8833	1181
8818	1133	1881	3388	8111
3381	8188	8811	1118	1833

0000	0808	3333	8080	8888
8033	8880	0088	0800	3308
0888	3300	8008	8833	0080
8808	0033	0880	3388	8000
3380	8088	8800	0008	0833

0000	0101	1010	1111	3333
1110	3311	0033	0100	1001
0133	1000	1101	3310	0011
3301	0010	0111	1033	1100
1011	1133	3300	0001	0110

7.2.1 Water Reflection

Let's write the above magic square with **vertical-flip**, i.e., **water-reflexive** images:

3325	2522	2255	5552	5233
2252	5533	5225	3322	2555
5222	3355	2552	2233	5525
2533	2225	5522	5255	3352
5555	5252	3333	2525	2222

3381	8188	8811	1118	1833
8818	1133	1881	3388	8111
1888	3311	8118	8833	1181
8133	8881	1188	1811	3318
1111	1818	3333	8181	8888

3380	8088	8800	0008	0833
8808	0033	0880	3388	8000
0888	3300	8008	8833	0080
8033	8880	0088	0800	3308
0000	0808	3333	8080	8888

1011	1133	3300	0001	0110
3301	0010	0111	1033	1100
0133	1000	1101	3310	0011
1110	3311	0033	0100	1001
0000	0101	1010	1111	3333

Let's check the sum of above magic squares:

Example 73. *A* ***pandiagonal*** *magic square of order 5 with digits 2, 3 and 5:*

		18887	18887	18887	18887	18887
	2222	2525	3333	5252	5555	18887
18887	5233	5552	2255	2522	3325	18887
18887	2555	3322	5225	5533	2252	18887
18887	5525	2233	2552	3355	5222	18887
18887	3352	5255	5522	2225	2533	18887
	18887	18887	18887	18887	18887	18887

Example 74. *A **pandiagonal** magic square of order 5 with digits 1, 3 and 8:*

		23331	23331	23331	23331	23331
	1111	1818	3333	8181	8888	23331
23331	8133	8881	1188	1811	3318	23331
23331	1888	3311	8118	8833	1181	23331
23331	8818	1133	1881	3388	8111	23331
23331	3381	8188	8811	1118	1833	23331
	23331	23331	23331	23331	23331	23331

Example 75. *A **pandiagonal** magic square of order 5 with digits 0, 3 and 8:*

		21109	21109	21109	21109	21109
	0000	0808	3333	8080	8888	21109
21109	8033	8880	0088	0800	3308	21109
21109	0888	3300	8008	8833	0080	21109
21109	8808	0033	0880	3388	8000	21109
21109	3380	8088	8800	0008	0833	21109
	21109	21109	21109	21109	21109	21109

Example 76. *A **pandiagonal** magic square of order 5 with digits 0, 1 and 3:*

		5555	5555	5555	5555	5555
	0000	0101	1010	1111	3333	5555
5555	1110	3311	0033	0100	1001	5555
5555	0133	1000	1101	3310	0011	5555
5555	3301	0010	0111	1033	1100	5555
5555	1011	1133	3300	0001	0110	5555
	5555	5555	5555	5555	5555	5555

Thus, all the above four **pandiagonal** magic squares of order 5 are **water-reflexive**.

7.3 2-Digits Magic Squares

Let's consider following three magic squares of order 5 with 2 digits.

Example 77. *Let's consider a **pandiagonal** magic square of order 5 with 2-digits 3 and 8 is given by*

		3383380	3383380	3383380	3383380	3383380
	383383	388388	838838	883883	888888	3383380
3383380	883838	888883	383888	388383	838388	3383380
3383380	388888	838383	883388	888838	383883	3383380
3383380	888388	383838	388883	838888	883383	3383380
3383380	838883	883888	888383	383388	388838	3383380
	3383380	3383380	3383380	3383380	3383380	3383380

Example 78. *Let's consider a **pandiagonal** magic square of order 5 with 2-digits 3 and 8 is given by*

		979979	979979	979979	979979	979979
	111111	113113	131131	311311	313313	979979
979979	311131	313311	111313	113111	131113	979979
979979	113313	131111	311113	313131	111311	979979
979979	313113	111131	113311	131313	311111	979979
979979	131311	311313	313111	111113	113131	979979
	979979	979979	979979	979979	979979	979979

