27.

De resolutione algebraica aequationis $X^{257}=1$, sive de divisione circuli per bisectionem anguli septies repetitam in partes 257 inter se aequales commentatio coronata.

(Cont. Diss. Vol. IX. Fasc. 1., 2. et 3.)

(Auct. *Richelot*, Doct. phil. Regiom.)

XII.

Transeamus ad hoc ultimum negotium, nimirum ad valores functionum f ex functionibus F determinandos. Numerus functionum f est $=128$, inter quas una est determinata semper eadem:

$$f_{128} = p_0 + p_1 + p_2 + \text{etc.} + p_{127} = f_{128m} = -1.$$

Iam vero ad ceteras 127 determinandas functiones f, inter eas ipsas hanc aequationem generalem constare, clarum est:

$$f_x . f_x = F^1 R^x . f_{2x} = F_x . f_{2x},$$

si loco ipsius $F^1 R^x$ signum hoc F_x introducitur brevitatis causa. Ibi pro x sensim numeris 1, 2, 4, 8, 16, 32, 64, 128 substitutis, oriuntur hae aequationes:

$$26. \quad \begin{cases} f_1^2 = F_1 . f_2, \quad f_2^2 = F_2 . f_4, \quad f_4^2 = F_4 . f_8, \quad f_8^2 = F_8 . f_{16}, \\ f_{16}^2 = F_{16} . f_{32}, \quad f_{32}^2 = F_{32} . f_{64}, \quad f_{64}^2 = F_{64} . f_{128}, \quad f_{128}^2 = F_{128} . f_{128}. \end{cases}$$

His vero aequationibus adhibitis fit

$$27. \quad f_1^{128} = (F_1)^{64} . (F_2)^{32} . (F_4)^{16} . (F_8)^8 . (F_{16})^4 . (F_{32})^2 . F_{64} . F_{128}.$$

Quae aequatio centisimi vigesimi octavi ordinis totidem secum fert valores quantitatis determinandae f_1 tales:

$$z, \quad zR, \quad zR^2, \quad zR^3, \quad \text{etc.} \quad zR^{127},$$

si z est una positiva realis radix aequationis (27.), sive:

$$z = \sqrt[128]{(F_1^{64} . F_2^{32} . F_4^{16} . F_8^8 . F_{16}^4 . F_{32}^2 . F_{64} . F_{128})}.$$

Sed functionem f_1 128 valoribus diversis uti debere, ex initio articuli huius facile intelligitur. Ibi invenitur:

$$f_1 = p_0 + p_1 R + p_1 R^2 + \text{etc.} + p_{127} R^{127},$$

ubi quantitates

$$p_0, \; p_1, \; p_2, \; \text{etc.} = [2, 1], = [2, 3], = [2, 3^2], \text{etc.}$$

adhuc in 128 suppositionibus stare potuerunt, prout fuit:

$$\sigma = \cos\frac{2\pi}{257} \pm i\sin\frac{2\pi}{257}, \quad \text{vel} \ = \cos\frac{4\pi}{257} \pm i\sin\frac{4\pi}{257}, \quad \text{etc.}$$
$$\text{vel} \ = \cos\frac{256\pi}{257} \pm i\sin\frac{256\pi}{257},$$

quippe quorum 128 valorum, σ indeterminatum quemlibet habente, etiam quinam 128 suorum valorum functioni f_1 tribuatur, prorsus arbitrio nostro permitti posse, concluditur.

Ponamus igitur $f_1 = z$. Inde derivantur, aequationibus (26.) rursus adhibitis, nec non theorematibus (6.) et (7.) revocatis:

$$28. \quad \begin{cases} f_1 = \sqrt[128]{(F_1^{64}.F_2^{32}.F_4^{16}.F_8^{8}.F_{16}^{4}.F_{32}^{2}.257)}, \\ f_2 = \dfrac{f_1^2}{F_2} = \sqrt[64]{(F_2^{32}.F_4^{16}.F_8^{8}.F_{16}^{4}.F_{32}^{2}.257)}, \\ f_4 = \dfrac{f_2^2}{F_2} = \sqrt[32]{(F_4^{16}.F_8^{8}.F_{16}^{4}.F_{32}^{2}.257)}, \\ f_8 = \dfrac{f_4^2}{F_4} = \sqrt[16]{(F_8^{8}.F_{16}^{4}.F_{32}^{2}.257)}, \\ f_{16} = \dfrac{f_8^2}{F_8} = \sqrt[8]{(F_{16}^{4}.F_{32}^{2}.257)}, \\ f_{32} = \dfrac{f_{16}^2}{F_{16}} = \sqrt[4]{(F_{32}.257)}, \\ f_{64} = \dfrac{f_{32}^2}{F_{32}} = \sqrt[2]{(257)}, \\ f_{128} = \dfrac{f_{64}^2}{F_{64}} = -1, \end{cases}$$

ubi signum radicale $\sqrt[h]{a}$ semper unam realem positivam radicem aequationis $X^h = a$ significat.

Ut ceterae 120 functiones adhuc indeterminatae, definiantur, primum earum numerus quam minimus reddatur. Quam ob rem theoremate (19.) revocato:

$$f_x\, f_{128-x} = 257;$$

iam colligitur, nonnisi functiones f_3 usque ad f_{63} remanere determinandas.

In aequationibus (28.), prima excepta, R^{2n+1} loco ipsius R posito, invenimus has:

$$28'. \quad f_{2(2n+1)} = \frac{f^2_{(2n+1)}}{F_{(2n+1)}}, \quad f_{4(2n+1)} = \frac{f^2_{2(2n+1)}}{F_{2(2n+1)}}, \quad \text{etc.}$$

Functione igitur $f_{(2n+1)}$ determinata, omnes functionum formae $f_{2^h(2n+1)}$ sine ulla ambiguitate inde derivantur. Itaque reductus est numerus determinandarum functionum ad 31; nempe remanserunt functiones

$$f_3, \ f_5, \ f_7, \ f_9, \ \text{etc.} \ f_{63}$$

determinandae.

Hae vero functiones, licet eadem ratione ac f_1 determinabiles, tamen non eidem arbitrio subiici possunt.

Ex aequatione (27.) enim quidem deduci potest, ubique R^{2n+1} pro R posito, haec:

$$28''.\quad f^{128}_{(2n+1)} = \{F^{64}_{(2n+1)}.F^{32}_{2(2n+1)}.F^{16}_{4(2n+1)}.F^{8}_{8(2n+1)}.F^{4}_{16(2n+1)}.F^{2}_{32(2n+1)}.F_{64(2n+1)}.F_{128(2n+1)}\}.$$

Tamen inter 128 valores functionis $f_{(2n+1)}$, hinc ortos, is est eligendus, qui cum suppositione antea de $f_{(1)}$ facta congruat; arbitrium enim inter valores functionum f, a quantitate σ dependens, ea nunc tali posita, qualis conditioni $f_1 = z$ satisfaciat, prorsus tollitur. Ad aliud igitur hîc confugiendum fuit, artificium novum, nunc exponendum.

Determinentur enim functionum f_3, f_7, f_{15}, f_{31}, f_{63} potestates, adhibitis solis aequationibus (28.) et (19.). Exempli causa in aequat. (28.) pro R, R^3, R^6, R^{12}, R^{24}, R^{48} posito, habemus respective

$$\begin{aligned} f_3^2 &= F_3\, f_6, \\ f_6^2 &= F_6\, f_{12}, \\ f_{12}^2 &= F_{12}\, f_{24}, \\ f_{24}^2 &= F_{24} f_{48}, \\ f_{48}^2 &= F_{48} f_{96} = \frac{257\, F_{48}}{f_{32}}. \end{aligned}$$

Inde derivatur:

$$f_3^{32} = \frac{F_3^{16}\, F_6^8\, F_{12}^4\, F_{24}^2\, F_{48}\ 257}{f_{32}}.$$

Similiter fiunt

$$f_7^{16} = \frac{F_7^8\, F_{14}^4\, F_{28}^2\, F_{56}\,.\,257}{f_{16}}, \qquad f_{31}^4 = \frac{F_{31}^2\, F_{62}\,.\,257}{f_4},$$

$$f_{15}^8 = \frac{F_{15}^4\, F_{30}^2\, F_{60}\,.\,257}{f_8}, \qquad f_{63}^2 = \frac{F_{63}\,.\,257}{f_2}.$$

Ex his theorematibus coniunctis cum prima aequationum (28.) struantur hae quantitates:

$$29.\quad \begin{cases} \left(\dfrac{f_1\, f_3}{f_4}\right)^{32} = \dfrac{F_1^{16}\, F_2^8\,.\,F_3^{16}\, F_6^8\, F_{12}^4\, F_{24}^2\, F_{48}}{F_4^{12}\, F_8^6\, F_{16}^3\, F_{32}^2}, \\[2ex] \left(\dfrac{f_1\, f_7}{f_8}\right)^{16} = \dfrac{F_1^8\, F_2^4\, F_4^2\, F_7^8\, F_{14}^4\, F_{28}^2\, F_{56}}{F_8^7\, F_{16}^4\, F_{32}^2}, \\[2ex] \left(\dfrac{f_1\, f_{15}}{f_{16}}\right)^8 = \dfrac{F_1^4\, F_2^2\, F_4\,.\,F_{15}^4\, F_{30}^2\, F_{60}}{F_{16}^4\, F_{32}^2}, \\[2ex] \left(\dfrac{f_1\, f_{31}}{f_{32}}\right)^4 = \dfrac{F_1^2\, F_2\,.\,F_{31}^2\, F_{62}}{F_{32}^2}, \\[2ex] \left(\dfrac{f_1\, f_{63}}{f_{64}}\right)^2 = F_1\, F_{63}. \end{cases}$$

