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Multivariate measures of skewness for the skew-normal distribution

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ABSTRACT

The main objective of this work is to calculate and compare different measures of multivariate skewness for the skew-normal family of distributions. For this purpose, we consider the Mardia (1970) [10], Malkovich and Afifi (1973) [9], Isogai (1982) [17], Srivastava (1984) [15], Song (2001) [14], Móri et al. (1993) [11], Balakrishnan et al. (2007) [3] and Kollo (2008) [7] measures of skewness. The exact expressions of all measures of skewness, except for Song's, are derived for the family of skew-normal distributions, while Song's measure of shape is approximated by the use of delta method. The behavior of these measures, their similarities and differences, possible interpretations, and their practical use in testing for multivariate normal are studied by evaluating their power in the case of some specific members of the multivariate skew-normal family of distributions.

1. Introduction

Skewness for a univariate distribution indicates deviations from symmetry and leans toward one side of the center. Even though skewness is conceptually simple, even in the univariate case, skewness is intrinsically connected to different features and characteristics of a distribution such as its tail behavior and the shape of the curve on the two directions from the mode. Many measures and indices of asymmetry have been proposed in the literature from different viewpoints; one may refer to Benjamini and Krieger [4] for an overview of these developments. A number of them are based on the moments of the distribution, some on quantiles, and some others on combinations of these quantities.

In the multivariate case, the situation is even more involved since different directions may be characterized by different univariate skewness measures; see, for example, Schwager [13]. Mardia [10] introduced one of the popular and commonly used measures of multivariate skewness of an arbitrary p-dimensional distribution F with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. If $\mathbf{X} = [X_1, \ldots, X_p]^T$ and $\mathbf{Y} = [Y_1, \ldots, Y_p]^T$ are two independent and identically distributed random vectors from this distribution, the Mardia measure of skewness is defined as

$$\gamma_{1,p} = \beta_{1,p} = E\left\{\left[(\mathbf{X}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]^3\right\}. \tag{1}$$

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Malkovich and Afifi [9] proposed a measure of multivariate skewness as a different type of generalization of the univariate measure. By denoting the unit p-dimensional sphere by $\emptyset_p = \{\mathbf{x} \in \Re^p : |\mathbf{x}| = 1\}$, for $\mathbf{u} \in \emptyset_p$, the usual univariate measure of skewness in the $\mathbf{u}$-direction is

$$\beta_1(\mathbf{u}) = \frac{\left[E\left(\mathbf{u}^T(\mathbf{Y} - E(\mathbf{Y}))\right)^3\right]^2}{\left[\mathrm{var}\left(\mathbf{u}^T\mathbf{Y}\right)\right]^3}, \tag{2}$$

and so the Malkovich–Afifi multivariate extension of it is simply defined as

$$\beta_1^* = \sup_{\mathbf{u}\in\emptyset_p} \beta_1(\mathbf{u}). \tag{3}$$

Both these measures give an overall view of skewness without any specific reference to the direction of skewness. For this reason, Balakrishnan et al. [3] modified the Malkovich–Afifi measure to produce an overall vectorial measure of skewness as

$$\mathbf{T} = \int_{\emptyset_p} \mathbf{u} c_1(\mathbf{u}) \mathrm{d}\lambda(\mathbf{u}), \tag{4}$$

where $c_1(\mathbf{u}) = E[\left(\mathbf{u}^T\mathbf{Z}\right)^3]$ is a signed measure of skewness of the standardized variable $\mathbf{Z} = \mathbf{\Sigma}^{-1/2}(\mathbf{Y} - E(\mathbf{Y}))$ in the direction of $\mathbf{u}$, and λ denotes the *rotationally invariant probability measure on the unit sphere*.

