Some Results in the Elementary Theory of Numbers.

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It is well known that if p is a prime number, all the coefficients of the equation

$$(x-1)(x-2)(x-3)\ldots(x-p+1)+1=0$$

are divisible by p, and that consequently the sum S_μ of the μ^{th} powers of the numbers 1, 2, 3, ... $p-1$ is divisible by p unless μ is a multiple of $p-1$ (see, *e.g.*, Serret, *Algèbre Supérieure*, Vol. II., p. 46). And it is clear in the same way that if $S_{-\mu}$, the sum of the inverse μ^{th} powers of the same numbers, be formed, the result of the addition will be a fraction whose numerator is divisible by p unless μ is a multiple of $p-1$. Since all the factors of the denominator of such fraction are prime to p, we may, for shortness, say that $S_{-\mu}$ is divisible by p unless μ is a multiple of $p-1$.

But this does not exhaust the whole truth. In fact, if μ is odd, S_μ and $S_{-\mu}$ are both divisible by p^2 with certain exceptions. In particular the sum of the reciprocals of 1, 2, 3, ... $p-1$ is a fraction whose numerator is always divisible by p^2 except when $p=2$ or $p=3$.* Results similar in character hold when p is replaced by any composite number N, and the numbers 1 to $p-1$ by the numbers less than N and prime to it. The object of the present paper is to prove the foregoing statements, and to examine into the divisibility by the various prime factors of any number N of the sum of the μ^{th} powers of the numbers less than N and prime to it. I confine myself chiefly to the inverse powers, as the results when μ is positive are of less interest and present less difficulty, since the actual value of S_μ can be found by summing the series.

1. Here, and in all that follows, μ will be supposed to be an *odd number*.

Using $M(p)$ to denote a multiple of p, and $\psi_\mu(p)$ to represent

* The remark that if the reciprocals of 1, 2, 3, ... $p-1$ are added together, the numerator of the resulting fraction is generally divisible not only by p, but by p^2, when p is a prime, was made to me some little time since by Mr. J. M. Dyer, M.A., of Eton College; and it is this remark which first led me to the subject of the present paper. I desire also to record my indebtedness to Mr. Morgan Jenkins, M.A., for some emendations and suggestions, especially in connection with § 2, which have proved very useful.

what has been hitherto called $S_{-\mu}$, we have

$$\psi_\mu(p) = \frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \ldots + \frac{1}{(p-1)^\mu}$$

$$= \tfrac{1}{2}\Sigma\left\{\frac{1}{x^\mu} + \frac{1}{(p-x)^\mu}\right\} \text{ from } x=1 \text{ to } x=p-1$$

$$= \frac{p}{2}\,\Sigma_1^{p-1}\,\frac{(p-x)^{\mu-1}-(p-x)^{\mu-2}x+\ldots+x^{\mu-1}}{x^\mu(p-x)^\mu} \text{ since } \mu \text{ is odd}$$

$$= \frac{p}{2}\left\{M(p) - \frac{\mu}{2}\,\Sigma_1^{p-1}\frac{x^{\mu-1}}{x^{2\mu}}\right\}$$

$$= \tfrac{1}{2}M(p^2) - \frac{\mu p}{2}\,\psi_{\mu+1}(p).$$

Therefore (excluding only the case $p=2$) it is seen that $\psi_\mu(p)$ is $M(p^2)$ if $\frac{\mu}{2}\psi_{\mu+1}(p)$ is $M(p)$, which is the case unless $\mu+1 = M(p-1)$ and at the same time μ is not $M(p)$. Now if $\mu+1 = M(p-1)$ and $\mu = M(p)$ at the same time, μ must be of the form $p\{(p-1)t-1\}$. The conclusion is, therefore, that if μ is odd, $\psi_\mu(p)$ is in general divisible by p^2; but if $\mu+1 = M(p-1)$, then $\psi_\mu(p)$ is only divisible by p, except in the cases where μ is of the form $p\{(p-1)t-1\}$, when $\psi_\mu(p)$ is divisible by p^2 as in general.

Exceptions.—If $p=2$, $\psi_\mu(p)$ reduces to 1 and is not divisible by p at all, as is indeed shown by the formula at once.

And if $p=3$, $\mu+1$, being even, must be $M(p-1)$; therefore $\psi_\mu(3)$ is only divisible by 3^2 when μ is an odd multiple of 3, and in all other cases $\psi_\mu(3)$ is divisible by 3, but not by 3^2.