Jam si theorema (16.), unde aequationes $F_{48}=F_{16}$, $F_{56}=F_8$, $F_{60}=F_4$, $F_{62}=F_2$ sequuntur, adhibeamus nec non in universum per $\sqrt[32]{a}$, $\sqrt[16]{b}$, $\sqrt[8]{c}$, $\sqrt[4]{d}$, $\sqrt[2]{f}$ unam realem positivam radicem respective aequationum:

$$X^{32}=a,\quad X^{16}=b,\quad X^8=c,\quad X^4=d,\quad X^2=f$$

significemus, omnes igitur radices respective per

$$\sqrt[32]{(a)}R^{4e_1},\quad \sqrt[16]{(b)}R^{8i_1},\quad \sqrt[8]{(c)}R^{16k_1},\quad \sqrt[4]{(d)}R^{32l_1},\quad \sqrt[2]{(e)}R^{64m_1}$$

exprimamus, ubi e_1 unus est numerorum 0, 1, 2, ... 31,

i_1	- - - -	0, 1, ... 15,
k_1	- - - -	0, 1, ... 7,
l_1	- - - -	0, 1, 2, 3,
m_1	- - - -	0, 1,

hae derivantur aequationes:

$$30.\quad \frac{f_1 f_3}{f_4} = \sqrt[16]{\left(\frac{F_1^8 F_2^4 F_3^8 F_6^4 F_{12}^2 F_4}{F_4^6 F_8^3 F_{16} F_{32}}\right)} R^{4e_1},$$

$$31.\quad \frac{f_1 f_7}{f_8} = \sqrt[8]{\left(\frac{F_1^4 F_2^2 F_4 F_7^4 F_{14}^2 F_{28}}{F_8^3 F_{16}^2 F_{32}}\right)} R^{8i_1},$$

$$32.\quad \frac{f_1 f_{15}}{f_{16}} = \sqrt[4]{\left(\frac{F_1^2 F_2 F_4 F_{15}^2 F_{30}}{F_{16}^2 F_{32}}\right)} R^{16k_1},$$

$$33.\quad \frac{f_1 f_{31}}{f_{32}} = \sqrt[2]{\left(\frac{F_1 F_2 F_{31}}{F_{32}}\right)} R^{32l_1},$$

$$34.\quad \frac{f_1 f_{63}}{f_{64}} = \sqrt[2]{(F_1 F_{63})}\, R^{64m_1}.$$

Ex quinque ultimis aequationibus, alias veras derivari ubique in functionibus f et F, R^{2n+1} pro R positis, ex utriusque functionis natura elucet. Nullo modo vero potestas illa ipsius R eidem substitutioni subiicitur, quippe qua substitutione proprie in aequationibus (29.) introducenda, ad novas aequationes puras ordinum 32, 16, 8, 4, 2 respective inter quarum radices verae sunt eligendae, venimus.

Hanc ob rem numeri $e_{(2n+1)}$, $i_{(2n+1)}$, $k_{(2n+1)}$, $l_{(2n+1)}$, $m_{(2n+1)}$, quos numeros e_1, i_1, k_1, l_1, m_1, in aequationibus (30.), (31.), (32.), (33.), (34.) R^{2n+1} loco ipsius R posito, fieri ponamus, prorsus segregato calculo determinandi videntur.

Iam vero inter aequationes, hac substitutione ortas, eas quibus ad sequentia utamur, proponamus.

Primum ex aequatione (34.) invenimus, si $4n+1<64$ est, 16 aequationes huius formae:

$$\frac{f_{(4n+1)}f_{63(4n+1)}}{f_{64(4n+1)}} = R^{64m_{(4n+1)}}\sqrt{(F_{(4n+1)}\cdot F_{63(4n+1)})}$$

sive quia

$$f_{63(4n+1)} = f_{64-(4n+1)} \quad \text{et} \quad F_{63(4n+1)} = F_{64-(4n+1)},$$

nec non

$$f_{64(2n+1)} = f_{64},$$

esse per se clarum est:

$$35. \quad \frac{f_{(4n+1)}f_{64-(4n+1)}}{f_{64}} = \sqrt[2]{(F_{4n+1}\cdot F_{64-(4n+1)})}\,R^{64m_{(4n+1)}}.$$

Ex aequatione (33.) invenimus, R^{4n+1} pro R substituto, ubi $4n+1<32$ sit, quia rursus $f_{31(4n+1)}=f_{(32-(4n+1))}$ nec non $f_{32}=f_{32(4n+1)}$ ponere licet, octo aequationes huius formae:

$$36. \quad \frac{f_{(4n+1)}f_{32-(4n+1)}}{f_{32}} = \sqrt[3]{\left(\frac{F_{(4n+1)}F_{2(4n+1)}F_{31(4n+1)}}{F_{32}}\right)}\,R^{32l_{(4n+1)}}.$$

Ex aequatione (32.), pro R respective R^5, R^9, R^{13} positis hae aequationes oriuntur:

$$37. \quad \begin{cases} \dfrac{f_5 f_{75}}{f_{80}} = \sqrt[4]{\left(\dfrac{F_5^2 F_{10} F_{20} F_{75}^2 F_{22}}{F_{80}^2 F_{32}}\right)}\,R^{16k_5}, \\[2ex] \dfrac{f_9 f_7}{f_{36}} = \sqrt[4]{\left(\dfrac{F_9^2 F_{18} F_{36} F_7^2 F_{14}}{F_{18}^2 F_{32}}\right)}\,R^{16k_9}, \\[2ex] \dfrac{f_{13} f_{67}}{f_{80}} = \sqrt[4]{\left(\dfrac{F_{13}^2 F_{26} F_{52} F_{67}^2 F_8}{F_{80}^2 F_{32}}\right)}\,R^{16k_{13}}. \end{cases}$$

Ex aequatione (31.), pro R, R^{19} posito, haec oritur:

$$38. \quad \frac{f_{19}f_5}{f_{24}} = \sqrt[8]{\left(\frac{F_5^2 F_{10} F_{20} F_{19}^4 F_{38}^2 F_{32}}{F_{24}^8 F_{16}^2 F_{32}}\right)}\,R^{8i_{19}}.$$

Loco ipsarum functionum f hanc formam: $\sqrt{(257)}(\cos\omega+i\sin\omega)$ introducere licere, clarum est, unde ex theorem. 19. sequitur haec aequatio:

$$39. \quad \omega_{128-x} = 360^0 - \omega_x.$$

Itaque in omnibus his expressionibus (28.), (30.), (31.), (32.), (33.), (34.), (35.), (36.), (37.), (38.) reducantur formae quantitatum f, F, R per angulos ω, ϑ et $\frac{\pi}{64}$ expressae; nec non anguli η, ι, $\varkappa$, λ, μ attrahantur. Quibus factis, nanciscimur adhibitis aequat. (39.), (22.), (23.), (24.), (25.) has aequationes:

ex aequat. (28.)

$$\omega_1 = \frac{32\vartheta_1+16\vartheta_2+8\vartheta_4+4\vartheta_8+2\vartheta_{16}+\vartheta_{32}}{64},$$

$$\omega_2 = \frac{16\vartheta_2+8\vartheta_4+4\vartheta_8+2\vartheta_{16}+\vartheta_{32}}{32},$$

$$\omega_4 = \frac{8\vartheta_4 + 4\vartheta_8 + 2\vartheta_{16} + \vartheta_{32}}{16},$$

$$\omega_8 = \frac{4\vartheta_8 + 2\vartheta_{16} + \vartheta_{32}}{8},$$

$$\omega_{16} = \frac{2\vartheta_{16} + \vartheta_{32}}{4},$$

$$\omega_{32} = \frac{\vartheta_{32}}{2},$$

$$\omega_{64} = 0,$$

ex aequat. (30.)

$$\omega_1 + \omega_3 - \omega_4 = \frac{8\vartheta_1 + 4\vartheta_2 + 8\vartheta_3 + 4\vartheta_6 + 2\vartheta_{12} + \vartheta_{24} - 6\vartheta_4 - 3\vartheta_8 - \vartheta_{16} - \vartheta_{32}}{16} + e_1 \frac{\pi}{16} = \eta_1,$$

ex aequat. (31.)

$$\omega_1 + \omega_7 - \omega_8 = \frac{4\vartheta_1 + 2\vartheta_2 + \vartheta_4 + 4\vartheta_7 + 2\vartheta_{14} + \vartheta_{28} - 3\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{8} + i_1 \frac{\pi}{8} = \iota_1,$$

$$\omega_5 + \omega_{19} - \omega_{24} = \frac{4\vartheta_5 + 2\vartheta_{10} + \vartheta_{20} + 4\vartheta_{19} + 2\vartheta_{26} - \vartheta_{12} - 3\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{8} + i_{19} \frac{\pi}{8} = \iota_{19},$$

ex aequat. (32.) et (37.):

$$\omega_1 + \omega_{15} - \omega_{16} = \frac{2\vartheta_1 + \vartheta_2 + \vartheta_4 + 2\vartheta_{15} + \vartheta_{30} - 2\vartheta_{16} - \vartheta_{32}}{4} + k_1 \frac{\pi}{4} = \varkappa_1,$$

$$\omega_5 + \omega_{48} - \omega_{53} = \frac{2\vartheta_5 + \vartheta_{10} + \vartheta_{20} + 2\vartheta_{16} + \vartheta_{22} - 2\vartheta_{11} - \vartheta_{32} + 90^\circ}{4} + k_5 \frac{\pi}{4} = \varkappa_5,$$

$$\omega_9 + \omega_7 - \omega_{16} = \frac{2\vartheta_9 + \vartheta_{18} + \vartheta_{28} + 2\vartheta_7 + \vartheta_{14} - 2\vartheta_{16} - \vartheta_{32}}{4} + k_9 \frac{\pi}{4} = \varkappa_9,$$

$$\omega_{13} + \omega_{48} - \omega_{61} = \frac{2\vartheta_{13} + \vartheta_{26} + \vartheta_{12} + 2\vartheta_{16} + \vartheta_6 - 2\vartheta_3 - \vartheta_{32} - 90^\circ}{4} + k_{13} \frac{\pi}{4} = \varkappa_{13},$$

ex aequat. (33.) et (36.)

$$\omega_1 + \omega_{31} - \omega_{32} = \frac{\vartheta_1 + \vartheta_2 + \vartheta_{31} - \vartheta_{32}}{2} + l_1 \frac{\pi}{2} = \lambda_1,$$

$$\omega_5 + \omega_{27} - \omega_{32} = \frac{\vartheta_5 + \vartheta_{10} + \vartheta_{27} - \vartheta_{32}}{2} + l_5 \frac{\pi}{2} = \lambda_5,$$

$$\omega_9 + \omega_{23} - \omega_{32} = \frac{\vartheta_9 + \vartheta_{18} + \vartheta_{23} - \vartheta_{32}}{2} + l_9 \frac{\pi}{2} = \lambda_9,$$

$$\omega_{13} + \omega_{19} - \omega_{32} = \frac{\vartheta_{13} + \vartheta_{26} + \vartheta_{19} - \vartheta_{32}}{2} + l_{13} \frac{\pi}{2} = \lambda_{13},$$

$$\omega_{17} + \omega_{15} - \omega_{32} = \frac{\vartheta_{17} + \vartheta_{30} + \vartheta_{15} - \vartheta_{32}}{2} + l_{17} \frac{\pi}{2} = \lambda_{17},$$

$$\omega_{21} + \omega_{11} - \omega_{32} = \frac{\vartheta_{21} + \vartheta_{22} + \vartheta_{11} - \vartheta_{32}}{2} + l_{21} \frac{\pi}{2} = \lambda_{21},$$

$$\omega_{25} + \omega_7 - \omega_{32} = \frac{\vartheta_{25} + \vartheta_{14} + \vartheta_7 - \vartheta_{32}}{2} + l_{25} \frac{\pi}{2} = \lambda_{25},$$

$$\omega_{29} + \omega_3 - \omega_{32} = \frac{\vartheta_{29} + \vartheta_6 + \vartheta_3 - \vartheta_{32}}{2} + l_{29} \frac{\pi}{2} = \lambda_{29}.$$

Aequatio vero (35.) theoremate (24.) determinatur, unde sequitur:

$$\mu_{(4n+1)} = \omega_{(4n+1)} + \omega_{64-(4n+1)} = \vartheta_{(4n+1)} + \frac{3\pi}{2},$$

et

$$\mu_{(4n+3)} = \omega_{(4n+3)} + \omega_{64-(4n+3)} = \vartheta_{(4n+3)} + \frac{\pi}{2}.$$

In ceteris vero aequationibus nunc allatis, si substituantur valores angulorum ϑ, η, ι, $\varkappa$, λ, μ, facillime eliciuntur valores numerorum e, i, k, l, m. Inde etiam clarum fit, angulos η, ι, $\varkappa$, λ, μ non tam stricte fuisse computandos, adeo suffecisse in quonam quadranto sint anguli k, determinare.