Isogai [17] proposed another overall measure of multivariate skewness, by generalizing the univariate Pearson index of skewness based on the standardized difference between mean and mode of the distribution, given by

$$S_I = (\boldsymbol{\mu} - \mathbf{M}_0)^T g^{-1}(\mathbf{\Sigma})(\boldsymbol{\mu} - \mathbf{M}_0), \tag{5}$$

where $\mathbf{M}_0$ is the mode of the distribution and $g(\mathbf{\Sigma})$ is an 'appropriate' function of the covariance matrix. From this measure, by choosing $g(\cdot)$ to be the identity function, the vector $\mathbf{\Sigma}^{-1/2}(\boldsymbol{\mu} - \mathbf{M}_0)$ becomes a natural choice for a measure of the direction of the asymmetry, where $\mathbf{\Sigma}^{1/2}$ is the Cholesky factorization of the covariance matrix.

Srivastava [15] proposed an overall measure of multivariate skewness based on the principal component method. Let $\mathbf{\Gamma} = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p)$ be an orthogonal matrix such that $\mathbf{\Gamma}^T\mathbf{\Sigma}\mathbf{\Gamma} = \mathbf{D}_\lambda$, where $\mathbf{D}_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and $\lambda_1, \ldots, \lambda_p$ are the characteristic roots of $\mathbf{\Sigma}$. Then the Srivastava measure is given by

$$\beta_{1p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{E(F_i - \theta_i)^3}{\lambda_i^{3/2}}\right\}^2,$$

where $F_i = \boldsymbol{\gamma}_i^T\mathbf{Y}$ and $\theta_i = \boldsymbol{\gamma}_i^T\boldsymbol{\mu}$.

Móri et al. [11] proposed a vectorial measure of skewness as

$$s(\mathbf{X}) = E(\|\mathbf{Z}\|^2\mathbf{Z}),$$

where $\mathbf{Z} = \mathbf{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$.

Kollo [7], by noting that not all third moments appear in the expression of the Móri et al. [11] measure, proposed a slight modification of it by including all the mixed moments of the third order as follows:

$$b(\mathbf{X}) = E\left\{\sum_{i,j}(\mathbf{Z}_i\mathbf{Z}_j)\mathbf{Z}\right\}.$$

Song [14] defined a general measure of the shape of a distribution, based on Rényi's entropy of order λ [12], as

$$\mathcal{S}(f) = \mathrm{var}[\log(f(\mathbf{X}))], \tag{6}$$

where f denotes the density function.

The primary aim of this paper is to analyze the specific deviation from symmetry of the multivariate skew-normal distribution of Azzalini and Capitanio [1], by determining the values of these skewness measures for that distribution, and by examining the role of the parameters of this family of distributions in producing the skewness in the model.

The rest of this paper is organized as follows. In Section 2, we give a brief description of the skew-normal (SN) distribution. In Section 3, we describe all the skewness measures that are studied here, and then derive their expressions specifically for the SN case. In Section 4, we apply these measures to the SN distribution for different choices of the model parameters and pass some comparative comments on them. Hypothesis tests and the performance of these skewness measures in this testing context are discussed in Section 5. Finally, some discussion of the obtained results is made in Section 6.

2. Multivariate skew-normal distribution

A p-dimensional random variable $\mathbf{Z}$ is said to have a multivariate SN distribution, denoted by $\mathbf{Z} \sim SN(\mathbf{0}, \bar{\mathbf{\Omega}}, \boldsymbol{\alpha})$, if it is continuous with density function

$$2\phi_p(\mathbf{z}; \bar{\mathbf{\Omega}})\Phi(\boldsymbol{\alpha}^T\mathbf{z}), \quad \mathbf{z} \in \Re^p,$$

where $\phi_p(\mathbf{z}; \bar{\mathbf{\Omega}})$ is the p-dimensional normal density with zero mean and correlation matrix $\bar{\mathbf{\Omega}}$, $\Phi(\cdot)$ is the univariate $N(0, 1)$ distribution function, and $\boldsymbol{\alpha}$ is a p-dimensional vector that regulates departure from symmetry. In general, we consider here the more general form, by introducing location and scale parameters,