Example 79. *Let's consider a* ***pandiagonal*** *magic square of order 5 with 2-digits 3 and 8 is given by*

		636636	636636	636636	636636	636636
	000000	003003	030030	300300	303303	636636
636636	300030	303300	000303	003000	030003	636636
636636	003303	030000	300003	303030	000300	636636
636636	303003	000030	003300	030303	300000	636636
636636	030300	300303	303000	000003	003030	636636
	636636	636636	636636	636636	636636	636636

Let's write the above magic squares with **special fonts**

383383	388388	838838	883883	888888
883838	888883	383888	388383	838388
388888	838383	883388	888838	383883
888388	383838	388883	838888	883383
838883	883888	888383	383388	388838

111111	113113	131131	311311	313313
311131	313311	111313	113111	131113
113313	131111	311113	313131	111311
313113	111131	113311	131313	311111
131311	311313	313111	111113	113131

000000	003003	030030	300300	303303
300030	303300	000303	003000	030003
003303	030000	300003	303030	000300
303003	000030	003300	030303	300000
030300	300303	303000	000003	003030

7.3.1 Water Reflection

Let's write the above magic squares with **water reflection** images:

838883	883888	888383	383388	388838
888388	383838	388883	838888	883383
388888	838383	883388	888838	383883
883838	888883	383888	388383	838388
383383	388388	838838	883883	888888

131311	311313	313111	111113	113131
313113	111131	113311	131313	311111
113313	131111	311113	313131	111311
311131	313311	111313	113111	131113
111111	113113	131131	311311	313313

030300	300303	303000	000003	003030
303003	000030	003300	030303	300000
003303	030000	300003	303030	000300
300030	303300	000303	003000	030003
000000	003003	030030	300300	303303

Let's check the sum of above 3 magic squares.

Example 80. *A **pandiagonal** magic square of order 5 with 2-digits 3 and 8 is given by*

		3383380	3383380	3383380	3383380	3383380
	838883	883888	888383	383388	388838	3383380
3383380	888388	383838	388883	838888	883383	3383380
3383380	388888	838383	883388	888838	383883	3383380
3383380	883838	888883	383888	388383	838388	3383380
3383380	383383	388388	838838	883883	888888	3383380
	3383380	3383380	3383380	3383380	3383380	3383380

Example 81. *A **pandiagonal** magic square of order 5 with 2-digits 3 and 8 is given by*

		979979	979979	979979	979979	979979
	131311	311313	313111	111113	113131	979979
979979	313113	111131	113311	131313	311111	979979
979979	113313	131111	311113	313131	111311	979979
979979	311131	313311	111313	113111	131113	979979
979979	111111	113113	131131	311311	313313	979979
	979979	979979	979979	979979	979979	979979

Example 82. *A **pandiagonal** magic square of order 5 with 2-digits 3 and 8 is given by*

		636636	636636	636636	636636	636636
	030300	300303	303000	000003	003030	636636
636636	303003	000030	003300	030303	300000	636636
636636	003303	030000	300003	303030	000300	636636
636636	300030	303300	000303	003000	030003	636636
636636	000000	003003	030030	300300	303303	636636
	636636	636636	636636	636636	636636	636636

8 Magic Squares of Order 6

Example 83. *Let's consider a magic square of order 6.*

						111
1	35	34	33	2	6	111
30	8	28	9	11	25	111
24	23	15	16	20	13	111
18	14	21	22	17	19	111
7	26	10	27	29	12	111
31	5	3	4	32	36	111
111	111	111	111	111	111	111

8.1 Palindromic Representations

Let's consider six letters a, b, c, d, e and f, where $a, b, c, d, e, f \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We can make exactly 36 palindromes of 3-digits with these six letters:

Table 8.1. *The palindromes are as follows:*

1	2	3	4	5	6	7	8	9	10	11	12
aaa	aba	aca	ada	aea	afa	bab	bbb	bcb	bdb	beb	bfb
13	14	15	16	17	18	19	20	21	22	23	24
cac	cbc	ccc	cdc	cec	cfc	dad	dbd	dcd	ddd	ded	dfd
25	26	27	28	29	30	31	32	33	34	35	36
eae	ebe	ece	ede	eee	efe	faf	fbf	fcf	fdf	fef	fff