Invenimus vero hac in via progredientes, has aequationes:

$$\text{I.} \quad \omega_1+\omega_3-\omega_4 = \frac{8\vartheta_1+4\vartheta_2+8\vartheta_3+4\vartheta_6+2\vartheta_{12}-\vartheta_{24}-6\vartheta_4-3\vartheta_8-\vartheta_{16}-\vartheta_{32}}{16}+\frac{3\pi}{8},$$
$$= 146^0\ 22'\ 30'',37.$$

$$\text{II.} \quad \omega_1+\omega_7-\omega_8 = \frac{4\vartheta_1+2\vartheta_2+\vartheta_4+\vartheta_{28}+2\vartheta_{14}+4\vartheta_7-3\vartheta_8-2\vartheta_{16}-\vartheta_{32}}{8}+\frac{3\pi}{4},$$
$$= 235^0\ \ 3'\ \ 8'',80.$$

$$\text{III.} \quad \omega_5+\omega_{19}-\omega_{24} = \frac{4\vartheta_5+2\vartheta_{10}+\vartheta_{20}+4\vartheta_{19}-\vartheta_{13}+2\vartheta_{26}-3\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{8}+\frac{3\pi}{2},$$
$$= 54^0\ \ 9'\ 49'',61.$$

$$\text{IV.} \quad \omega_1+\omega_{15}-\omega_{16} = \frac{2\vartheta_1+2\vartheta_{15}+\vartheta_2+\vartheta_{30}+\vartheta_4-2\vartheta_{16}-\vartheta_{32}}{4}+\pi,$$
$$= 19^0\ 25'\ \ 5'',35.$$

$$\text{V.} \quad \omega_5+\omega_{48}-\omega_{53} = \frac{2\vartheta_5+\vartheta_{10}+\vartheta_{20}+\vartheta_{22}+2\vartheta_{16}-2\vartheta_{11}-\vartheta_{32}}{4}+\frac{\pi}{2},$$
$$= 39^0\ 53'\ 54'',24.$$

$$\text{VI.} \quad \omega_9+\omega_7-\omega_{16} = \frac{2\vartheta_9+2\vartheta_7+\vartheta_{18}+\vartheta_{14}+\vartheta_{28}-2\vartheta_{16}-\vartheta_{32}}{4}+\frac{3\pi}{2},$$
$$= 310^0\ \ 0'\ 13'',07.$$

$$\text{VII.} \quad \omega_{13}+\omega_{48}-\omega_{61} = \frac{2\vartheta_{13}+\vartheta_{26}+\vartheta_{12}+\vartheta_6+2\vartheta_{16}-2\vartheta_3-\vartheta_{32}}{4}+\frac{3\pi}{2},$$
$$= 110^0\ 53'\ 45'',41.$$

$$\text{VIII.} \quad \omega_1+\omega_{31}-\omega_{32} = \frac{\vartheta_1+\vartheta_{31}+\vartheta_2-\vartheta_{32}}{2}+\frac{3\pi}{2},$$
$$= 173^0\ 55'\ \ 7'',44.$$

$$\text{IX.} \quad \omega_5+\omega_{27}-\omega_{32} = \frac{\vartheta_5+\vartheta_{27}+\vartheta_{10}-\vartheta_{32}}{2}+\frac{\pi}{2},$$
$$= 91^0\ 38'\ 37'',43.$$

$$\text{X.} \quad \omega_9+\omega_{23}-\omega_{32} = \frac{\vartheta_9+\vartheta_{23}+\vartheta_{18}-\vartheta_{32}}{2}+\frac{3\pi}{2},$$
$$= 31^0\ 39'\ 54'',08.$$

$$\text{XI.}\quad \omega_{13}+\omega_{19}-\omega_{32}=\frac{\vartheta_{13}+\vartheta_{19}+\vartheta_{26}-\vartheta_{32}}{2}+\frac{3\pi}{2},$$
$$=348^0\ 31'\ 7'',48.$$

$$\text{XII.}\quad \omega_{17}+\omega_{15}-\omega_{32}=\frac{\vartheta_{17}+\vartheta_{15}+\vartheta_{30}-\vartheta_{32}}{2}+\frac{\pi}{2},$$
$$=151^0\ 47'\ 42'',74.$$

$$\text{XIII.}\quad \omega_{21}+\omega_{11}-\omega_{32}=\frac{\vartheta_{21}+\vartheta_{11}+\vartheta_{22}-\vartheta_{32}}{2}+\frac{\pi}{2},$$
$$=34^0\ 35'\ 31'',74.$$

$$\text{XIV.}\quad \omega_{25}+\omega_{7}-\omega_{32}=\frac{\vartheta_{25}+\vartheta_{7}+\vartheta_{14}-\vartheta_{32}}{2}+\frac{3\pi}{2},$$
$$=118^0\ 47'\ 30'',09.$$

$$\text{XV.}\quad \omega_{29}+\omega_{3}-\omega_{32}=\frac{\vartheta_{29}+\vartheta_{3}+\vartheta_{6}-\vartheta_{32}}{2}+\frac{3\pi}{2},$$
$$=139^0\ 57'\ 41'',59.$$

$$\text{XVI.}\quad \omega_1+\omega_{63}=\mu_1=\vartheta_1+\frac{3\pi}{2},$$
$$\text{XVII.}\quad \omega_{31}+\omega_{33}=\mu_{31}=\vartheta_{31}+\frac{\pi}{2},$$
$$\text{XVIII.}\quad \omega_5+\omega_{59}=\mu_5=\vartheta_5+\frac{3\pi}{2},$$
$$\text{XIX.}\quad \omega_{27}+\omega_{37}=\mu_{27}=\vartheta_{27}+\frac{\pi}{2},$$
$$\text{XX.}\quad \omega_9+\omega_{55}=\mu_9=\vartheta_9+\frac{3\pi}{2},$$
$$\text{XXI.}\quad \omega_{23}+\omega_{41}=\mu_{23}=\vartheta_{23}+\frac{\pi}{2},$$
$$\text{XXII.}\quad \omega_{13}+\omega_{51}=\mu_{13}=\vartheta_{13}+\frac{3\pi}{2},$$
$$\text{XXIII.}\quad \omega_{19}+\omega_{45}=\mu_{19}=\vartheta_{19}+\frac{\pi}{2},$$
$$\text{XXIV.}\quad \omega_{17}+\omega_{47}=\mu_{17}=\vartheta_{17}+\frac{3\pi}{2},$$
$$\text{XXV.}\quad \omega_{15}+\omega_{49}=\mu_{15}=\vartheta_{15}+\frac{\pi}{2},$$
$$\text{XXVI.}\quad \omega_{21}+\omega_{43}=\mu_{21}=\vartheta_{21}+\frac{3\pi}{2},$$
$$\text{XXVII.}\quad \omega_{11}+\omega_{53}=\mu_{11}=\vartheta_{11}+\frac{\pi}{2},$$
$$\text{XXVIII.}\quad \omega_{25}+\omega_{39}=\mu_{25}=\vartheta_{25}+\frac{3\pi}{2},$$
$$\text{XXIX.}\quad \omega_{7}+\omega_{57}=\mu_{7}=\vartheta_{7}+\frac{\pi}{2},$$
$$\text{XXX.}\quad \omega_{29}+\omega_{35}=\mu_{29}=\vartheta_{29}+\frac{3\pi}{2},$$
$$\text{XXXI.}\quad \omega_{3}+\omega_{61}=\mu_{3}=\vartheta_{3}+\frac{\pi}{2}.$$

Ad quas aequationes triginta et unam adiiciamus adhuc has, quae ex antecedentibus, theoremate (14.) adhibito facillime derivari possunt:

$$\left\{\begin{array}{llll}\text{XXXII.} & \omega_2=2\omega_1-\vartheta_1, & \text{XXXIII.} & \omega_4=2\omega_2-\vartheta_2,\\ \text{XXXIV.} & \omega_8=2\omega_4-\vartheta_4, & \text{XXXV.} & \omega_{16}=2\omega_8-\vartheta_8,\\ \text{XXXVI.} & \omega_{32}=2\omega_{16}-\vartheta_{16}, & \text{XXXVII.} & \omega_{64}=2\omega_{32}-\vartheta_{32}.\end{array}\right.$$

$$\text{XXXVIII.}\quad \omega_6=2\omega_3-\vartheta_3,\qquad \text{XXXIX.}\quad \omega_{12}=2\omega_6-\vartheta_6,$$
$$\text{XL.}\quad \omega_{24}=2\omega_{12}-\vartheta_{12},\qquad \text{XLI.}\quad \omega_{48}=2\omega_{24}-\vartheta_{24}.$$

$$\begin{cases}\text{XLII. } \omega_{10}=2\omega_5-\vartheta_5, & \text{XLIII. } \omega_{20}=2\omega_{10}-\vartheta_{10}, & \text{XLIV. } \omega_{40}=2\omega_{20}-\vartheta_{20}.\\ \text{XLV. } \omega_{14}=2\omega_7-\vartheta_7, & \text{XLVI. } \omega_{28}=2\omega_{14}-\vartheta_{14}, & \text{XLVII. } \omega_{56}=2\omega_{28}-\vartheta_{28}.\end{cases}$$

$$\begin{cases}\text{XLVIII. } \omega_{18}=2\omega_9-\vartheta_9, & \text{XLVIX. } \omega_{36}=2\omega_{18}-\vartheta_{18}.\\ \text{L. } \omega_{22}=2\omega_{11}-\vartheta_{11}, & \text{LI. } \omega_{44}=2\omega_{22}-\vartheta_{22}.\\ \text{LII. } \omega_{26}=2\omega_{13}-\vartheta_{13}, & \text{LIII. } \omega_{52}=2\omega_{26}-\vartheta_{26}.\\ \text{LIV. } \omega_{30}=2\omega_{15}-\vartheta_{15}, & \text{LV. } \omega_{60}=2\omega_{30}-\vartheta_{30}.\end{cases}$$

$$\begin{array}{ll}\text{LVI. } \omega_{34}=2\omega_{17}-\vartheta_{17}. & \text{LX. } \omega_{50}=2\omega_{25}-\vartheta_{25}.\\ \text{LVII. } \omega_{38}=2\omega_{19}-\vartheta_{19}. & \text{LXI. } \omega_{54}=2\omega_{27}-\vartheta_{27}.\\ \text{LVIII. } \omega_{42}=2\omega_{21}-\vartheta_{21}. & \text{LXII. } \omega_{58}=2\omega_{29}-\vartheta_{29}.\\ \text{LVIX. } \omega_{46}=2\omega_{23}-\vartheta_{23}. & \text{LXIII. } \omega_{62}=2\omega_{31}-\vartheta_{31}.\end{array}$$