$$\mathbf{Y} = \boldsymbol{\xi} + \boldsymbol{\omega}\mathbf{Z},$$

where $\boldsymbol{\xi}$ is a location parameter and $\boldsymbol{\omega}$ is a diagonal matrix of scale parameters such that $\mathbf{\Omega} = \boldsymbol{\omega}\bar{\mathbf{\Omega}}\boldsymbol{\omega}$ is the covariance matrix. It is known that the mean vector and covariance matrix of $\mathbf{Y}$ are then given by

$$E(\mathbf{Y}) = \boldsymbol{\xi} + \sqrt{\frac{2}{\pi}}\,\boldsymbol{\omega}\boldsymbol{\delta} \quad \text{and} \quad \text{var}(\mathbf{Y}) = \mathbf{\Sigma} = \mathbf{\Omega} - \frac{2}{\pi}\,\boldsymbol{\omega}\boldsymbol{\delta}\boldsymbol{\delta}^T\boldsymbol{\omega},$$

where $\boldsymbol{\delta} = \frac{1}{(1+\boldsymbol{\alpha}^T\bar{\mathbf{\Omega}}\boldsymbol{\alpha})^{1/2}}\,\bar{\mathbf{\Omega}}\boldsymbol{\alpha}$ is a vector whose i-th element lies in the interval $(-1, 1)$.

3. Skewness indices for skew-normal distribution

3.1. Mardia index

Mardia's [10] measure is one of the popular and commonly used measures of multivariate skewness and, for a p-dimensional random vector $\mathbf{X}$, it is as defined in (1) and can be rewritten as

$$\gamma_{1,p} = \beta_{1,p} = \sum_{i,j,k}\sum_{i',j',k'} \sigma^{ii'}\sigma^{jj'}\sigma^{kk'}\mu_{i,j,k}\mu_{i',j',k'},$$

where $\sigma^{ii'}$ is the (i, i')-th entry of $\mathbf{\Sigma}^{-1}$, the inverse of the covariance matrix $\mathbf{\Sigma}$, $\boldsymbol{\mu} = [\mu_1, \dots, \mu_p]^T$ is the mean vector, and $\mu_{i,j,k}$ for $i, j, k = 1, \dots, p$, are the third central moments defined by $\mu_{i,j,k} = E[(X_i - \mu_i)(X_j - \mu_j)(X_k - \mu_k)]$. Kollo and Srivastava [8] showed that Mardia's skewness measure may be expressed as $\gamma_{1,p} = tr(\mathbf{M}_3'\mathbf{M}_3)$, where $\mathbf{M}_3$ is the third moment matrix.

This measure is location and scale invariant, and has been evaluated for the SN case by Azzalini and Capitanio [1] to be

$$\gamma_{1,p} = \beta_{1,p} = \left(\frac{4-\pi}{2}\right)^2 \left(\boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^3.$$

3.2. Malkovich–Afifi measure

Malkovich and Afifi [9] introduced a different measure of multivariate skewness which is also location and scale invariant. They defined the measure as in (2) and (3) and showed that if $\mathbf{Z}$ is the standardized variable $\mathbf{Z} = \mathbf{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$, an equivalent version of β_1^* is

$$\beta_1^* = \sup_{\mathbf{u}\in\emptyset_p} \left(E\left[\left(\mathbf{u}^T\mathbf{Z}\right)^3\right]\right)^2.$$

For obtaining this measure for the SN case, it is convenient to consider the canonical form of this distribution (see [1]). They established that if $\mathbf{Y}$ possesses a multivariate SN distribution, then there exists a linear transform $\mathbf{Z}^* = \mathbf{A}^*\mathbf{Y}$ such that $\mathbf{Z}^*$ is still distributed as a multivariate SN but in which at most one component is skewed.