Replacing the above values with their respective palindromes in a magic square of order 6 given in Example 100 , we get a **palindromic** grid given below:

Grid 8. *Using six letters a, b, c, d, e and f, we have only 36 palindromes of 3-digits. This allows us to write as the following palindromic grid:*

aaa	*fef*	*fdf*	*fcf*	*aba*	*afa*
efe	*bbb*	*ede*	*bcb*	*beb*	*eae*
dfd	*ded*	*ccc*	*cdc*	*dbd*	*cac*
cfc	*cbc*	*dcd*	*ddd*	*cec*	*dad*
bab	*ebe*	*bdb*	*ece*	*eee*	*bfb*
faf	*aea*	*aca*	*ada*	*fbf*	*fff*

where $a, b, c, d, e, f \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

For all $a, b, c, d, e, f \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the grid given in 8 represents a palindromic magic square of order 6, if exists, then its sum is given by

$$S_{6\times 6}(a, b, c, d, e, f) := (a + b + c + d + e + f) \times 111.$$

Let's see below some examples:

Example 84. *For $a = 1, b = 2, c = 3, d = 4, e = 5$ and $f = 6$ in Grid 8, the 3-digits **palindromic** magic square of order 6 with magic sum $S_{6\times 6}(1, 2, 3, 4, 5, 6) := (1 + 2 + 3 + 4 + 5 + 6) \times 111 = 2331$ is given by.*

						2331
111	656	646	636	121	161	2331
565	222	545	232	252	515	2331
464	454	333	343	424	313	2331
363	323	434	444	353	414	2331
212	525	242	535	555	262	2331
616	151	131	141	626	666	2331
2331	2331	2331	2331	2331	2331	2331

Here we are not sure whether we shall have a magic square of order 6 for non-sequential numbers. See the example below:

Example 85. *For $a = 1, b = 2, c = 3, d = 5, e = 7$ and $f = 9$ in Grid 8, we following grid of order 6:*

						2997
111	979	959	939	121	191	3300
797	222	757	232	272	717	2997
595	575	333	353	525	313	2694
393	323	535	555	373	515	2694
212	727	252	737	777	292	2997
919	171	131	151	929	999	3300
3027	2997	2997	2997	2997	3027	2997

Thus we observe that considering no consecutive entries, we are unable to bring a magic square. Below is another example of consecutive entries giving a magic square of order 6.

Example 86. *For $a=3, b=4, c=5, d=6, e=7$ and $f=8$ in Grid 8, the 3-digits* ***palindromic*** *magic square of order 6 with magic sum $S_{6\times 6}(1,2,3,4,5,6) := (3+4+5+6+7+8)\times 111 = 3663$ is given by*

						3663
333	878	868	858	343	383	3663
787	444	767	454	474	737	3663
686	676	555	565	646	535	3663
585	545	656	666	575	636	3663
434	747	464	757	777	484	3663
838	373	353	363	848	888	3663
3663	3663	3663	3663	3663	3663	3663

8.2 Composite Magic Squares

Grid 9. *Eliminating the third values in Grid 8, and then splitting in two single digits grids, we get*

a	f	f	f	a	a
e	b	e	b	b	e
d	d	c	c	d	c
c	c	d	d	c	d
b	e	b	e	e	b
f	a	a	a	f	f
			A		

a	e	d	c	b	f
f	b	d	c	e	a
f	e	c	d	b	a
f	b	c	d	e	a
a	b	d	c	e	f
a	e	c	d	b	f
			B		

aa	fe	fd	fc	ab	af
ef	bb	ed	bc	be	ea
df	de	cc	cd	db	ca
cf	cb	dc	dd	ce	da
ba	eb	bd	ec	ee	bf
fa	ae	ac	ad	fb	ff
			A		

The grid AB can be written as

$$AB := 10 \times A + B$$

Example 87. *In particular for $a = 1, b = 2, c = 3, d = 4, e = 5$ and $e = 6$, we get*