Habemus igitur sexaginta et tres aequationes, in quibus determinandae sunt

$$\omega_2,\ \omega_3,\ \omega_4 \text{ etc. } \quad \omega_{64}$$

quantitates, anguli autem $\vartheta_1, \vartheta_2, \vartheta_3, \ldots. \vartheta_{32}$ nec non angulus ω_1 quantitates cognitae sunt, ita ut illae inde possint determinari. Adhuc vero adiiciendum est, omnes illos angulos ω ipsos ex aequationibus propositis prodire posse ullo coefficiente carentes, unde fit, ut angulus quisque ω sine ulla ambiguitate nonnisi uno positivo valore gaudeat.

Quos valores ita invenimus.

Habemus:

$$\omega_1=\frac{32\vartheta_1+16\vartheta_2+8\vartheta_4+4\vartheta_8+2\vartheta_{16}+\vartheta_{32}}{64}, \qquad =139^0\ 36'\ 39'',09,$$

quo valore in aequat. XXXII. usque ad XXXVIII. adhibito, fiunt:

$$\omega_2=\frac{16\vartheta_2+8\vartheta_4+4\vartheta_8+2\vartheta_{16}+\vartheta_{32}}{32}, \qquad =214^0\ 34'\ 36'',45,$$

$$\omega_4=\frac{8\vartheta_4+4\vartheta_8+2\vartheta_{16}+\vartheta_{32}}{16}, \qquad =176^0\ 29'\ 15'',67,$$

$$\omega_8=\frac{4\vartheta_8+2\vartheta_{16}+\vartheta_{32}}{8}, \qquad =147^0\ 32'\ 59'',19,$$

$$\omega_{16}=\frac{2\vartheta_{16}+\vartheta_{32}}{4}, \qquad =193^0\ 3'\ 45'',94,$$

$$\omega_{32}=\frac{\vartheta_{32}}{2}, \qquad =46^0\ 47'\ 17'',40,$$

$$\omega_{64}=0, \qquad =0,$$

hisque valoribus in aequationibus I., II., IV., VIII., XVI. substitutis inde emergunt hi valores:

$$\omega_3=\frac{32\vartheta_3+16\vartheta_6+8\vartheta_{12}+4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{64}+\frac{3\pi}{8}=183^0\ 15'\ 6'',96,$$

$$\omega_7 = \frac{32\vartheta_7 + 16\vartheta_{14} + 8\vartheta_{28} + 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{3\pi}{4} = 242^0\ 59'\ 28'',91,$$

$$\omega_{15} = \frac{32\vartheta_{15} + 16\vartheta_{30} + 8\vartheta_4 - 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \pi = 72^0\ 52'\ 12'',20,$$

$$\omega_{31} = \frac{32\vartheta_{31} + 16\vartheta_2 - 8\vartheta_4 - 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{3\pi}{2} = 81^0\ 5'\ 45'',75,$$

$$\omega_{63} = \frac{32\vartheta_1 - 16\vartheta_2 - 8\vartheta_4 - 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{3\pi}{2} = 195^0\ 2'\ 2'',64,$$

unde ex aequationibus XXXVIII., XXXIX., XL., XLI. sequuntur:

$$\omega_6 = \frac{16\vartheta_6 + 8\vartheta_{12} + 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{32} + \frac{3\pi}{4} = 241^0\ 5'\ 58'',20,$$

$$\omega_{12} = \frac{8\vartheta_{12} + 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{16} + \frac{3\pi}{2} = 21^0\ 20'\ 34'',52,$$

$$\omega_{24} = \frac{4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{8} + \pi = 331^0\ 11'\ 18'',21,$$

$$\omega_{48} = \frac{2\vartheta_{16} - \vartheta_{32}}{4} = 146^0\ 16'\ 28'',54,$$

ex aequat. XLV., XLVI., XLVII.:

$$\omega_{14} = \frac{16\vartheta_{14} + 8\vartheta_{28} + 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{32} + \frac{3\pi}{2} = 42^0\ 41'\ 24'',45,$$

$$\omega_{28} = \frac{8\vartheta_{28} + 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{16} + \pi = 221^0\ 52'\ 55'',53,$$

$$\omega_{56} = \frac{4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{8} = 314^0\ 29'\ 13'',25.$$

Valore anguli ω_1 supposito, atque in aequationibus XXXII. usque ad XXXVIII. substituto inde sensim sensimque emergunt:

$$\omega_2,\ \omega_4,\ \omega_8,\ \omega_{16},\ \omega_{32},\ \omega_{64},$$

quibus valoribus in aequat. I., II., IV., VIII., XVI. substitutis proveniunt:

$$\omega_3,\ \omega_7,\ \omega_{15},\ \omega_{31},\ \omega_{63}.$$

Unde ex aequationibus XXXVIII., XXXIX., XL., XLI., XLV., XLVI., XXVII., LIV., LV. et LXIII. sequuntur:

$$\omega_6,\ \omega_{12},\ \omega_{24},\ \omega_{48},\ \omega_{14},\ \omega_{28},\ \omega_{56},\ \omega_{30},\ \omega_{60}\ \text{et}\ \omega_{62},$$

ex valoribus ω_3 et ω_{32} et aequat. XV. et XXI., XXX. et LXII.:

$$\omega_{29},\ \omega_{61},\ \omega_{35},\ \omega_{58},$$

ex ω_7 et aequat. VI., XIV., XXIX. sequuntur:

$$\omega_9,\ \omega_{25},\ \omega_{57},$$

unde per aequat. XLVIII., XLXIX., XXVIII. et LX., XX., X., LVI. et XXI.:

$$\omega_{18},\ \omega_{36},\ \omega_{39},\ \omega_{50},\ \omega_{55},\ \omega_{23},\ \omega_{46},\ \omega_{41}.$$

Ex ω_{15}, ω_{31} et aequat. XII., XXV., XXXIII., LVI., XXIV. fluunt:

ω_{17}, ω_{49}, ω_{33}, ω_{34}, ω_{47}.

Ex ω_{61} et aequat. VII. tum vero LII., LIII., XXII., XI., LVII. fluunt:

ω_{13}, ω_{26}, ω_{52}, ω_{51}, ω_{19}, ω_{38}.

Ex ω_{19} et aequat. III., XLII., XLIII., XLIV. et XXIII. fluunt:

ω_{5}, ω_{10}, ω_{20}, ω_{40}, ω_{45}.

Ex ω_{5} et aequat. V., XVIII., IX., LXI., XIX. deducuntur:

ω_{53}, ω_{59}, ω_{27}, ω_{54}, ω_{37}.

Ex ω_{53} et aequat. XXVII., L., LI., XIII., LVIII., XXVI. sequuntur:

ω_{11}, ω_{22}, ω_{44}, ω_{21}, ω_{42}, ω_{43}.

Valores vero ipsi angulorum ω hi sunt:

Angulus: $\omega_{64}=0$.

Angulus: $\omega_{32}=\frac{\vartheta_{32}}{2}=46^0\ 47'\ 17'',40$.

Anguli formae: $\omega_{16(2n+1)}$

$$\omega_{16}=\frac{2\vartheta_{16}+\vartheta_{32}}{4}=193^0\ 3'\ 45'',94, \quad \omega_{48}=\frac{2\vartheta_{16}-\vartheta_{32}}{4}=146^0\ 16'\ 28'',54.$$

Anguli formae: $\omega_{8(2n+1)}$

$$\omega_{8}=\frac{4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{8}=147^0\ 32'\ 59'',19,$$

$$\omega_{56}=\frac{4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{8}=314^0\ 29'\ 13'',25,$$

$$\omega_{24}=\frac{4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{8}+\pi=331^0\ 11'\ 18'',21,$$

$$\omega_{40}=\frac{4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{8}+\pi=184^0\ 54'\ 49'',67.$$

Anguli formae: $\omega_{4(2n+1)}$

$$\omega_{4}=\frac{8\vartheta_{4}+4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{16}=176^0\ 29'\ 15'',67,$$

$$\omega_{60}=\frac{8\vartheta_{4}-4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{16}=28^0\ 56'\ 16'',48,$$

$$\omega_{12}=\frac{8\vartheta_{12}+4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{16}+\frac{3\pi}{2}=21^0\ 20'\ 34'',52,$$

$$\omega_{52}=\frac{8\vartheta_{12}-4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{16}+\frac{\pi}{2}=50^0\ 9'\ 16'',31,$$

$$\omega_{20}=\frac{8\vartheta_{20}+4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{16}+\frac{\pi}{2}=265^0\ 20'\ 47'',34,$$

$$\omega_{44}=\frac{8\vartheta_{20}-4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{16}+\frac{3\pi}{2}=80^0\ 25'\ 57'',67,$$

$$\omega_{28}=\frac{8\vartheta_{28}+4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{16}+\pi=221^0\ 52'\ 55'',53,$$

$$\omega_{36}=\frac{8\vartheta_{28}-4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{16}+\pi=267^0\ 23'\ 42'',29.$$