This means that the Malkovich and Afifi index, which is the maximum of the univariate skewness measures among all the directions of the unit sphere, will be, for $\mathbf{Z}^*$, the index of asymmetry in the only (if there is) skew direction (without loss of any generality, we take the first component of $\mathbf{Z}^*$ to be skewed and denote it $\mathbf{Y}^*$):

$$\beta_1^* = \beta_1^*(\mathbf{u}) = \frac{\left[E\left(\mathbf{u}^T(\mathbf{Y} - E(\mathbf{Y}))\right)^3\right]^2}{\left[\text{var}\left(\mathbf{u}^T\mathbf{Y}\right)\right]^3} = \frac{\left[E\left(\mathbf{Y}^* - E(\mathbf{Y}^*)\right)^3\right]^2}{[\text{var}(\mathbf{Y}^*)]^3} = \gamma_1^2$$
$$= \left(\frac{4-\pi}{2}\right)^2 \left(\boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^3. \qquad (7)$$

Here, we have used γ_1 to denote the univariate skewness measure of the unique (if any) skewed component of the canonical form $\mathbf{Z}^*$.

Since this measure is location and scale invariant, it is invariant for linear transforms and consequently (7) is also the Malkovich–Afifi measure for $\mathbf{Y}$. Hence, this measure is the same as the Mardia index for the case of multivariate SN distribution.

3.3. Balakrishnan–Brito–Quiroz measure

In the univariate case, it is customary to indicate the direction of the skewness (right or left); both the Mardia and Malkovich and Afifi measures give an overall view of skewness, but without indication of the direction of skewness, and so it would be useful to have an indication of the direction of skewness by means of a vectorial notion. In addition to this drawback, these measures also have the problem of taking on the same skewness value for distributions with very different shapes with the difference being not only rotational. For these reasons, Balakrishnan et al. [3] modified the Malkovich–Afifi index to produce such a vectorial measure of skewness.

They observed that in $\beta_1^* = \sup_{\mathbf{u}\in\emptyset_p}\left(E\left[\left(\mathbf{u}^T\mathbf{Z}\right)^3\right]\right)^2$, $c_1(\mathbf{u}) = E\left[\left(\mathbf{u}^T\mathbf{Z}\right)^3\right]$ can be seen as a signed measure of skewness of the standardized variable $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{Y}-\boldsymbol{\mu})$ in the direction of $\mathbf{u}$. If $c_1(\mathbf{u})$ is negative, it indicates skewness in the direction of $-\mathbf{u}$, while $\mathbf{u}c_1(\mathbf{u})$ provides a vectorial index of skewness in the $\mathbf{u}$ (or $-\mathbf{u}$) direction. Summation of these vectors over $\mathbf{u}$ (in the form of an integral) will then yield an overall vectorial measure of skewness presented earlier in (4).

For obtaining a single numerical measure (for inferential purposes), they proposed the quantity $Q = \mathbf{T}^T\boldsymbol{\Sigma}_\mathbf{T}\mathbf{T}$, where $\mathbf{T}$ is as defined in (4) and $\boldsymbol{\Sigma}_\mathbf{T}$ is the covariance matrix of $\mathbf{T}$. As we will see, the computation of $\mathbf{T}$ is feasible for the multivariate SN distributions, while the covariance matrix $\boldsymbol{\Sigma}_\mathbf{T}$ depends on moments of sixth order (see [3]) and for the SN case they are not available yet in an explicit form. For the calculation of Q, therefore, we proceed by Monte Carlo simulations. However, a slight modification of the overall Balakrishnan–Brito–Quiroz measure, obtained upon replacing $\boldsymbol{\Sigma}_\mathbf{T}$ by $\boldsymbol{\Sigma}_\mathbf{Z}$ yields the index $Q^* = \mathbf{T}^T\boldsymbol{\Sigma}_\mathbf{Z}\mathbf{T}$. This provides a reasonable measure of overall multivariate skewness, and here the use of $\boldsymbol{\Sigma}_\mathbf{Z}$ is because of the ease of its computation.