1	6	6	6	1	1
5	2	5	2	2	5
4	4	3	3	4	3
3	3	4	4	3	4
2	5	2	5	5	2
6	1	1	1	6	6
			A		

1	5	4	3	2	6
6	2	4	3	5	1
6	5	3	4	2	1
6	2	3	4	5	1
1	2	4	3	5	6
1	5	3	4	2	6
			B		

11	65	64	63	12	16
56	22	54	23	25	51
46	45	33	34	42	31
36	32	43	44	35	41
21	52	24	53	55	26
61	15	13	14	62	66
			AB		

Note 8.1. *Applying $6 \times (A - 1) + B$ over the elements of A and B given above, we get a magic square of order 6 given in Example 100. We observe that both the grid A and B don't obey very much the Latin square rules. Below is another example, where at least one of the grid is diagonal Latin square.*

Example 88. *Alternative way of composite magic square is given by*

2	1	4	5	6	3
1	3	6	4	5	2
3	6	5	2	4	1
4	2	1	6	3	5
5	4	2	3	1	6
6	5	3	1	2	4
			A		

5	2	3	4	2	5
1	3	5	5	1	6
1	3	3	2	6	6
2	4	3	4	6	2
6	4	3	2	5	1
6	5	4	4	1	1
			B		

25	12	43	54	62	35
11	33	65	45	51	26
31	63	53	22	46	16
42	24	13	64	36	52
56	44	23	32	15	61
66	55	34	14	21	41
			AB		

Note 8.2. *In this example, the grid A is diagonal Latin square. The Example 87 helps in bringing block-wise magic squares multiple of $6k$ (ref. Taneja [24]).*

Below is exceptional example of **composite** magic square in terms of three letters 2, 5 and 8.

Example 89. *For $a = 25, b = 28, c = 52, d = 58, e = 82$ and $f = 85$ in Grid 9, we get following* ***upside down*** *magic square of order 6:*

						33330
2525	*8582*	*8558*	*8552*	*2528*	*2585*	*33330*
8285	*2828*	*8258*	*2852*	*2882*	*8225*	*33330*
5885	*5882*	*5252*	*5258*	*5828*	*5225*	*33330*
5285	*5228*	*5852*	*5858*	*5282*	*5825*	*33330*
2825	*8228*	*2858*	*8252*	*8282*	*2885*	*33330*
8525	*2582*	*2552*	*2558*	*8528*	*8585*	*33330*
33330	*33330*	*33330*	*33330*	*33330*	*33330*	*33330*

Below is **digital** version of above magic square:

2525	8582	8558	8552	2528	2585
8285	2828	8258	2852	2882	8225
5885	5882	5252	5258	5828	5225
5285	5228	5852	5858	5282	5825
2825	8228	2858	8252	8282	2885
8525	2582	2552	2558	8528	8585

This magic square is **upside-down** but not **mirror looking**. See below:

- 180^0 **Rotation**

Rotating above magic square in 180^o, we get following magic square:

5858	8258	8552	2552	2852	5258
5882	2828	2528	8582	8228	5282
5285	2825	8585	2585	8225	5825
5225	8285	8525	2525	2885	5885
5228	2882	2582	8528	8282	5828
5852	8252	2558	8558	2858	5252

Note 8.3. *Let's check below that it is a magic square:*

						33330
5858	8258	8552	2552	2852	5258	33330
5882	2828	2528	8582	8228	5282	33330
5285	2825	8585	2585	8225	5825	33330
5225	8285	8525	2525	2885	5885	33330
5228	2882	2582	8528	8282	5828	33330
5852	8252	2558	8558	2858	5252	33330
33330	33330	33330	33330	33330	33330	33330

• **Mirror Looking**

2825	8525	5228	8228	5828	2525
2558	5885	5285	8258	8585	2858
2552	8582	8252	5252	5882	2882
2582	5852	8282	5282	8552	2852
2885	5858	5258	8285	8558	2585
2828	8528	8225	5225	5825	2528

Note 8.4. *As we have seen above the magic square is* ***upside down****. It is not* ***mirror looking*** *magic square. Let's check it by calculating sums of row, columns and principal diagonals:*

						27927
2825	8525	5228	8228	5828	2525	33159
2558	2558	5885	8258	8585	2858	30702
2552	2552	8582	5252	5882	2882	27702
2582	2582	5852	5282	8552	2852	27702
2885	2885	5858	8285	8558	2585	31056
2828	2828	8528	5225	5825	2528	27762
16230	21930	39933	40530	43230	16230	30333

Thus we observe that it is not a magic square.