Anguli formae: $\omega_{2(2n+1)}$

$$\omega_{2}=\frac{16\vartheta_{2}+8\vartheta_{4}+4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{32}=214^0\ 34'\ 36'',45,$$

$$\omega_{62}=\frac{16\vartheta_{2}-8\vartheta_{4}-4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{32}+\pi=218^0\ 5'\ 20'',78,$$

$$\omega_{6}=\frac{16\vartheta_{6}+8\vartheta_{12}+4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{32}+\frac{3\pi}{4}=241^0\ 5'\ 58'',20,$$

$$\omega_{58}=\frac{16\vartheta_{6}-8\vartheta_{12}-4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{32}+\frac{\pi}{4}=39^0\ 45'\ 23'',68,$$

$$\omega_{10}=\frac{16\vartheta_{10}+8\vartheta_{20}+4\vartheta_{24}-2\vartheta_{16}-\vartheta_{32}}{32}+\frac{5\pi}{4}=70^0\ 43'\ 22'',70,$$

$$\omega_{54}=\frac{16\vartheta_{10}-8\vartheta_{20}-4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{32}+\frac{7\pi}{4}=345^0\ 22'\ 35'',37,$$

$$\omega_{14}=\frac{16\vartheta_{14}+8\vartheta_{28}+4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{32}+\frac{3\pi}{2}=42^0\ 41'\ 24'',45,$$

$$\omega_{50}=\frac{16\vartheta_{14}-8\vartheta_{28}-4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{32}+\frac{3\pi}{2}=0^0\ 48'\ 28'',91,$$

$$\omega_{18}=\frac{16\vartheta_{18}+8\vartheta_{28}-4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{32}+\frac{3\pi}{2}=25^0\ 15'\ 29'',65,$$

$$\omega_{46}=\frac{16\vartheta_{18}-8\vartheta_{28}+4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{32}+\frac{3\pi}{2}=297^0\ 51'\ 47'',37,$$

$$\omega_{22}=\frac{16\vartheta_{22}+8\vartheta_{20}-4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{32}+\frac{3\pi}{4}=266^0\ 31'\ 10'',12,$$

$$\omega_{42}=\frac{16\vartheta_{22}-8\vartheta_{20}+4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{32}+\frac{\pi}{4}=6^0\ 5'\ 12'',45,$$

$$\omega_{26}=\frac{16\vartheta_{26}+8\vartheta_{12}-4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{32}+\frac{5\pi}{4}=286^0\ 27'\ 9'',72,$$

$$\omega_{38}=\frac{16\vartheta_{26}-8\vartheta_{12}+4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{32}+\frac{7\pi}{4}=56^0\ 17'\ 53'',41,$$

$$\omega_{30}=\frac{16\vartheta_{30}+8\vartheta_{4}-4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{32}=151^0\ 40'\ 1'',75,$$

$$\omega_{34}=\frac{16\vartheta_{30}-8\vartheta_{4}+4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{32}+\pi=302^0\ 43'\ 45'',27.$$

Anguli formae: ω_{2n+1}

$$\omega_{1}=\frac{32\vartheta_{1}+16\vartheta_{2}+8\vartheta_{4}+4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{64}=139^0\ 36'\ 39'',09,$$

$$\omega_{63}=\frac{32\vartheta_{1}-16\vartheta_{2}-8\vartheta_{4}-4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{64}+\frac{12\pi}{8}=195^0\ 2'\ 2'',64,$$

$$\omega_3 = \frac{32\vartheta_3 + 16\vartheta_6 + 8\vartheta_{12} + 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{3\pi}{8} = 183^0\ 15'\ \ 6'',96,$$

$$\omega_{61} = \frac{32\vartheta_3 - 16\vartheta_6 - 8\vartheta_{12} - 4\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{\pi}{8} = 32^0\ \ 9'\ \ 8'',76,$$

$$\omega_5 = \frac{32\vartheta_5 + 16\vartheta_{10} + 8\vartheta_{20} + 4\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{5\pi}{8} = 346^0\ 49'\ \ 8'',58,$$

$$\omega_{59} = \frac{32\vartheta_5 - 16\vartheta_{10} - 8\vartheta_{20} - 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{7\pi}{8} = 186^0\ \ 5'\ 45'',78,$$

$$\omega_7 = \frac{32\vartheta_7 + 16\vartheta_{14} + 8\vartheta_{28} + 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{6\pi}{8} = 242^0\ 59'\ 28'',91,$$

$$\omega_{57} = \frac{32\vartheta_7 - 7\vartheta_{14} - 8\vartheta_{28} - 4\vartheta_8 + 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{14\pi}{8} = 290^0\ 18'\ \ 4'',45,$$

$$\omega_9 = \frac{32\vartheta_9 + 16\vartheta_{18} + 8\vartheta_{28} - 4\vartheta_8 + 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{6\pi}{8} = 260^0\ \ 4'\ 30'',10,$$

$$\omega_{55} = \frac{32\vartheta_9 - 16\vartheta_{18} - 8\vartheta_{28} + 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{6\pi}{8} = 144^0\ 49'\ \ 0'',45,$$

$$\omega_{11} = \frac{32\vartheta_{11} + 16\vartheta_{22} + 8\vartheta_{20} - 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{3\pi}{8} = 269^0\ 42'\ 53'',00,$$

$$\omega_{53} = \frac{32\vartheta_{11} - 16\vartheta_{22} - 8\vartheta_{20} + 4\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{\pi}{8} = 93^0\ 11'\ 42'',88,$$

$$\omega_{13} = \frac{32\vartheta_{13} + 16\vartheta_{26} + 8\vartheta_{12} - 4\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{13\pi}{8} = 356^0\ 46'\ 25'',63,$$

$$\omega_{51} = \frac{32\vartheta_{13} - 16\vartheta_{26} - 8\vartheta_{12} + 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{15\pi}{8} = 340^0\ 19'\ 15'',91,$$

$$\omega_{15} = \frac{32\vartheta_{15} + 16\vartheta_{30} + 8\vartheta_4 - 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{8\pi}{8} = 72^0\ 52'\ 12'',20,$$

$$\omega_{49} = \frac{32\vartheta_{15} - 16\vartheta_{30} - 8\vartheta_4 + 4\vartheta_8 + 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{12\pi}{8} = 11^0\ 12'\ 10'',45,$$

$$\omega_{17} = \frac{32\vartheta_{17} + 16\vartheta_{30} - 8\vartheta_4 + 4\vartheta_8 + 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{12\pi}{8} = 125^0\ 42'\ 47'',94,$$

$$\omega_{47} = \frac{32\vartheta_{17} - 16\vartheta_{30} + 8\vartheta_4 - 4\vartheta_8 - 2\vartheta_{16} - \vartheta_{32}}{64} = 92^0\ 59'\ \ 2'',67,$$

$$\omega_{19} = \frac{32\vartheta_{19} + 16\vartheta_{26} - 8\vartheta_{12} + 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{15\pi}{8} = 38^0\ 31'\ 59'',25,$$

$$\omega_{45} = \frac{32\vartheta_{19} - 16\vartheta_{26} + 8\vartheta_{17} - 4\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{5\pi}{8} = 72^0\ 14'\ \ 5'',84,$$

$$\omega_{21} = \frac{32\vartheta_{21} + 16\vartheta_{22} - 8\vartheta_{20} + 4\vartheta_{24} - 2\vartheta_{16} + \vartheta_{32}}{64} + \frac{\pi}{8} = 171^0\ 39'\ 56'',14,$$

$$\omega_{43} = \frac{32\vartheta_{21} - 16\vartheta_{22} + 8\vartheta_{20} - 4\vartheta_{24} + 2\vartheta_{16} - \vartheta_{32}}{64} + \frac{11\pi}{8} = 75^0\ 34'\ 43'',69,$$

$$\omega_{23}=\frac{32\vartheta_{23}+16\vartheta_{18}-8\vartheta_{28}+4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{64}+\frac{6\pi}{8}=178^0\,22'\,41'',38,$$

$$\omega_{41}=\frac{32\vartheta_{23}-16\vartheta_{18}+8\vartheta_{28}-4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{64}+\frac{14\pi}{8}=330^0\,30'\,54'',01,$$

$$\omega_{25}=\frac{32\vartheta_{25}+16\vartheta_{14}-8\vartheta_{28}-4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{64}+\frac{6\pi}{8}=282^0\,35'\,18'',58,$$

$$\omega_{39}=\frac{32\vartheta_{25}-16\vartheta_{14}+8\vartheta_{28}+4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{64}+\frac{6\pi}{8}=191^0\,46'\,59'',67,$$

$$\omega_{27}=\frac{32\vartheta_{27}+16\vartheta_{10}-8\vartheta_{20}-4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{64}+\frac{15\pi}{8}=151^0\,36'\,46'',25$$

$$\omega_{37}=\frac{32\vartheta_{27}-16\vartheta_{10}+8\vartheta_{20}+4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{64}+\frac{5\pi}{8}=256^0\,14'\,10'',89,$$

$$\omega_{29}=\frac{32\vartheta_{29}+16\vartheta_{6}-8\vartheta_{12}-4\vartheta_{24}-2\vartheta_{16}+\vartheta_{32}}{64}+\frac{9\pi}{8}=3^0\,29'\,52'',03,$$

$$\omega_{35}=\frac{32\vartheta_{29}-16\vartheta_{6}+8\vartheta_{12}+4\vartheta_{24}+2\vartheta_{16}-\vartheta_{32}}{64}+\frac{3\pi}{8}=233^0\,44'\,28'',35,$$

$$\omega_{31}=\frac{32\vartheta_{31}+16\vartheta_{2}-8\vartheta_{4}-4\vartheta_{8}-2\vartheta_{16}-\vartheta_{32}}{64}+\frac{12\pi}{8}=81^0\,5'\,45'',75,$$

$$\omega_{33}=\frac{32\vartheta_{31}-16_{2}+8\vartheta_{4}+4\vartheta_{8}+2\vartheta_{16}+\vartheta_{32}}{64}+\frac{8\pi}{8}=313^0\,0'\,24'',97,$$

Priusquam angulos ω relinquamus, haud erit inutile generalem eorum formam afferre, quae et valoribus nunc expositis, et ex theorematibus (28.″) et (28.′) deduci potest.