2. Let now N be any number, and let $f_\mu(N, n)$ denote the sum of the inverse μ^{th} powers of the numbers, prime to N, which lie between nN and $(n+1)N$, where μ is odd as before. Then, since if $nN+x$ is such a number, $(n+1)N-x$ is another,

$$f_\mu(N, n) = \tfrac{1}{2}\Sigma\left\{\frac{1}{(nN+x)^\mu} + \frac{1}{(nN+N-x)^\mu}\right\},$$

the summation taking place for every positive integral value of x less than N and prime to N; and the coefficient $\frac{1}{2}$ being required because $nN+x$ and $nN+N-x$ are repeated and interchanged when x is changed to $N-x$. Since μ is odd, $f_\mu(N, n)$ is evidently divisible by

$nN+x+nN+N-x$, that is, by $(2n+1)N$. So

$$f_\mu(N,0)=\tfrac{1}{2}\Sigma\left\{\frac{1}{x^\mu}+\frac{1}{(N-x)^\mu}\right\},$$

therefore
$$\frac{1}{2n+1}f_\mu(N,n)-f_\mu(N,0)$$

$$=\tfrac{1}{2}\Sigma\left\{\frac{(nN+x)^{-\mu}+(nN+N-x)^{-\mu}}{2n+1}-\frac{x^{-\mu}+(N-x)^{-\mu}}{1}\right\}$$

$$=\tfrac{1}{2}\Sigma x^{-\mu}\left\{\frac{\left(1+\frac{nN}{x}\right)^{-\mu}+(-1)^\mu\left(1-\overline{n+1}\frac{N}{x}\right)^{-\mu}}{2n+1}-\frac{1+(-1)^\mu\left(1-\frac{N}{x}\right)^{\mu}}{1}\right\}$$

$$=\Sigma\tfrac{1}{2}x^{-\mu}\left\{\frac{\left[1-\mu n\frac{N}{x}+\frac{\mu(\mu+1)}{1.2}n^2\frac{N^2}{x^2}-\dots\right]-\left[1+\mu(n+1)\frac{N}{x}+\frac{\mu(\mu+1)}{1.2}(n+1)^2\frac{N^2}{x^2}+\dots\right]}{2n+1}\right.$$
$$\left.-1+\left[1+\mu\frac{N}{x}+\frac{\mu(\mu+1)}{1.2}\frac{N^2}{x^2}+\dots\right]\right\}$$

The constant term inside the brackets vanishes, as also does the coefficient of $\frac{N}{x}$; the coefficient of the general term $\left(\frac{N}{x}\right)^{m+1}$ being

$$-\frac{\mu(\mu+1)\dots(\mu+m)}{1.2.3\dots(m+1)}\left\{\frac{(n+1)^{m+1}-(-n)^{m+1}}{2n+1}-1\right\}$$

It is easily seen that this coefficient is always divisible by $n(n+1)$, and that $(n+1)^{m+1}-(-n)^{m+1}$ is always divisible by $2n+1$.

We have then

$$\frac{1}{2n+1}f_\mu(N,n)-f_\mu(N,0)=-\tfrac{1}{2}\Sigma x^{-\mu}\left\{\frac{\mu(\mu+1)(\mu+2)}{1.2.3}\frac{N^3}{x^3}(n^2+n)\right.$$
$$\left.+\frac{\mu(\mu+1)(\mu+2)(\mu+3)}{1.2.3.4}\frac{N^4}{x^4}2(n^2+n)+\dots\right\}$$

$$=-\frac{N^3}{2}(n^2+n)\frac{\mu(\mu+1)(\mu+2)}{1.2.3}\left\{\Sigma x^{-(\mu+3)}+\frac{\mu+3}{2}N\Sigma x^{-(\mu+4)}+\dots\right\}$$

$$=-\frac{N^3}{2}(n^2+n)\frac{\mu(\mu+1)(\mu+2)}{6}\left\{f_{\mu+3}(N,0)\right.$$
$$\left.+\frac{\mu+3}{2}Nf_{\mu+4}(N,0)+\dots\right\}\dots\dots\dots\dots(1)$$

3. Next, let $\psi_\mu(N)$ denote the sum of the inverse μ^{th} powers of all the numbers less than N, and prime to it [so that $\psi_\mu(N)$ is the same thing as $f_\mu(N, 0)$]. Further, let $a, b, c, \ldots$ be the prime factors of N, so that $N = a^l b^m c^n \ldots$ say, where $l, m, n, \ldots$ are supposed each greater than unity; and write $\frac{N}{a}$ for $a^{l-1} b^m c^n \ldots$. Then, since any number prime to N is prime to $\frac{N}{a}$,

$$\psi_\mu(N) = f_\mu\left(\frac{N}{a}, 0\right) + f_\mu\left(\frac{N}{a}, 1\right) + f_\mu\left(\frac{N}{a}, 2\right) + \ldots + f_\mu\left(\frac{N}{a}, a-1\right)$$

$$= f_\mu\left(\frac{N}{a}, 0\right)$$

$$+3\left[f_\mu\left(\frac{N}{a}, 0\right) - 1\,.\,2\,\frac{\mu(\mu+1)(\mu+2)}{12}\,\frac{N^3}{a^3}\left\{f_{\mu+3}\left(\frac{N}{a}, 0\right)\right.\right.$$
$$\left.\left. + \frac{\mu+3}{2}\,\frac{N}{a}\,f_{\mu+4}\left(\frac{N}{a}, 0\right) + M\left(\frac{N^2}{a^2}\right)\right\}\right]$$

$$+5\left[f_\mu\left(\frac{N}{a}, 0\right) - 2\,.\,3\,\frac{\mu(\mu+1)(\mu+2)}{12}\,\frac{N^3}{a^3}\left\{f_{\mu+3}\left(\frac{N}{a}, 0\right)\right.\right.$$
$$\left.\left. + \frac{\mu+3}{2}\,\frac{N}{a}\,f_{\mu+4}\left(\frac{N}{a}, 0\right) + M\left(\frac{N^2}{a^2}\right)\right\}\right]$$