Let's see another example of **upside-down**.

Example 90. *For* $a = 15, b = 19, c = 51, d = 59, e = 91$ *and* $f = 95$ *in Grid 9, we get following* ***upside down*** *magic square of order 6:*

						33330
1515	9591	9559	9551	1519	1595	33330
9195	1919	9159	1951	1991	9115	33330
5995	5991	5151	5159	5919	5115	33330
5195	5119	5951	5959	5191	5915	33330
1915	9119	1959	9151	9191	1995	33330
9515	1591	1551	1559	9519	9595	33330
33330	33330	33330	33330	33330	33330	33330

Below is a ***digital version*** *of above magic square:*

1515	9591	9559	9551	1519	1595
9195	1919	9159	1951	1991	9115
5995	5991	5151	5159	5919	5115
5195	5119	5951	5959	5191	5915
1915	9119	1959	9151	9191	1995
9515	1591	1551	1559	9519	9595

Note 8.5. *We can easily check that it is not an* ***upside-down****. The* ***upside-down*** *version of above magic square is given by*

5656	6156	6551	1551	1651	5156
5661	1616	1516	6561	6116	5161
5165	1615	6565	1565	6115	5615
5115	6165	6515	1515	1665	5665
5116	1661	1561	6516	6161	5616
5651	6151	1556	6556	1656	5151

Let's check the sums of row, columns and principal diagonals:

						26664
5656	6165	6551	1551	1651	5156	26730
5661	1616	1516	6561	6116	5161	26631
5165	1615	6565	1565	6115	5615	26640
5115	6165	6515	1515	1665	5665	26640
5116	1661	1561	6516	6161	5616	26631
5651	6151	1665	6556	1656	5151	26830
32364	23373	24373	24264	23364	32364	26664

It is no more a magic square.

Finally, there is only magic square of order 6, we found as **upside-down** given in Example 89.

8.3 2-Digits Entries

8.3.1 The Numbers 1 and 8

Example 91. *For* $a = 111, b = 118, c = 181, d = 818, e = 881$ *and* $f = 888$ *in Grid 9, we get following* ***universal*** *magic square of order 6 for 2-digits 1 and 8:*

						2999997
888888	111818	111881	111118	888181	888111	2999997
181111	818181	181881	818118	818818	181888	2999997
188111	188818	811118	811881	188181	811888	2999997
811111	811181	188118	188881	811818	188888	2999997
818888	181181	818881	181118	181818	818111	2999997
111888	888818	888118	888881	111181	111111	2999997
2999997	2999997	2999997	2999997	2999997	2999997	2999997

Let's write the above magic square without sums in ***special fonts***

888888	111818	111881	111118	888181	888111
181111	818181	181881	818118	818818	181888
188111	188818	811118	811881	188181	811888
811111	811181	188118	188881	811818	188888
818888	181181	818881	181118	181818	818111
111888	888818	888118	888881	111181	111111

- 180^o **Rotation**

111111	181111	188888	811888	818888	888111
111818	818181	811181	188818	181181	888818
888881	818118	188881	811881	181118	111118
888118	181881	188118	811118	818881	111881
888181	818818	811818	188181	181818	111181
111888	181888	811111	188111	818111	888888

Let's check the sums of rows,columns and of principal diagonals:

						2999997
111111	*181111*	*188888*	*811888*	*818888*	*888111*	*2999997*
111818	*818181*	*811181*	*188818*	*181181*	*888818*	*2999997*
888881	*818118*	*188881*	*811881*	*181118*	*111118*	*2999997*
888118	*181881*	*188118*	*811118*	*818881*	*111881*	*2999997*
888181	*818818*	*811818*	*188181*	*181818*	*111181*	*2999997*
111888	*181888*	*811111*	*188111*	*818111*	*888888*	*2999997*
2999997	*2999997*	*2999997*	*2999997*	*2999997*	*2999997*	*2999997*