Inde enim sequuntur, eadem significatione signi radicalis ac antea adhibita hae aequationes:

$$f_{(2n+1)}=R^{y(2n+1)}\sqrt[128]{\{F^{64}_{(2n+1)}.F^{32}_{2(2n+1)}.F^{16}_{4(2n+1)}.F^{8}_{8(2n+1)}.F^{4}_{16(2n+1)}.F^{2}_{32(2n+1)}.257\}},$$

$$f_{2(2n+1)}=\frac{f^2_{(2n+1)}}{F_{(2n+1)}}=R^{2y(2n+1)}\sqrt[64]{\{F^{32}_{2(2n+1)}.F^{16}_{4(2n+1)}.F^{8}_{8(2n+1)}.F^{4}_{16(2n+3)}.F^{2}_{32(2n+1)}.257\}},$$

$$f_{4(2n+1)}=\frac{f^2_{2(2n+1)}}{F_{2(2n+1)}}=R^{4y(2n+1)}\sqrt[32]{\{F^{16}_{4(2n+1)}.F^{8}_{8(2n+1)}.F^{4}_{16(2n+1)}.F^{2}_{32(2n+1)}.257\}},$$

$$f_{8(2n+1)}=\frac{f^2_{4(2n+1)}}{F_{4(2n+1)}}=R^{8y(2n+1)}\sqrt[16]{\{F^{8}_{8(2n+1)}.F^{4}_{16(2n+1)}.F^{2}_{32(2n+1)}.257\}},$$

$$f_{16(2n+1)}=\frac{f^2_{8(2n+1)}}{F_{8(2n+1)}}=R^{16y(2n+1)}\sqrt[8]{\{F^{4}_{16(2n+1)}.F^{2}_{32(2n+1)}.257\}},$$

$$f_{32(2n+1)}=\frac{f^2_{16(2n+1)}}{F_{16(2n+1)}}=R^{32y(2n+1)}\sqrt[4]{\{F^{2}_{32(2n+1)}.257\}},$$

$$f_{64(2n+1)}=\frac{f^2_{32(2n+)}}{F_{32(2n+1)}}=R^{64y(2n+1)}\sqrt[2]{(257)},$$

ubi numerus integer y adhuc determinetur necesse est, inter numeros 2, 4, 6, etc. 128.

Hinc deducuntur hae generales formae angulorum ω:

$$40.\quad \begin{cases} \omega_{(2n+1)} = \dfrac{32\vartheta_{(2n+1)}+16\vartheta_{2(2n+1)}+8\vartheta_{4(2n+1)}+4\vartheta_{8(2n+1)}+2\vartheta_{16(2n+1)}+\vartheta_{32(2n+1)}}{64}+\dfrac{y_{(2n+1)}\cdot\pi}{64}, \\ \omega_{2(2n+1)} = \dfrac{16\vartheta_{2(2n+1)}+8\vartheta_{4(2n+1)}+4\vartheta_{8(2n+1)}+2\vartheta_{16(2n+1)}+\vartheta_{32(2n+1)}}{32}+\dfrac{y_{(2n+1)}\cdot\pi}{32}, \\ \omega_{4(2n+1)} = \dfrac{8\vartheta_{4(2n+1)}+4\vartheta_{8(2n+1)}+2\vartheta_{16(2n+1)}+\vartheta_{32(2n+1)}}{16}+\dfrac{y_{(2n+1)}\cdot\pi}{16}, \\ \omega_{8(2n+1)} = \dfrac{4\vartheta_{8(2n+1)}+2\vartheta_{16(2n+1)}+\vartheta_{32(2n+1)}}{8}+\dfrac{y_{(2n+1)}\cdot\pi}{8}, \\ \omega_{16(2n+1)} = \dfrac{2\vartheta_{16(2n+1)}+\vartheta_{32(2n+1)}}{4}+\dfrac{y_{(2n+1)}\cdot\pi}{4}, \\ \omega_{32(2n+1)} = \dfrac{\vartheta_{32(2n+1)}}{2}+\dfrac{y_{(2n+1)}\cdot\pi}{2}, \end{cases}$$

ubique multiplis ipsius 2π omissis.

Quas formas in tribus et sexaginta angulis allatis reperimus, simulac revocemus theoremata de angulis ϑ: (22.) et (23.):

$$\vartheta_{128+x}=\vartheta_x,\quad \vartheta_{128-x}=-\vartheta_x,\quad \vartheta_{64-2x}=\vartheta_{2x},\quad \vartheta_{64-(2x+1)}=\pi+\vartheta_{(2x+1)},$$

multiplis ipsius 2π desumtis.

Iam inde clarum fit, in antecedentibus nonnisi numeros y esse determinatos, ita ut haec tabula valorum ipsius $y_{(2n+1)}$, ad angulos ω ipsos cognoscendos, suficiat:

$y_1 = 0$,	$y_{17} = 96$,	$y_{63} = 64$,	$y_{47} = 32$,
$y_3 = 24$,	$y_{19} = 120$,	$y_{61} = 104$,	$y_{45} = 8$,
$y_5 = 40$,	$y_{21} = 8$,	$y_{59} = 88$,	$y_{43} = 120$,
$y_7 = 44$,	$y_{23} = 48$,	$y_{57} = 80$,	$y_{41} = 80$,
$y_9 = 48$,	$y_{25} = 48$,	$y_{55} = 16$,	$y_{39} = 80$,
$y_{11} = 24$,	$y_{27} = 120$,	$y_{53} = 40$,	$y_{37} = 72$,
$y_{13} = 104$,	$y_{29} = 72$,	$y_{51} = 88$,	$y_{35} = 56$,
$y_{15} = 64$,	$y_{31} = 96$,	$y_{49} = 0$,	$y_{33} = 96$,

qui numeris pro $y_{(2n+1)}$ in formulis substituti, ibi ubique introductis aequationibus $\vartheta_{128+x}=\vartheta_x$, $\vartheta_{128-x}=\vartheta_x$ veros angulorum ω valores ex angulis ϑ_1 usque ad ϑ_{63} compositos efficiunt.

Quae omnia cum ita sint, quomodo anguli ω adhibeantur, ut ostendatur restat. Quem ad finem in iis aequationibus, quas in articulo IX. attulimus $m=128$, ponentes, nec non loco quantitatum:

$$x_1,\ x_2,\ \text{etc.}\ x_m,\quad \text{has:}\quad p_0,\ p_1,\ p_2,\ p_3,\ \text{etc.}\ p_{127},$$

introducentes, habemus:

$$f_1 = p_0 + p_1 R + p_2 R^2 + p_3 R^3 + \text{etc.} + p_{127} R^{127},$$
$$f_2 = p_0 + p_1 R^2 + p_2 R^4 + p_3 R^6 + \text{etc.} + p_{127} R^{127},$$
$$f_3 = p_0 + p_1 R^3 + p_2 R^6 + p_3 R^9 + \text{etc.} + p_{127} R^{127},$$
etc.
$$f_n = p_0 + p_1 R^n + p_2 R^{2n} + p_3 R^{3n} + \text{etc.} + p_{127} R^{n.127},$$
etc.
$$f_{127} = p_0 + p_1 R^{127} + p_2 R^{127.2} + p_3 R^{127.3} + \text{etc.} + p_{127} R^{127.127},$$
$$f_{128} = p_0 + p_1 + p_2 + p_3 + \text{etc.} + p_{127}.$$

Introducamus in sequentibus brevitatis causa hoc signum:

$$\Sigma^{(z)}(R^m f_n) = R^m f_n + R^{2m} f_{2n} + R^{3m} f_{3n} + \text{etc.} + R^{mz} f_{nz}.$$

Quo adhibito, theorematibusque idoneis de quantitatibus R, R^2, R^3, etc. revocatis, facile derivantur hae aequationes:

$$f_{64} + f_{128} = (2p_0 + 2p_2 + 2p_4 + \text{etc.} + 2p_{126})$$
$$= 2\{[2,1] + [2,3^2] + [2,3^4] + \text{etc.} + [2,3^{126}]\} = 2[128,1],$$

atque in universum:

$$\Sigma^{(2)}(R^{64n} f_{64}) = 2[128,3^n],$$
$$f_{32} + f_{64} + f_{96} + f_{128} = 4(p_0 + p_4 + p_8 + \text{etc.} \; p_{12n})$$
$$= 4\{[2,1] + [2,3^4] + [2,3^8] + \text{etc.} + [2,3^{124}]\} = 4[64,1],$$

atque in universum:

$$\Sigma^{(4)}(R^{32n} f_{32}) = 4[64,3^n],$$
$$\Sigma^{(8)}(f_{16}) = 8(p_0 + p_8 + \text{etc.} + p_{120})$$
$$= 8\{[2,1] + [2,3^8] + \text{etc.} + [2,3^{120}]\} = 8[32,1],$$

atque:

$$\Sigma^{(8)}(R^{16n} f_{16}) = 8(32,3^n),$$
$$\Sigma^{(16)}(f_8) = 16(p_0 + p_{16} + p_{32} + p_{48} + p_{64} + p_{80} + p_{96} + p_{112})$$
$$= 16\{[2,1] + [2,3^{16}] + \text{etc.} + [2,3^{112}]\} = 16[16,1],$$

atque:

$$\Sigma^{(16)}(R^{8n} f_8) = 16[16,3^n],$$
$$\Sigma^{(32)}(f_4) = 32(p_0 + p_{32} + p_{64} + p_{96})$$
$$= 32\{[2,1] + [2,3^{32}] + [2,3^{64}] + [2,3^{96}]\} = 32[8,1],$$

atque:

$$\Sigma^{(32)}(R^{4n} f_4) = 32[8,3^n],$$
$$\Sigma^{64}(f_2) = 64(p_0 + p_{64})$$
$$= 64\{[2,1] + [2,3^{64}]\} = 64[4,1],$$

atque:

$$\Sigma^{(64)}(R^{2n}f_{(2)}) = 64[4,3^n],$$
$$\Sigma^{(128)}(f_1) = 128p_0 = 128[2,1] = 128.2\cos\frac{2\mu\pi}{257},$$

atque:

$$\Sigma^{128}(R^n f_1) = 128[2,3^n] = 128.2\cos\frac{2\mu n\pi}{257}.$$

In quibus aequationibus si valores quantitatum f et potestatum ipsius R, per angulos expressi, introducantur, agregataque inde orta computentur, nanciscimur:

$$41.\quad \begin{cases} [128,1] = \dfrac{-1+\sqrt{(257)}}{2} = \\ [64,1] = \dfrac{-1+\sqrt{(257)}(1+2\cos\omega_{32})}{4}, \\ [32,1] = \dfrac{-1+\sqrt{(257)}(1+2\Sigma^3\cos\omega_{16})}{8}, \\ [16,1] = \dfrac{-1+\sqrt{(257)}(1+2\Sigma^7\cos\omega_8)}{16}, \\ [8,1] = \dfrac{-1+\sqrt{(257)}(1+2\Sigma^{15}\cos\omega_4)}{32}, \\ [4,1] = \dfrac{-1+\sqrt{(257)}(1+2\Sigma^{31}\cos\omega_2)}{64}, \\ [2,1] = \dfrac{-1+\sqrt{(257)}(1+2\Sigma^{63}\cos\omega_1)}{128}, \end{cases}$$

ubi rursus haec significatio adhibita est generalis:

$$\Sigma^{(z)}\cos\left(\omega_m+\frac{(n)\pi}{q}\right) =$$
$$\cos\left(\omega_m+\frac{n\pi}{q}\right)+\cos\left(\omega_{2m}+\frac{2n\pi}{q}\right)+\cos\left(\omega_{3m}+\frac{3n\pi}{q}\right)+\text{etc.}+\cos\left(\omega_{zm}+\frac{zn\pi}{q}\right).$$