+ &c.

$$+(2a-1)\left[f_\mu\left(\frac{N}{a}, 0\right) - (a-1)\,a\,\frac{\mu(\mu+1)(\mu+2)}{12}\,\frac{N^3}{a^3}\left\{f_{\mu+3}\left(\frac{N}{a}, 0\right)\right.\right.$$
$$\left.\left. + \frac{\mu+3}{2}\,\frac{N}{a}\,f_{\mu+4}\left(\frac{N}{a}, 0\right) + M\left(\frac{N^2}{a^2}\right)\right\}\right]$$

by equation (1) of § 2;

$$= (1+3+5+\ldots+\overline{2a-1})\,\psi_\mu\left(\frac{N}{a}\right)$$
$$-(1\,.\,2\,.\,3 + 2\,.\,3\,.\,5 + \ldots + \overline{2a-1}\,.\,\overline{a-1}\,.\,a)\,\frac{\mu(\mu+1)(\mu+2)}{12}$$
$$\left\{\frac{N^3}{a^3}\,\psi_{\mu+3}\left(\frac{N}{a}\right) + \frac{\mu+3}{2}\,M\left(\frac{N^4}{a^4}\right)\right\}$$

$$= a^2\psi_\mu\left(\frac{N}{a}\right) - \frac{\mu(\mu+1)(\mu+2)}{24}\,a^2(a^2-1)\left\{\frac{N^3}{a^3}\,\psi_{\mu+3}\left(\frac{N}{a}\right)\right.$$
$$\left. + \frac{\mu+3}{2}\,M\left(\frac{N^4}{a^4}\right)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$$\therefore\ \psi_\mu(N)-a^2\psi_\mu\left(\frac{N}{a}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$
$$\times\left\{a^{3l-1}b^{3m}c^{2n}\ldots\psi_{\mu+3}\left(\frac{N}{a}\right)+\frac{\mu+3}{2}M(a^{4l-2}b^{4m}c^{4n}\ldots)\right\}$$

Similarly,

$$\psi_\mu\left(\frac{N}{a}\right)-a^2\psi_\mu\left(\frac{N}{a^2}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$
$$\times\left\{a^{3l-4}b^{3m}c^{3n}\ldots\psi_{\mu+3}\left(\frac{N}{a^2}\right)+\frac{\mu+3}{2}M(a^{4l-6}b^{4m}c^{4n}\ldots)\right\},$$

and so on, until finally

$$\psi_\mu\left(\frac{N}{a^{l-2}}\right)-a^2\psi_\mu\left(\frac{N}{a^{l-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$
$$\times\left\{a^5b^{3m}c^{3n}\ldots\psi_{\mu+3}\left(\frac{N}{a^{l-1}}\right)+\frac{\mu+3}{2}M(a^6b^{4m}c^{4n}\ldots)\right\},$$

from which, by multiplying by 1, a^2, a^4, ... a^{2l-4}, and adding

$$\psi_\mu(N)-a^{2l-2}\psi_\mu\left(\frac{N}{a^{l-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(a^2-1)$$
$$\times\left\{M(a^{2l+1}b^{3m}c^{3n}\ldots)+\frac{\mu+3}{2}M(a^{2l+2}b^{4m}c^{4n}\ldots)\right\}.$$

Similarly,

$$\psi_\mu\left(\frac{N}{a^{l-1}}\right)-b^{2m-2}\psi_\mu\left(\frac{N}{a^{l-1}b^{m-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}(b^2-1)$$
$$\times\left\{M(a^3b^{2m+1}c^{3n}\ldots)+\frac{\mu+3}{2}M(a^4b^{2m+2}c^{4n}\ldots)\right\};$$

$$\therefore\ \psi_\mu(N)-a^{2l-2}b^{2m-2}\psi_\mu\left(\frac{N}{a^{l-1}b^{m-1}}\right)=-\frac{\mu(\mu+1)(\mu+2)}{24}$$
$$\times\left\{M(a^{2l+1}b^{2m+1}c^{3n}\ldots)+\frac{\mu+3}{2}M(a^{2l+2}b^{2m+2}c^{4n}\ldots)\right\},$$

and by proceeding in the same manner we finally arrive at

$$\psi_\mu(N)-a^{2l-2}b^{2m-2}c^{2n-2}\ldots\psi_\mu\left(\frac{N}{a^{l-1}b^{m-1}c^{n-1}\ldots}\right)$$
$$=-\frac{\mu(\mu+1)(\mu+2)}{24}\left\{M(a^{2l+1}b^{2m+1}c^{2n+1}\ldots)+\frac{\mu+3}{2}M(a^{2l+2}b^{2m+2}c^{2n+2}\ldots)\right\}$$
$$\ldots\ldots\ldots\ldots\ldots(3).$$