- **Mirror Looking**

111888	181888	811111	188111	818111	888888
888181	818818	811818	188181	181818	111181
888118	181881	188118	811118	818881	111881
888881	818118	188881	811881	181118	111118
111818	818181	811181	188818	181181	888818
111111	181111	188888	811888	818888	888111

Let's check the sums of rows, columns and of principal diagonals:

						2999997
111888	181888	811111	188111	818111	888888	2999997
888181	818818	811818	188181	181818	111181	2999997
888118	181881	188118	811118	818881	111881	2999997
888881	818118	188881	811881	181118	111118	2999997
111818	818181	811181	188818	181181	888818	2999997
111111	181111	188888	811888	818888	888111	2999997
2999997	2999997	2999997	2999997	2999997	2999997	2999997

Thus, we have a **universal** magic square of order 6 for the digits 1 and 8. In all the three cases, the magic sums is same, i.e., $S_{6\times 6}(1,8) := 2999997$

8.3.2 The Numbers 2 and 5

Example 92. *For $a = 222, b = 252, c = 255, d = 522, e = 525$ and $f = 555$ in Grid 9, we get following **universal** magic square of order 6 for 2-digits 2 and 5:*

						2333331
555555	222525	222552	222225	555252	555222	2333331
252222	525252	252552	525225	525525	252555	2333331
255222	255525	522225	522552	255252	522555	2333331
522222	522252	255225	255552	522525	255555	2333331
525555	252252	525552	252225	252525	525222	2333331
222555	555525	555225	555552	222252	222222	2333331
2333331	2333331	2333331	2333331	2333331	2333331	2333331

*Let's write the above magic square without sums in **digital fonts***

555555	222525	222552	222225	555252	555222
252222	525252	252552	525225	525525	252555
255222	255525	522225	522552	255252	522555
522222	522252	255225	255552	522525	255555
525555	252252	525552	252225	252525	525222
222555	555525	555225	555552	222252	222222

- 180^o **Rotation**

111111	181111	188888	811888	818888	888111
111818	818181	811181	188818	181181	888818
888881	818118	188881	811881	181118	111118
888118	181881	188118	811118	818881	111881
888181	818818	811818	188181	181818	111181
111888	181888	811111	188111	818111	888888

Let's check the sums of rows,columns and of principal diagonals:

						2333331
222222	*252222*	*255555*	*522555*	*525555*	*555222*	*2333331*
222525	*525252*	*522252*	*255525*	*252252*	*555525*	*2333331*
555552	*525225*	*255552*	*522552*	*252225*	*222225*	*2333331*
555225	*252552*	*255225*	*522225*	*525552*	*222552*	*2333331*
555252	*525525*	*522525*	*255252*	*252525*	*222252*	*2333331*
222555	*252555*	*522222*	*255222*	*525222*	*555555*	*2333331*
2333331	*2333331*	*2333331*	*2333331*	*2333331*	*2333331*	*2333331*

- **Mirror Looking**

555222	525222	255555	522555	252555	222222
222525	252252	255252	522525	525252	555525
222552	525225	522552	255552	252225	555225
222225	252552	522225	255225	525552	555552
555252	252525	255525	522252	525525	222252
555555	525555	522222	255222	252222	222555

Let's check the sums of rows,columns and of principal diagonals:

						2333331
555222	525222	255555	522555	252555	222222	2333331
222525	252252	255252	522525	525252	555525	2333331
222552	525225	522552	255552	252225	555225	2333331
222225	252552	522225	255225	525552	555552	2333331
555252	252525	255525	522252	525525	222252	2333331
555555	525555	522222	255222	252222	222555	2333331
2333331	2333331	2333331	2333331	2333331	2333331	2333331

Thus, we have a **universal** magic square of order 6 for the digits 2 and 5. In all the three cases, the magic sums is same, i.e., $S_{6\times6}(2,5) := 2333331$

8.3.3 The Numbers 6 and 9

Example 93. *For* $a = 666, b = 696, c = 699, d = 966, e = 969$ *and* $f = 999$ *in Grid 9, we get following* ***universal*** *magic square of order 6 for 2-digits 6 and 9:*