Eadem significatione utentes habemus has formulas generales:

$$42.\quad [128,3^n] = \frac{-1\pm\sqrt{(257)}}{2},$$

$$43.\quad [64,3^n] = \frac{-1\pm\sqrt{(257)}\left\{1\pm 2\cos\left(\omega_{32}+\frac{(n)\pi}{2}\right)\right\}}{4},$$

$$44.\quad [32,3^n] = \frac{-1\pm\sqrt{(257)}\left\{1\pm 2\Sigma^3\cos\left(\omega_{16}+\frac{(n)\pi}{4}\right)\right\}}{8},$$

$$45.\quad [16,3^n] = \frac{-1\pm\sqrt{(257)}\left\{1\pm 2\Sigma^7\cos\left(\omega_8+\frac{(n)\pi}{8}\right)\right\}}{16},$$

$$46.\quad [8,3^n] = \frac{-1\pm\sqrt{(257)}\left\{1\pm 2\Sigma^{15}\cos\left(\omega_4+\frac{(n)\pi}{16}\right)\right\}}{32},$$

$$47.\qquad [4,3^n] = \frac{-1 \pm \sqrt{(257)}\left\{1 \pm 2\Sigma^{31}\cos\left(\omega_2 + \frac{(n)\pi}{32}\right)\right\}}{64},$$

$$48.\qquad [2,3^n] = \frac{-1 \pm \sqrt{(257)}\left\{1 \pm 2\Sigma^{63}\cos\left(\omega_1 + \frac{(n)\pi}{64}\right)\right\}}{128},$$

ubi signa superiora, si n est numerus par, inferiora, si n est impar, quia $R^{64n}f_{64}$ illic $= +\sqrt{(257)}$, hic $= -\sqrt{(257)}$ esse constat, ponantur.

Ut cognoscamus, quanam radice aequationis $\frac{X^{257}-1}{X-1} = 0$, $= \sigma$ supposita, quantitates [128, 1], [64, 1] etc. [2, 1] valeant, ultima formularum (41.) computetur necesse est.

Iam vero facile invenimus:

$\cos\omega_{32} = 0{,}684698.$		
$\cos\omega_{16} = -0{,}974123,$	$\cos\omega_{48} = -0{,}831708.$	
$\cos\omega_{8} = -0{,}843858,$	$\cos\omega_{24} = 0{,}876209,$	$\cos\omega_{40} = -0{,}996325,$
$\cos\omega_{56} = 0{,}700748.$		
$\cos\omega_{4} = -0{,}998121,$	$\cos\omega_{12} = 0{,}931419,$	$\cos\omega_{20} = -0{,}081130,$
$\cos\omega_{28} = -0{,}744520,$	$\cos\omega_{36} = -0{,}045449,$	$\cos\omega_{44} = 0{,}166206,$
$\cos\omega_{52} = 0{,}640719,$	$\cos\omega_{60} = 0{,}875145.$	
$\cos\omega_{2} = -0{,}82338,$	$\cos\omega_{6} = -0{,}48329,$	$\cos\omega_{10} = 0{,}33023,$
$\cos\omega_{14} = 0{,}73504,$	$\cos\omega_{18} = 0{,}90440,$	$\cos\omega_{22} = -0{,}06071,$
$\cos\omega_{26} = 0{,}28323,$	$\cos\omega_{30} = -0{,}88020,$	$\cos\omega_{34} = 0{,}54066,$
$\cos\omega_{38} = 0{,}55487,$	$\cos\omega_{42} = 0{,}99438,$	$\cos\omega_{46} = 0{,}46736,$
$\cos\omega_{50} = 0{,}99990,$	$\cos\omega_{54} = 0{,}96760,$	$\cos\omega_{58} = 0{,}76878,$
$\cos\omega_{62} = 0{,}78707.$		
$\cos\omega_{1} = -0{,}76166,$	$\cos\omega_{3} = -0{,}99839,$	$\cos\omega_{5} = 0{,}97366,$
$\cos\omega_{7} = -0{,}45412,$	$\cos\omega_{9} = -0{,}17236,$	$\cos\omega_{11} = -0{,}00498,$
$\cos\omega_{13} = 0{,}99841,$	$\cos\omega_{15} = 0{,}29454,$	$\cos\omega_{17} = -0{,}58373,$
$\cos\omega_{19} = 0{,}78224,$	$\cos\omega_{21} = -0{,}98944,$	$\cos\omega_{23} = -0{,}99961,$
$\cos\omega_{25} = 0{,}21795,$	$\cos\omega_{27} = -0{,}87975,$	$\cos\omega_{29} = 0{,}99814,$
$\cos\omega_{31} = 0{,}15478,$	$\cos\omega_{33} = 0{,}68209,$	$\cos\omega_{35} = -0{,}59144,$
$\cos\omega_{37} = -0{,}23792,$	$\cos\omega_{39} = -0{,}97893,$	$\cos\omega_{41} = 0{,}87048,$
$\cos\omega_{43} = 0{,}24905,$	$\cos\omega_{45} = 0{,}30511,$	$\cos\omega_{47} = -0{,}05206,$
$\cos\omega_{49} = 0{,}98094,$	$\cos\omega_{51} = 0{,}94159,$	$\cos\omega_{53} = -0{,}05574,$
$\cos\omega_{55} = -0{,}81732,$	$\cos\omega_{57} = 0{,}34695,$	$\cos\omega_{59} = -0{,}99433,$
$\cos\omega_{61} = 0{,}84664,$	$\cos\omega_{63} = -0{,}96579.$	

Quorum cosinuum summa est $= 2{,}97671$. Unde sequitur:

$$[2,1] = \frac{-1+\sqrt{(257)}\,[1+2\,(2{,}97671)]}{128} = 0{,}863044.$$

Ope vero logarithmorum sinuum et cosinuum tabula invenitur:

$$0{,}863044 = 2\cos 64^0\,26', \qquad 8{,}87 = 2\cos\frac{46.2\pi}{257}.$$

Unde sequitur radicem σ suppositam esse:

$$\sigma = \cos\frac{46.2\pi}{257} \pm i\sin\frac{46.2\pi}{257}.$$

Si vero signa (2,), (4,), (8,) etc. in articulis primis adhibita pro radice aequationis $\frac{X^{257}-1}{X-1} = 0$, $\varrho = \cos\frac{2\pi}{257} \pm i\sin\frac{2\pi}{257}$ valere, in articulo VI. inventum esse revocemus, tabula secunda apte adhibita nanciscimur:

$$49.\quad \begin{cases} [2,1] = (2,46) = (2,3^{76}), \\ [4,1] = (4,46) = (4,35) = (4,3^{12}), \\ [8,1] = (8,46) = (8,35) = (8,3^{12}), \\ [16,1] = (16,46) = (15,35) = (16,3^{12}), \\ [32,1] = (32,46) = (32,81) = (32,3^4), \\ [64,1] = (64,46) = (64,1), \\ [128,1] = (128,1), \end{cases}$$

quae aequationes etiam comprobantur computatione. Inde vero prodeunt has aequationes:

$$\begin{aligned} (2,3^x) &= (2,3^{128+x}) = (2,3^{76+52+x}) = [2,3^{52+x}], \\ (4,3^x) &= (4,3^{64+x}) = (4,3^{12+52+x}) = [4,3^{52+x}], \\ (8,3^x) &= (8,3^{32+x}) = (8,3^{12+20+x}) = [8,3^{20+x}], \\ (16,3^x) &= (16,3^{16+x}) = (16,3^{12+4+x}) = [16,3^{4+x}], \\ (32,3^x) &= (32,3^{8+x}) = (32,3^{4+4+x}) = [32,3^{4+x}], \\ (64,3^x) &= [64,3^x], \\ (128,3^x) &= [128,3^x]. \end{aligned}$$

Itaque in formulis (48.), (47.), (46.), (45.), (44.), (43.), (42.) respective posito $n = 2+x = 4+x = 4+x = 4+x = 20+x = 52+x = 52+x$ hae generales proveniunt formulae:

$$50.\quad (128,3^x) = \frac{-1\pm\sqrt{(257)}}{2},$$

$$51.\quad (64,3^x) = \frac{-1\pm\sqrt{(257)}\left\{1\pm 2\cos\left(\omega_{1,} + \frac{(x)\pi}{2}\right)\right\}}{4},$$

$$52.\qquad (32,3^x)=\frac{-1\pm\sqrt{(257)}\left\{1\pm2\Sigma^{(3)}\cos\left(\omega_{16}+\frac{(4+x)\pi}{4}\right)\right\}}{8},$$

$$53.\qquad (16,3^x)=\frac{-1\pm\sqrt{(257)}\left\{1\pm2\Sigma^{(7)}\cos\left(\omega_{8}+\frac{(4+x)\pi}{8}\right)\right\}}{16},$$

$$54.\qquad (8,3^x)=\frac{-1\pm\sqrt{(257)}\left\{1\pm2\Sigma^{(15)}\cos\left(\omega_{4}+\frac{(20+x)\pi}{16}\right)\right\}}{32},$$

$$55.\qquad (4,3^x)=\frac{-1\pm\sqrt{(257)}\left\{1\pm2\Sigma^{(31)}\cos\left(\omega_{2}+\frac{(52+x)\pi}{32}\right)\right\}}{64},$$

$$56.\qquad (2,3^x)=\frac{-1\pm\sqrt{(257)}\left\{1\pm2\Sigma^{(63)}\cos\left(\omega_{1}+\frac{(52+x)\pi}{64}\right)\right\}}{128},$$

superiori signo rursus pro numero pari x, inferiore pro impare x valente. Inde haec totius problematis fluit solutio:

„Si quaeritur formula quantitatem $2\cos\frac{2t\pi}{257}$ sive $(2,t)$ exprimens; ex „tabula quarta facillime quaeratur is index τ quantitatis p sive ea po„testas τ radicis primitivae 3, quae ad t pertinet, qua loco ipsius x „in formula (56.) substituta invenitur:

$$57.\quad (2,3^\tau)=(2,t)=\frac{1}{128}\left[-1\pm\sqrt{(257)}\left(1\pm2\left\{\begin{array}{l}\cos\left(\omega_1+\frac{(52+\tau)\pi}{64}\right)+\cos\left(\omega_2+\frac{2(52+\tau)\pi}{64}\right)+\text{etc.}\\ \qquad+\cos\left(\omega_{31}+\frac{31(52+\tau)\pi}{64}\right)+\cos\left(\omega_{32}+\frac{\tau\pi}{2}\right)\\ \cos\left(\omega_{63}-\frac{(52+\tau)\pi}{64}\right)+\cos\left(\omega_{62}-\frac{2(52+\tau)\pi}{64}\right)+\text{etc.}\\ \qquad+\cos\left(\omega_{33}-\frac{31(52+\tau)\pi}{64}\right)\end{array}\right\}\right)\right]$$

„ubi anguli ω valoribus antea expositis utuntur. Sin quantitates $(4,t)$, „$(8,t)$, $(16,t)$, $(32,t)$, $(64,t)$, $(128,t)$ quaerantur, substituatur nume„rus idem τ in formulis (55.), (54.), (53.), (52.), (51.), (50.), loco „numeri x unde prodeunt hae formulae:

$$(4,\;)=\frac{1}{64}\left[-1\pm\sqrt{(257)}\left(1\pm2\left\{\begin{array}{l}\cos\left(\omega_2+\frac{(52+\tau)\pi}{32}\right)+\cos\left(\omega_4+\frac{2(52+\tau)\pi}{32}\right)+\text{etc.}\\ \qquad+\cos\left(\omega_{30}+\frac{15(52+\tau)\pi}{32}\right)+\cos\left(\omega_{32}+\frac{\tau\pi}{2}\right)\\ \cos\left(\omega_{62}-\frac{(52+\tau)\pi}{32}\right)+\cos\left(\omega_{60}-\frac{2(52+\tau)\pi}{32}\right)+\text{etc.}\\ \qquad+\cos\left(\omega_{34}-\frac{15(52+\tau)\pi}{32}\right)\end{array}\right\}\right)\right],$$

$$(8,t)=\frac{1}{32}\left[-1\pm\sqrt{(257)}\left(1\pm2\left\{\begin{array}{l}\cos\left(\omega_4+\frac{(20+\tau)\pi}{16}\right)+\cos\left(\omega_8+\frac{2(20+\tau)\pi}{12}\right)+\text{etc.}\\ \qquad+\cos\left(\omega_{28}+\frac{7(20+\tau)\pi}{16}\right)+\cos\left(\omega_{32}+\frac{\tau\pi}{2}\right)\\ \cos\left(\omega_{60}-\frac{(20+\tau)\pi}{16}\right)+\cos\left(\omega_{56}-\frac{2(20+\tau)\pi}{16}\right)+\text{etc.}\\ \qquad+\cos\left(\omega_{36}-\frac{7(20+\tau)\pi}{16}\right)\end{array}\right\}\right)\right],$$

$$(16,t)=\frac{1}{16}\left[-1\pm\sqrt{(257)}\left(1\pm2\left\{\begin{array}{l}\cos\left(\omega_8+\frac{(4+\tau)\pi}{8}\right)+\cos\left(\omega_{16}+\frac{2(4+\tau)\pi}{8}\right)\\ \quad+\cos\left(\omega_{24}+\frac{3(4+\tau)\pi}{8}\right)+\cos\left(\omega_{32}+\frac{\tau\pi}{2}\right)\\ \cos\left(\omega_{56}-\frac{(4+\tau)\pi}{8}\right)+\cos\left(\omega_{48}-\frac{2(4+\tau)\pi}{8}\right)\\ \quad+\cos\left(\omega_{40}-\frac{3(4+\tau)\pi}{8}\right)\end{array}\right\}\right)\right],$$

$$(32,t)=\frac{1}{8}\left[-1\pm\sqrt{(257)}\left(1\pm2\left\{\begin{array}{l}\cos\left(\omega_{16}+\frac{(4+\tau)\pi}{4}\right)+\cos\left(\omega_{32}+\frac{\tau\pi}{2}\right)\\ \cos\left(\omega_{48}-\frac{(4+\tau)\pi}{4}\right)\end{array}\right\}\right)\right],$$

$$(64,t)=\frac{1}{4}\left[-1\pm\sqrt{(257)}\left(1\pm2\cos\omega_{32}+\frac{\tau\pi}{2}\right)\right],$$

$$(128,t)=\frac{1}{2}\left[-1\pm\sqrt{(257)}\right],$$

„in quibus omnibus formulis signum superius pro τ formae $2h$, in-
„ferius pro τ formae $2h+1$ valet."

Ut denique exemplo omnia illustremus, ponamus $t=1$. Numerus τ ad $t=1$ pertinens e tabula quarta invenitur $=0$, quo valore in formula (57.) substituto, invenimus:

$$256\cos\left(\frac{2\pi}{257}\right) = -1+\sqrt{(257)}\times$$

$$\left(\begin{array}{l}
\left\{\begin{array}{l}
\left\{\begin{array}{l}
2\Big(\cos\left(\omega_1+\frac{13\pi}{16}\right)+\cos\left(\omega_3+\frac{7\pi}{16}\right)+\cos\left(\omega_5+\frac{\pi}{16}\right)+\cos\left(\omega_7-\frac{5\pi}{16}\right)+\cos\left(\omega_9-\frac{11\pi}{16}\right)\\
\qquad +\cos\left(\omega_{11}+\frac{15\pi}{16}\right)+\cos\left(\omega_{13}+\frac{9\pi}{16}\right)+\cos\left(\omega_{15}+\frac{3\pi}{16}\right)\Big)\\
2\Big(\cos\left(\omega_{63}-\frac{13\pi}{16}\right)+\cos\left(\omega_{61}-\frac{7\pi}{16}\right)+\cos\left(\omega_{59}-\frac{\pi}{16}\right)+\cos\left(\omega_{57}+\frac{5\pi}{16}\right)+\cos\left(\omega_{55}+\frac{11\pi}{16}\right)\\
\qquad +\cos\left(\omega_{53}-\frac{15\pi}{16}\right)+\cos\left(\omega_{51}-\frac{9\pi}{16}\right)+\cos\left(\omega_{49}-\frac{3\pi}{16}\right)\Big)
\end{array}\right\}\\
\left\{\begin{array}{l}
2\Big(\cos\left(\omega_{17}-\frac{3\pi}{16}\right)+\cos\left(\omega_{19}-\frac{9\pi}{16}\right)+\cos\left(\omega_{21}-\frac{15\pi}{16}\right)+\cos\left(\omega_{23}+\frac{11\pi}{16}\right)+\cos\left(\omega_{25}+\frac{5\pi}{16}\right)\\
\qquad +\cos\left(\omega_{27}-\frac{\pi}{16}\right)+\cos\left(\omega_{29}-\frac{7\pi}{16}\right)+\cos\left(\omega_{31}-\frac{13\pi}{16}\right)\Big)\\
2\Big(\cos\left(\omega_{47}+\frac{3\pi}{16}\right)+\cos\left(\omega_{45}+\frac{9\pi}{16}\right)+\cos\left(\omega_{43}+\frac{15\pi}{16}\right)+\cos\left(\omega_{41}-\frac{11\pi}{16}\right)+\cos\left(\omega_{39}-\frac{5\pi}{16}\right)\\
\qquad +\cos\left(\omega_{37}+\frac{\pi}{16}\right)+\cos\left(\omega_{35}+\frac{7\pi}{16}\right)+\cos\left(\omega_{33}+\frac{13\pi}{16}\right)\Big)
\end{array}\right\}
\end{array}\right\}\\
\left\{\begin{array}{l}
2\Big(\cos\left(\omega_2-\frac{3\pi}{8}\right)+\cos\left(\omega_6+\frac{7\pi}{8}\right)+\cos\left(\omega_{10}+\frac{\pi}{8}\right)+\cos\left(\omega_{14}-\frac{5\pi}{8}\right)+\cos\left(\omega_{18}+\frac{5\pi}{8}\right)\\
\qquad +\cos\left(\omega_{22}-\frac{\pi}{8}\right)+\cos\left(\omega_{26}-\frac{7\pi}{8}\right)+\cos\left(\omega_{30}+\frac{3\pi}{8}\right)\Big)\\
2\Big(\cos\left(\omega_{62}+\frac{3\pi}{8}\right)+\cos\left(\omega_{58}-\frac{7\pi}{8}\right)+\cos\left(\omega_{54}-\frac{\pi}{8}\right)+\cos\left(\omega_{50}+\frac{5\pi}{8}\right)+\cos\left(\omega_{46}-\frac{5\pi}{8}\right)\\
\qquad +\cos\left(\omega_{42}+\frac{\pi}{8}\right)+\cos\left(\omega_{38}+\frac{7\pi}{8}\right)+\cos\left(\omega_{34}-\frac{3\pi}{8}\right)\Big)
\end{array}\right\}\\
\left\{\begin{array}{l}
2\Big(\cos\left(\omega_4-\frac{3\pi}{4}\right)+\cos\left(\omega_{12}-\frac{\pi}{4}\right)+\cos\left(\omega_{20}+\frac{\pi}{4}\right)+\cos\left(\omega_{28}+\frac{3\pi}{4}\right)\Big)\\
2\Big(\cos\left(\omega_{60}+\frac{3\pi}{4}\right)+\cos\left(\omega_{52}+\frac{\pi}{4}\right)+\cos\left(\omega_{44}-\frac{\pi}{4}\right)+\cos\left(\omega_{36}-\frac{3\pi}{4}\right)\Big)
\end{array}\right\}\\
\left\{\begin{array}{l}
2\Big(\cos\left(\omega_8+\frac{\pi}{2}\right)+\cos\left(\omega_{24}+\frac{3\pi}{2}\right)\Big)\\
2\Big(\cos\left(\omega_{56}-\frac{\pi}{2}\right)+\cos\left(\omega_{40}-\frac{3\pi}{2}\right)\Big)
\end{array}\right\}
+\left\{\begin{array}{l}
2\,(\cos(\omega_{16}+\pi))\\
2\,(\cos(\omega_{48}-\pi))
\end{array}\right\}
+2\cos(\omega_{32})+1.
\end{array}\right)$$

Quia vero anguli ω ex angulis ϑ componuntur, quippe qui bisectione peripheriae septies repetita construi possunt, iam etiam anguli $=\frac{2\pi}{257}$ constructionem, ad bisectionem circuli septies repetitam reductam esse, clarum est.

Haecce fuerunt quae de problemate proposito scribenda mihi videbantur.

Scripsi Regiomonti, nonis Decembribus 1830.