If N_0 be written for the number $abc\ldots$, this last result is

$$\psi_\mu(N)=\frac{N^2}{N_0^2}\psi_\mu(N_0)-\frac{\mu(\mu+1)(\mu+2)}{24}\left\{M(N^2N_0)+\frac{\mu+3}{2}M(N^2N_0^2)\right\} \quad\ldots\ldots\ldots\ldots\ldots(4).$$

The determination of the divisibility of $\psi_\mu(N)$ by the prime factors of N is therefore reduced to the determination of the same thing with regard to $\psi_\mu(N_0)$.

4. Let p be a prime, and N any number prime to p. To obtain the numbers less than Np and prime to it we must deduct from the numbers less than Np and prime to N those which are multiples of p. But these last are just p multiplied by the numbers less than N and prime to N; accordingly $\psi_\mu(Np)$ is equal to

$$f_\mu(N,0)+f_\mu(N,1)+f_\mu(N,2)+\ldots+f(N,p-1)-\frac{1}{p^\mu}f_\mu(N,0).$$

Making use of equation (1) of § 2 to transform the expression on the right-hand side, by a method similar to that of § 3, we obtain

$$\begin{aligned}\psi_\mu(Np)&=p^2\psi_\mu(N)-\frac{\mu(\mu+1)(\mu+2)}{24}p^2(p^2-1)\\&\qquad\times\left\{N^3\psi_{\mu+3}(N)+\frac{\mu+3}{2}M(N^4)\right\}-\frac{1}{p^\mu}f_\mu(N,0)\\&=\frac{p^{\mu+2}-1}{p^\mu}\psi_\mu(N)-\frac{\mu(\mu+1)(\mu+2)}{24}p^2(p^2-1)\\&\qquad\times\left\{N^3\psi_{\mu+3}(N)+\frac{\mu+3}{2}M(N^4)\right\}\quad\ldots\ldots\ldots\ldots\ldots(5).\end{aligned}$$

By means of this formula we can reduce the determination of the divisibility of $\psi_\mu(abc\ldots k)$ to the determination of the same thing with regard to $\psi_\mu(bc\ldots k)$, and so, by successive applications of the formula, to the determination of it with regard to $\psi_\mu(k)$, which question has been solved in § 1. By means therefore of (3) or (4) and (5), it can be found how many times each of the prime factors of N divides $\psi_\mu(N)$. It will be seen presently that $\psi_{\mu+3}(N)$, where $\mu+3$ is an even number, is in general a multiple of N, so that in (5) we may often write $M(N)$ for $\psi_{\mu+3}(N)$; but there are several exceptions to this (see below, § 6). If N is a prime number it has been seen in § 1 that $\psi_{\mu+3}(N)$ is a multiple of N unless $\mu+3 = M(N-1)$; thus, if a and b are two primes,

$$\psi_\mu(ab) = \frac{b^{\mu+2}-1}{b^\mu}\psi_\mu(a) - \frac{\mu(\mu+1)(\mu+2)}{24}b^2(b^2-1)M(a^4) \ldots\ldots (6),$$

unless $\mu+3 = M(a-1)$, when $M(a^3)$ must be substituted in place of $M(a^4)$.

5. In any case, whether $\psi_{\mu+3}(N)$ is or is not divisible by N, we have always

$$\psi_\mu(Np) = \frac{p^{\mu+2}-1}{p^\mu}\psi_\mu(N) - \frac{\mu(\mu+1)(\mu+2)}{24}p^2(p^2-1)M(a^3),$$

so that, if $a, b, c \ldots k$ are prime numbers,

$$\psi_\mu(abc \ldots k) = \frac{a^{\mu+2}-1}{a^\mu}\psi_\mu(bc \ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24}M(b^3c^3 \ldots k^3)$$

$$= \frac{a^{\mu+2}-1}{a^\mu}\left\{\frac{b^{\mu+2}-1}{b^\mu}\psi_\mu(c \ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24}M(c^3 \ldots k^3)\right\}$$

$$- \frac{\mu(\mu+1)(\mu+2)}{24}M(b^3c^3 \ldots k^3)$$

$$= \frac{(a^{\mu+2}-1)(b^{\mu+2}-1)}{a^\mu b^\mu}\psi_\mu(c \ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24}M(c^3 \ldots k^3)$$

$$= \&c.$$

$$= \frac{(a^{\mu+2}-1)(b^{\mu+2}-1)(c^{\mu+2}-1)\ldots}{a^\mu b^\mu c^\mu \ldots}\psi_\mu(k) - \frac{\mu(\mu+1)(\mu+2)}{24}M(k^3) \quad \ldots\ldots\ldots\ldots\ldots (7).$$