						4999995
999999	666969	666996	666669	999696	999666	4999995
696666	969696	696996	969669	969969	696999	4999995
699666	699969	966669	966996	699696	966999	4999995
966666	966696	699669	699996	966969	699999	4999995
969999	696696	969996	696669	696969	969666	4999995
666999	999969	999669	999996	666696	666666	4999995
4999995	4999995	4999995	4999995	4999995	4999995	4999995

Let's write the above magic square with ***special fonts***

999999	666969	666996	666669	999696	999666
696666	969696	696996	969669	969969	696999
699666	699969	966669	966996	699696	966999
966666	966696	699669	699996	966969	699999
969999	696696	969996	696669	696969	969666
666999	999969	999669	999996	666696	666666

- 180^o **Rotation**

999999	969999	966666	699666	696666	666999
999696	696969	699969	966696	969969	666696
666669	696996	966669	699669	969996	999996
666996	969669	966996	699996	696669	999669
666969	696696	699696	966969	969696	999969
999666	969666	699999	966999	696999	666666

Let's check the sums of rows, columns and principal diagonals:

						4999995
999999	969999	966666	699666	696666	666999	4999995
999696	696969	699969	966696	969969	666696	4999995
666669	696996	966669	699669	969996	999996	4999995
666996	969669	966996	699996	696669	999669	4999995
666969	696696	699696	966969	969696	999969	4999995
999666	969666	699999	966999	696999	666666	4999995
4999995	4999995	4999995	4999995	4999995	4999995	4999995

Thus, we have an **upside-down** of order 6 for 2-digits 6 and 9 with **magic sum** $S_{6\times 6}(6,9) := 4999995$.

9 Water Reflection Magic Squares of Order 6

9.1 3-Digits Magic Squares

Let's consider following magic square of order 6 in 3 digits 1, 2 and 3.

Example 94. *Let's consider a magic square of order 6.*

						13332
1212	*3231*	*3223*	*3221*	*1213*	*1232*	*13332*
3132	*1313*	*3123*	*1321*	*1331*	*3112*	*13332*
2332	*2331*	*2121*	*2123*	*2313*	*2112*	*13332*
2132	*2113*	*2321*	*2323*	*2131*	*2312*	*13332*
1312	*3113*	*1323*	*3121*	*3131*	*1332*	*13332*
3212	*1231*	*1221*	*1223*	*3213*	*3232*	*13332*
13332	*13332*	*13332*	*13332*	*13332*	*13332*	*13332*

Let's write the above magic square with **digital fonts**:

1212	3231	3223	3221	1213	1232
3132	1313	3123	1321	1331	3112
2332	2331	2121	2123	2313	2112
2132	2113	2321	2323	2131	2312
1312	3113	1323	3121	3131	1332
3212	1231	1221	1223	3213	3232

Remark 9.1. *Even though the above magic square is possible to write in digital fonts, but it is not **water-reflexive**. The reason is that when we make **vertical flip**, 2 becomes 5 and the block of order 6×6 with digits 1, 3 and 5 is no more magic square. But we have examples of **water-reflexive** of order 6 with 2-digits given in following subsection.*

9.2 2-Digits Magic Squares

Let's consider below three magic squares in 2-digit using the digits 0, 1, 3 and 8.

Example 95. *Let's consider a magic square of order 6 with 2 digits 3 and 8.*

						3666663
333333	888883	888838	888383	333338	333888	3666663
838888	383338	838838	383383	383883	838333	3666663
833888	833883	388383	388838	833338	388333	3666663
388888	388338	833383	833838	388883	833333	3666663
383333	838338	383838	838383	838883	383888	3666663
888333	333883	333383	333838	888338	888888	3666663
3666663	3666663	3666663	3666663	3666663	3666663	3666663

Example 96. *Let's consider a magic square of order 6 with 2 digits 1 and 3.*

						1333332
111111	333331	333313	333131	111113	111333	1333332
313333	131113	313313	131131	131331	313111	1333332
311333	311331	133131	133313	311113	133111	1333332
133333	133113	311131	311313	133331	311111	1333332
131111	313113	131313	313131	313331	131333	1333332
333111	111331	111131	111313	333113	333333	1333332
1333332	1333332	1333332	1333332	1333332	1333332	1333332