Evaluation of $\mathbf{T}$ becomes feasible by using the integrals of some monomials over the unit sphere Ø:

$$J_4 = \int_{\emptyset_p} x_j^4 p\lambda(x) = \frac{3}{p(p+2)}, \qquad J_{2,2} = \int_{\emptyset_p} x_j^2 x_i^2 p\lambda(x) = \frac{1}{p(p+2)},$$

for $j \neq i$, $1 \leq j, i \leq p$. Note that these integrals do not depend on the particular choices of j and i. Therefore, the r-th coordinate of $\mathbf{T}$ is simply

$$T_r = J_4 E(X_r^3) + 3\sum_{i\neq r} J_{2,2} E(X_i^2 X_r).$$

For the case of multivariate SN distribution, we can obtain $E(X_i^3)$ and $E(X_i^2X_j)$ from the moment generating function given by Azzalini and Dalla Valle [2] and the general expressions of Genton et al. [6]:

$$\begin{aligned}\mathbf{M}_3 =\ & \boldsymbol{\Omega}\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes\boldsymbol{\Omega} + \mathrm{vec}(\boldsymbol{\Omega})\otimes\boldsymbol{\xi}^T + \boldsymbol{\xi}\otimes\boldsymbol{\xi}^T\otimes\boldsymbol{\xi} + \sqrt{\frac{2}{\pi}}\left[\boldsymbol{\delta}\otimes\boldsymbol{\Omega} + \mathrm{vec}(\boldsymbol{\Omega})\boldsymbol{\delta}^T\right.\\ & \left. +\ (\mathbf{I}_p\otimes\boldsymbol{\delta})\boldsymbol{\Omega} - \boldsymbol{\delta}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\delta} + \boldsymbol{\delta}\otimes\boldsymbol{\xi}^T\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes\boldsymbol{\delta}^T\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes\boldsymbol{\xi}^T\otimes\boldsymbol{\delta}\right].\end{aligned} \tag{8}$$

The specific moments of interest are then given by

$$\begin{aligned} E(X_i^3) &= \mathbf{M}_3[(i-1)p+i, i] = 3\boldsymbol{\omega}_{ii}\boldsymbol{\xi}_i + \boldsymbol{\xi}_i^3 + \sqrt{\frac{2}{\pi}}\left[3\boldsymbol{\delta}_i\boldsymbol{\omega}_{ii} - \boldsymbol{\delta}_i^3 + 3\boldsymbol{\delta}_i\boldsymbol{\xi}_i^2\right]\\ &= \boldsymbol{\xi}_i(3\boldsymbol{\omega}_{ii} + \boldsymbol{\xi}_i^2) + \sqrt{\frac{2}{\pi}}\boldsymbol{\delta}_i\left[3\boldsymbol{\omega}_{ii} - \boldsymbol{\delta}_i^2 + 3\boldsymbol{\xi}_i^2\right],\\ E(X_i^2X_j) &= \mathbf{M}_3[(i-1)p+i, j] = 2\boldsymbol{\omega}_{ij}\boldsymbol{\xi}_i + \boldsymbol{\omega}_{ii}\boldsymbol{\xi}_j + \boldsymbol{\xi}_i^2\boldsymbol{\xi}_j\\ &\quad + \sqrt{\frac{2}{\pi}}\left[\boldsymbol{\delta}_j\boldsymbol{\omega}_{ii} + 2\boldsymbol{\delta}_i\boldsymbol{\omega}_{ij} - \boldsymbol{\delta}_i^2\boldsymbol{\delta}_j + 2\boldsymbol{\delta}_i\boldsymbol{\xi}_j\boldsymbol{\xi}_i + \boldsymbol{\delta}_j\boldsymbol{\xi}_i^2\right].\end{aligned}$$

where $\mathbf{M}_3[\cdot,\cdot]$ indicates the elements of matrix $\mathbf{M}_3$.