But by (3)

$$\psi_\mu(a^l b^m c^n \ldots) = a^{2l-2}b^{2m-2}c^{2n-2}\ldots\psi_\mu(abc \ldots k) - \frac{\mu(\mu+1)(\mu+2)}{24}M(a^{2l+1}b^{2m+1}c^{m+1}\ldots) \quad \ldots\ldots\ldots\ldots (8).$$

Combining this with (7), after interchanging in the latter formula a and k for convenience,

$$\psi_\mu(a^l b^m c^n \ldots) = b^{2m+\mu-2}c^{2n+\mu-2}\ldots(b^{\mu+2}-1)(c^{\mu+2}-1)\ldots a^{2l-2}\psi_\mu(a) - \frac{\mu(\mu+1)(\mu+2)}{24}M(a^{2l+1}) \quad \ldots\ldots\ldots\ldots (9),$$

where a is any one of the prime factors of N.

Having regard to what has been proved in § 1 as to $\psi_\mu(a)$, and remembering that $\mu(\mu+1)(\mu+2)$ is always divisible by 6, we deduce from (8) the following conclusions:—

If a is any prime except 2 or 3, $\psi_\mu(N)$ is in general divisible by a^{2l}; but if $\mu+1 = M(a-1)$, then only by a^{2l-1}, unless at the same time $\mu = M(a)$, when $\psi_\mu(N)$ is divisible by a^{2l} as in general.

If a is 3, $\psi_\mu(N)$ is divisible by a^{2l} when μ is an odd multiple of 3, but in all other cases only by a^{2l-1}.

If a is 2, $\psi_\mu(N)$ is divisible by a^{2l-1}, except when the number of prime factors of N is less than 2 (*i.e.*, when $N = a^l$ simply), in which case $\psi_\mu(N)$ is divisible by a^{2l-2} only. This follows since $\mu+2$ is odd, and therefore $b^{\mu+2}-1$, $c^{\mu+2}-1$, &c. are each divisible by 2.

Should any of the expressions $b^{\mu+2}-1$, $c^{\mu+2}-1$, &c., or $\mu(\mu+1)(\mu+2)$ be a multiple of a, or of a power of a, then $\psi_\mu(N)$ may be divisible by a higher power of a than that given above; and this will frequently be the case.

6. By proceeding exactly as in § 1, it is seen that

$$\psi_\mu(N) = \tfrac{1}{2}M(N^2) - \frac{\mu}{2}N\psi_{\mu+1}(N).$$

Thus the divisibility of $\psi_{\mu+1}(N)$, where $\mu+1$ is *even*, by any prime factor a of N, depends on the divisibility of $\psi_\mu(N)$ by a, and this has been determined in § 5.

In particular, let $\mu = 3$; then

$$\psi_4(N) = \tfrac{1}{3}M(N) - \frac{2}{3N}\psi_3(N).$$

Now by § 5, $\psi_3(N)$ will be divisible in general by a^{2l}, and this will be so even if $a = 3$; but if $a = 5$ it is divisible by a^{2l-1} only, and if $a = 2$ by a^{2l-1} in all cases except where N is equal merely to a^l, when $\psi_3(N)$ will be divisible only by a^{2l-2}.

Accordingly, $\psi_4(N)$ is in general divisible by a^l, the exceptions being when a is 2, 3, or 5.

If $a = 5$, $\psi_4(N)$ is divisible by a^{l-1}.

If $a = 3$, $\psi_4(N)$ is divisible by a^{l-1}.

If $a = 2$, $\psi_4(N)$ follows the rule of being divisible by a^l unless $N = a^l$ merely, in which case $\psi_4(N)$ is divisible only by a^{l-1}.

The conclusion is therefore that $\psi_4(N) = \dfrac{1}{3\,.\,5}M(N)$ in every case except where N is simply a power of 2, when $\psi_4(N) = \tfrac{1}{2}M(N)$.

7. I proceed to consider in somewhat greater detail the case of $\mu = 1$, *i.e.*, that of the sum of the reciprocals of the numbers less than a number N and prime to it. When N is a prime number, p suppose, the result corresponding to that of § 1 may be proved conveniently as follows:—