Example 97. *Let's consider a magic square of order 6 with 2 digits 0 and 3.*

						999999
000000	333330	333303	333030	000003	000333	999999
303333	030003	303303	030030	030330	303000	999999
300333	300330	033030	033303	300003	033000	999999
033333	033003	300030	300303	033330	300000	999999
030000	303003	030303	303030	303330	030333	999999
333000	000330	000030	000303	333003	333333	999999
999999	999999	999999	999999	999999	999999	999999

Let's write the above magic squares with **special fonts**:

333333	888883	888838	888383	333338	333888
838888	383338	838838	383383	383883	838333
833888	833883	388383	388838	833338	388333
388888	388338	833383	833838	388883	833333
383333	838338	383838	838383	838883	383888
888333	333883	333383	333838	888338	888888

111111	333331	333313	333131	111113	111333
313333	131113	313313	131131	131331	313111
311333	311331	133131	133313	311113	133111
133333	133113	311131	311313	133331	311111
131111	313113	131313	313131	313331	131333
333111	111331	111131	111313	333113	333333

000000	333330	333303	333030	000003	000333
303333	030003	303303	030030	030330	303000
300333	300330	033030	033303	300003	033000
033333	033003	300030	300303	033330	300000
030000	303003	030303	303030	303330	030333
333000	000330	000030	000303	333003	333333

9.2.1 Water Reflection

Let's write the above magic square with **vertical-flip**, i.e., **water-reflexive** images:

888333	333883	333383	333838	888338	888888
383333	838338	383838	838383	838883	383888
388888	388338	833383	833838	388883	833333
833888	833883	388383	388838	833338	388333
838888	383338	838838	383383	383883	838333
333333	888883	888838	888383	333338	333888

333111	111331	111131	111313	333113	333333
131111	313113	131313	313131	313331	131333
133333	133113	311131	311313	133331	311111
311333	311331	133131	133313	311113	133111
313333	131113	313313	131131	131331	313111
111111	333331	333313	333131	111113	111333

333000	000330	000030	000303	333003	333333
030000	303003	030303	303030	303330	030333
033333	033003	300030	300303	033330	300000
300333	300330	033030	033303	300003	033000
303333	030003	303303	030030	030330	303000
000000	333330	333303	333030	000003	000333

Let's check the sum of above magic squares.

Example 98. *Let's consider a magic square of order 6 with 2 digits 3 and 8.*

						3666663
888333	333883	333383	333838	888338	888888	3666663
383333	838338	383838	838383	838883	383888	3666663
388888	388338	833383	833838	388883	833333	3666663
833888	833883	388383	388838	833338	388333	3666663
838888	383338	838838	383383	383883	838333	3666663
333333	888883	888838	888383	333338	333888	3666663
3666663	3666663	3666663	3666663	3666663	3666663	3666663

Example 99. *Let's consider a magic square of order 6 with 2 digits 1 and 3.*

						1333332
333111	111331	111131	111313	333113	333333	1333332
131111	313113	131313	313131	313331	131333	1333332
133333	133113	311131	311313	133331	311111	1333332
311333	311331	133131	133313	311113	133111	1333332
313333	131113	313313	131131	131331	313111	1333332
111111	333331	333313	333131	111113	111333	1333332
1333332	1333332	1333332	1333332	1333332	1333332	1333332

Example 100. *Let's consider a magic square of order 6 with 2 digits 0 and 3.*

						999999
333000	000330	000030	000303	333003	333333	999999
030000	303003	030303	303030	303330	030333	999999
033333	033003	300030	300303	033330	300000	999999
300333	300330	033030	033303	300003	033000	999999
303333	030003	303303	030030	030330	303000	999999
000000	333330	333303	333030	000003	000333	999999
999999	999999	999999	999999	999999	999999	999999

Thus, we have 3 magic squares of order 6 with 2-digits having **water-reflexive** property.

10 Author's Contributions to Magic Squares

For author's contribution to **magic squares** and **recreation numbers** please see the links below:

- **Inder J. Taneja**, Magic Squares,
  1. *https://inderjtaneja.wordpress.com/2019/06/27/publications-magic-squares/*
  2. *https://numbers-magic.com/?p=668*
- **Inder J. Taneja**, Recreation of Numbers,
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