By making use of these expressions, we get the elements of $\mathbf{T}$ as

$$\begin{aligned} T_r &= J_4 E(X_r^3) + 3\sum_{i\neq r} J_{2,2} E(X_i^2 X_r)\\ &= J_4\left\{\boldsymbol{\xi}_i(3\boldsymbol{\omega}_{ii} + \boldsymbol{\xi}_i^2) + \sqrt{\frac{2}{\pi}}\boldsymbol{\delta}_i\left[3\boldsymbol{\omega}_{ii} - \boldsymbol{\delta}_i^2 + 3\boldsymbol{\xi}_i^2\right]\right\}\\ &\quad + 3\sum_{i\neq r} J_{2,2}\left\{2\boldsymbol{\omega}_{ij}\boldsymbol{\xi}_i + \boldsymbol{\omega}_{ii}\boldsymbol{\xi}_j + \boldsymbol{\xi}_i^2\boldsymbol{\xi}_j + \sqrt{\frac{2}{\pi}}\left[\boldsymbol{\delta}_j\boldsymbol{\omega}_{ii} + 2\boldsymbol{\delta}_i\boldsymbol{\omega}_{ij} - \boldsymbol{\delta}_i^2\boldsymbol{\delta}_j + 2\boldsymbol{\delta}_i\boldsymbol{\xi}_j\boldsymbol{\xi}_i + \boldsymbol{\delta}_j\boldsymbol{\xi}_i^2\right]\right\}.\end{aligned}$$

3.4. Isogai measure

Capitanio [5] showed that, once the identity function is chosen for $g(\cdot)$, the Isogai measure in (5) for the multivariate SN becomes

$$S_I = \frac{\left(\sqrt{\frac{2}{\pi}}\delta^* - m_0\right)^2}{1 - \frac{2}{\pi}\delta^{*2}},$$

where $\delta^* = \left(\boldsymbol{\delta}^T \bar{\boldsymbol{\Omega}}^{-1} \boldsymbol{\delta}\right)^{1/2}$ and m_0 is the mode of the single scalar SN in the canonical form of the distribution. This index is essentially the Mahalanobis distance between the null vector and the vector $E(\mathbf{Y}) - \mathbf{M}_0$, and it is location and scale invariant. Capitanio [5] also pointed out that the vector $S_C = \boldsymbol{\omega}^{-1}(\boldsymbol{\mu} - \mathbf{M}_0)$ is then a natural choice for characterizing the direction of the asymmetry of the multivariate SN distribution, leading to the index

$$S_C = \left(\sqrt{\frac{2}{\pi}} - \frac{m_0}{\delta^*}\right)\boldsymbol{\delta}.$$

Hence, for the multivariate SN distribution, the direction of $\boldsymbol{\delta}$, which aligns mean and mode of the distribution, may be viewed as a measure of vectorial skewness.

3.5. Srivastava measure

Srivastava measure of skewness is based on the principal components $\mathbf{F} = \boldsymbol{\Gamma}\mathbf{Y}$ of the multivariate variable $\mathbf{Y}$, where $\boldsymbol{\Gamma}$ is the matrix of eigenvectors of the covariance matrix $\boldsymbol{\Sigma}$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$. Srivastava measure may then be written as

$$\beta_{1p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{E(F_i - \theta_i)^3}{\lambda_i^{3/2}}\right\}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{E[\boldsymbol{\gamma}_i^T(\mathbf{Y} - \boldsymbol{\mu})]^3}{\lambda_i^{3/2}}\right\}^2.$$

This measure is, thus, based on central moments of third order $E[\boldsymbol{\gamma}_i^T(\mathbf{Y} - \boldsymbol{\mu})]^3$. We know that for any multivariate random variable $\mathbf{X}$, the central moments of third order (let us denote them by $\overline{\mathbf{M}}_3(\mathbf{X})$) are related to the non-central moments ($\mathbf{M}_3(X)$) by the relationship (see, for example, Kollo and Srivastava [8]):

$$\overline{\mathbf{M}}_3(\mathbf{Y}) = \mathbf{M}_3(\mathbf{Y}) - \mathbf{M}_2(\mathbf{Y}) \otimes E(\mathbf{Y}) - E(\mathbf{Y}) \otimes \mathbf{M}_2(\mathbf{Y}) - vec(\mathbf{M}_2(\mathbf{Y}))E(\mathbf{Y})^T + 2E(\mathbf{Y})E(\mathbf{Y})^T \otimes E(\mathbf{Y}),$$

where $\mathbf{M}_2(\mathbf{X})$ denotes the non-central moment of second order.