The congruence

$$\{x^2+1(p-1)\}\{x^2+2(p-2)\}\dots\left\{x^2+\frac{p-1}{2}\,\frac{p+1}{2}\right\}-(x^{p-1}-1)\equiv 0 \pmod{p},$$

is satisfied by $p-1$ values of x, viz., $1, 2, 3 \dots p-1$. For $x^2+r(p-r)$ becomes pr when $x=r$ and $p(p-r)$ when $x=p-r$, so that it is divisible by p in either case; and $x^{p-1}-1$ is divisible by p when x has any one of the stated values, by Fermat's theorem. Now, since the congruence is only of degree $p-3$, it must be identical; therefore the numbers $1(p-1)$, $2(p-2)$, $3(p-3)$, ... $\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)$ are such that their sum, product two and two, ... $\frac{p-3}{2}$ and $\frac{p-3}{2}$ together, are all divisible by p. The last of these gives

$$\underline{|p-1}\left\{\frac{1}{1(p-1)}+\frac{1}{2(p-2)}+\dots+\frac{1}{\left(\frac{p+1}{2}\right)\left(\frac{p+1}{2}\right)}\right\},$$

or
$$\frac{1}{p}\,\underline{|p-1}\left\{\frac{1}{1}+\frac{1}{p-1}+\frac{1}{2}+\frac{1}{p-2}+\dots+\frac{1}{\frac{p-1}{2}}+\frac{1}{\frac{p+1}{2}}\right\},$$

divisible by p; so that

$$\underline{|p-1}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p-1}\right)$$

is divisible by p^2

Since the congruence is of degree $p-3$, p must be greater than 3, and the argument does not hold for $p=2$ or $p=3$. In fact, if $p=3$, $\psi_1(p)$ is divisible only by p; and if $p=2$, $\psi_1(p)$ reduces to 1 and is not divisible by p at all.

8. For the case of $\mu = 1$ the formula (5) becomes

$$\psi_1(Np)=\frac{p^3-1}{p}\psi_1(N)-\tfrac{1}{4}p^2(p^2-1)\{N^3\psi_4(N)+2M(N^4)\}\quad\dots(10),$$

which in accordance with what has been proved in § 6 may be written,

$$\psi_1(Np)=\frac{p^3-1}{p}\psi_1(N)-\frac{p^2(p^2-1)}{3.4.5}M(N^4)\quad\dots\dots\dots(11),$$

unless N is simply a power of 2, say 2^n, in which case

$$\psi_1(Np) = \frac{p^3-1}{p}\psi_1(N) - \frac{p^2(p^2-1)}{8}M(N^4),$$

or

$$\psi_1(2^n p) = \frac{p^3-1}{p}\psi_1(2^n) - p^2(p^2-1)M(2^{4n-3}) \quad \ldots\ldots\ldots(12).$$

9. Writing in (9), $N = a$, $p = b$, where a and b are primes,

$$\psi_1(ab) = \frac{b^3-1}{b}\psi_1(a) - \frac{b^2(b^2-1)}{60}M(a^4).$$

Again, writing $N = ab$, $p = c$, where a, b, c are primes,

$$\psi_1(abc) = \frac{c^3-1}{c}\psi_1(ab) - \frac{c^2(c^2-1)}{60}M(a^4b^4)$$

$$= \frac{(b^3-1)(c^3-1)}{bc}\psi_1(a) - \frac{b^2(b^2-1)(c^3-1)}{60c}M(a^4) - \frac{b^4c^2(c^2-1)}{60}M(a^4)$$

$$= \frac{(b^3-1)(c^3-1)}{bc}\psi_1(a) - \frac{b^2(c-1)}{60c}M(a^4).$$

Then, writing $N = abc$, $p = d$, where a, b, c, d are primes,

$$\psi_1(abcd) = \frac{(b^3-1)(c^3-1)(d^3-1)}{bcd}\psi_1(a) - \frac{d-1}{60cd}M(a^4),$$

and by proceeding in the same manner we obtain

$$\psi_1(abcd\ldots k) = \frac{(b^3-1)(c^3-1)(d^3-1)\ldots(k^3-1)}{bcd\ \ldots\ k}\psi_1(a) + \frac{k-1}{60cd\ldots k}M(a^4),$$

where k stands for any of the prime factors of $N = abcd\ldots k$. From this formula conclusions may be drawn with regard to $\psi_1(N)$ in the same manner as has been done in § 5 with regard to $\psi_\mu(N)$. It may be noticed that the numerator of $\psi_1(N)$ is divisible by $k-1$, and so by each of the numbers $a-1$, $b-1$, &c.; but the denominator of $\psi_1(N)$ will not in general be prime to these numbers. In working out any numerical case in practice the best method will be to make use of (8) and of successive applications of (9) or (10).

10. Formulæ similar in character to the foregoing may in the same way be proved for the sum [say $\phi_\mu(N)$] of the μ^{th} powers of the numbers less than N and prime to it, μ being a positive odd integer. For the reasons explained at the beginning of the paper it may be sufficient to give a statement of the results for this case, merely working out the formula corresponding to (1).

If $F_\mu(N, n)$ denote the sum of the μ^{th} powers of the numbers prime to N which lie between nN and $(n+1)N$, then exactly as in § 2, using a similar notation,

$$\frac{1}{2n+1} F_\mu(N,n) - F_\mu(N,0)$$

$$= \tfrac{1}{2}\Sigma \left\{ \frac{(nN+x)^\mu + (nN+N-x)^\mu}{2n+1} - \frac{x^\mu + (N-x)^\mu}{1} \right\}$$

$$= \tfrac{1}{2}\Sigma x^\mu \left\{ \frac{\left(1+\frac{nN}{x}\right)^\mu + (-1)^\mu \left(1+\frac{nN+N}{x}\right)^\mu}{2n+1} - 1 + (-1)^\mu \left(1-\frac{N}{x}\right)^\mu \right\}$$

$$= \frac{N^3}{2}(n^2+n)\frac{\mu(\mu-1)(\mu-2)}{6} \left\{ \phi_{\mu-3}(N) - \frac{\mu-3}{2}\phi_{\mu-4}(N) + \dots \right\},$$

the coefficient of the general term $\phi_{\mu-m-1}(N)$ being

$$\frac{N^3}{2}\frac{\mu(\mu-1)\dots(\mu-m)}{1.2\dots(m+1)} \left\{ \frac{(n+1)^{m+1} - (-n)^{m+1}}{2n+1} - 1 \right\}.$$

This is the formula corresponding in this case to (1). Corresponding to the results of § 1, we have precisely similar ones with $\mu-1$ written in place of the $\mu+1$ which occurs there. Corresponding to (2), (3), (5), (7), (9), respectively, we find

$$\phi_\mu(N) = a^2 \phi_\mu\left(\frac{N}{a}\right)$$

$$+ \frac{\mu(\mu-1)(\mu-2)}{24} a^2(a^2-1) \left\{ \frac{N^3}{a^3}\phi_{\mu-3}\left(\frac{N}{a}\right) - \frac{\mu+3}{2} M\left(\frac{N^4}{a^4}\right) \right\},$$

$$\phi_\mu(N) = a^{2l-2} b^{2m-2} c^{2n-2} \dots \phi_\mu\left(\frac{N}{a^{l-1} b^{m-1} c^{n-1}\dots}\right)$$

$$+ \frac{\mu(\mu-1)(\mu-2)}{24} \left\{ M(a^{2l+1} b^{2m+1} c^{2n+1}\dots) - \frac{\mu-3}{2} M(a^{2l+2} b^{2m+2} c^{2n+2}\dots) \right\},$$

$$\phi_\mu(Np) = -p^2(p^{\mu-2}-1)\phi_\mu(N)$$

$$+ \frac{\mu(\mu-1)(\mu-2)}{24} p^3(p^2-1) \left\{ N^3\phi_{\mu-3}(N) - \frac{\mu-3}{2} M(N^4) \right\},$$

$$\phi_\mu(abc\dots) = a^2b^2c^2\dots(1-a^{\mu-2})(1-b^{\mu-2})(1-c^{\mu-2})\dots\phi_\mu(k)$$

$$+ \frac{\mu(\mu-1)(\mu-2)}{24} M(k^3),$$

and $\phi_\mu (a^l b^m c^n \ldots) = b^{2m} c^{2n} \ldots (1-a^{\mu-2})(1-b^{\mu-2})(1-c^{\mu-2}) \ldots a^{2l-2} \phi_\mu (a)$

$$+\frac{\mu(\mu-1)(\mu-2)}{24} M(a^{2l+1}).$$

If therefore $-\mu$ be written for μ in the conclusions as to $\psi_\mu (N)$ given in § 5, these will apply in the case of $\phi_\mu (N)$.

11. It is necessary in applying the foregoing formulæ to bear in mind their precise meaning, as otherwise wrong conclusions may easily be drawn from them. Such a formula as (11), for instance, shows that, if $N = 2^\alpha 3^\beta 5^\gamma a^l b^m c^n \ldots$, the numerator of the expression

$$\frac{p^3-1}{p} \psi_1 (N) - \psi_1 (Np)$$

is divisible by $2^{4\alpha-2} 3^{4\beta-1} 5^{4\gamma-1} a^{4l} b^{4m} c^{4n} \ldots$ in any case, and by any further power of any prime factor of N which may happen to be contained in p^2-1. The formula must not be interpreted as affirming anything concerning the divisibility of the numerator of the above-mentioned expression by p^3 or p^3-1, because p^2 and p^2-1 need not be prime to 3, 4, or 5, nor to the denominator of the expression denoted in (11) by $M(N^4)$. For example, if $N = 2^2$ and $p = 3$, formula (12) shows that

$$\begin{aligned} \frac{26}{3} \psi_1 (4) - \psi_1 (12) &= 9 \,.\, 8 M(2^5) \\ &= 9 \,.\, M(2^8), \end{aligned}$$

and the conclusion is that the expression on the left-hand side is divisible by 2^8, which is correct, since it is equal to

$$\begin{aligned} \frac{26}{3}\left(\frac{1}{1}+\frac{1}{3}\right)-\left(\frac{1}{1}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}\right) &= \frac{35072}{1155} \\ &= \frac{2^8 \,.\, 137}{1155}. \end{aligned}$$

But it would be wrong to infer that the expression is divisible by 9, as in fact can be seen at once on going back to formula (10) from which (12) was deduced. For $\psi_4 (N)$, which occurs on the right-hand side of (10), is in this case $\frac{1}{1}+\frac{1}{81}$ or $\frac{82}{81}$, and so involves a power of 9 in its denominator which cancels against the 9 arising from the factor p^2-1.

12. I add a few numerical examples of the application of the formulæ.

Ex. 1. $N=21$, $\mu=1$.

By (11), $\psi_1(21)=\frac{342}{7}\psi_1(3)-\frac{49.48}{60}M(3^4)=M(3^3)$,

and $$\psi_1(21)=\frac{26}{3}\psi_1(7)-\frac{9.8}{60}M(7^4)=M(7^2),$$

therefore $$\psi_1(21)=M(3^3.7^2),$$

and then, by (8),

$$\psi_1(3^l.7^m)=3^{2l-2}.7^{2m-2}\psi_1(21)-\tfrac{1}{4}M(3^{2l+1}.7^{2m+1})$$
$$=M(3^{2l+1}.7^{2m}).$$

Ex. 2. $N=505$, $\mu=1$.

By (11),

$$\psi_1(505)=\frac{1030300}{101}\psi_1(5)-\frac{10201.10200}{60}M(5^4)=M(5^4),$$

and $$\psi_1(505)=\frac{124}{5}\psi_1(101)-\frac{24.25}{60}M(101^4)=M(101^2),$$

therefore $$\psi_1(505)=M(5^4.101^2).$$

Ex. 3. $N=78$, $\mu=1$.

Here $$\psi_1(78)=\frac{26}{3}\psi_1(26)-\frac{9.8}{60}M(26^4),$$

$$\psi_1(78)=\frac{2196}{13}\psi_1(6)-\frac{169.168}{60}M(6^4),$$

$$\psi_1(26)=\frac{7}{2}\psi_1(13)-\frac{4.3}{60}M(13^4),$$

$$\psi_1(6)=\frac{26}{3}\psi_1(2)-9.8\,M(2),\text{ by (12)},$$

$$\psi_1(6)=\frac{7}{2}\psi_1(3)-\frac{4.3}{60}M(3^4),$$

by (11). Therefore

$$\psi_1(6) = M(2), \quad \psi_1(6) = M(3), \quad \psi_1(26) = M(13^2);$$

and $$\psi_1(78) = M(2^3 \,.\, 3^3 \,.\, 13^3) = M(78^3).$$

Ex. 4. $N = 28$, $\mu = 13$.

By (3),

$$\psi_{13}(28) = 2^2\psi_{13}(14) - \frac{13 \,.\, 14 \,.\, 15}{24}\left\{M(2^4 \,.\, 7^2) + 8M(2^5 \,.\, 7^3)\right\}$$

$$= 2^2\psi_{13}(14) - M(2^2 \,.\, 7^3),$$

and, by (6),

$$\psi_{13}(14) = \frac{2^{15}-1}{2^{13}}\psi_{13}(7) - \frac{13 \,.\, 14 \,.\, 15}{24} 4 \,.\, 3\, M(7^4) = M(7^3),$$

$$\psi_{13}(14) = \frac{7^{15}-1}{7^{13}}\psi_{13}(2) - \frac{13 \,.\, 14 \,.\, 15}{24} \,.\, 49 \,.\, 48\, M(2^3) = M(2),$$

therefore $$\psi_{13}(28) = M(2^3 \,.\, 7^3) - M(2^2 \,.\, 7^3)$$

$$= M(2^2 \,.\, 7^3).$$

Ex. 5. $N = 56$, $\mu = 11$.

By (3),

$$\psi_{11}(56) = 2^4\psi_{11}(14) - \frac{11 \,.\, 12 \,.\, 13}{24}\left\{M(2^7 \,.\, 7^3) + 7M(2^8 \,.\, 7^4)\right\},$$

and, by (6),

$$\psi_{11}(14) = \frac{2^{13}-1}{2^{11}}\psi_{11}(7) - \frac{11 \,.\, 12 \,.\, 13}{24} 4 \,.\, 3M(7^4),$$

$$\psi_{11}(14) = \frac{7^{13}-1}{7^{11}}\psi_{11}(2) - \frac{11 \,.\, 12 \,.\, 13}{24} 49 \,.\, 48\, M(2^3).$$

Now $\psi_{11}(7)$ is divisible by 7 (not by 7^2), since $11+1 = M(6)$; and $2^{13}-1 = 8191$, which is a prime number; therefore $\psi_{11}(14) = M(7)$. Again, $7^{13}-1$ is divisible by 2 (not by 2^2); $\therefore\ \psi_{11}(14) = M(2)$.

Therefore $$\psi_{11}(56) = M(2^5 \,.\, 7) - \tfrac{1}{2}M(2^7 \,.\, 7^3)$$

$$= M(2^5 \,.\, 7).$$