Therefore, for obtaining this measure for the SN distribution, we only need to obtain the non-central moments up to third order. From Genton et al. [6], we have second and third order non-central moments for the SN distribution, and by using the relations for affine transformations of moments, we obtain

$$\begin{aligned}
\mathbf{M}_1(\mathbf{F}) &= E(\mathbf{F}) = E(\boldsymbol{\gamma}_i^T\mathbf{Y}) = \boldsymbol{\gamma}_i^T E(\mathbf{Y}),\\
\mathbf{M}_2(\mathbf{F}) &= E(\mathbf{F}\mathbf{F}^T) = E[(\boldsymbol{\gamma}_i^T\mathbf{Y})^2] = \boldsymbol{\gamma}_i^T E(\mathbf{Y}\mathbf{Y}^T)\boldsymbol{\gamma}_i,\\
\mathbf{M}_3(\mathbf{F}) &= E\left\{(\mathbf{F}\otimes\mathbf{F})\mathbf{F}^T\right\} = E\left\{vec(\mathbf{F}\mathbf{F}^T)\mathbf{F}^T\right\} = (\boldsymbol{\gamma}_i^T \otimes \boldsymbol{\gamma}_i^T)\mathbf{M}_3(\mathbf{Y})\boldsymbol{\gamma}_i.
\end{aligned}$$

Finally, we obtain the third central moment of $\mathbf{F}$ to be

$$\begin{aligned}
E[\boldsymbol{\gamma}_i^T(\mathbf{Y} - \boldsymbol{\mu})]^3 = \overline{\mathbf{M}}_3(\mathbf{F}) &= (\boldsymbol{\gamma}_i^T \otimes \boldsymbol{\gamma}_i^T)\mathbf{M}_3(\mathbf{Y})\boldsymbol{\gamma}_i - [\boldsymbol{\gamma}_i^T E(\mathbf{Y}\mathbf{Y}^T)\boldsymbol{\gamma}_i] \otimes [\boldsymbol{\gamma}_i^T E(\mathbf{Y})] - \boldsymbol{\gamma}_i^T E(\mathbf{Y}) \otimes [\boldsymbol{\gamma}_i^T E(\mathbf{Y}\mathbf{Y}^T)\boldsymbol{\gamma}_i]\\
&\quad - \mathrm{vec}(\boldsymbol{\gamma}_i^T E(\mathbf{Y}\mathbf{Y}^T)\boldsymbol{\gamma}_i)E(\mathbf{Y})^T\boldsymbol{\gamma}_i + 2[\boldsymbol{\gamma}^T E(\mathbf{Y})E(\mathbf{Y})^T\boldsymbol{\gamma}_i] \otimes [\boldsymbol{\gamma}_i^T E(\mathbf{Y})],
\end{aligned}$$

where $\mathbf{M}_3(\mathbf{Y})$ is as given in (8), and $\mathbf{M}_2(\mathbf{Y}) = E(\mathbf{Y}\mathbf{Y}^T) = \boldsymbol{\Omega} + \boldsymbol{\xi}\boldsymbol{\xi}^T + \sqrt{\frac{2}{\pi}}(\boldsymbol{\xi}\boldsymbol{\delta}^T\boldsymbol{\omega} + \boldsymbol{\omega}\boldsymbol{\delta}\boldsymbol{\xi}^T)$ while $E(\mathbf{Y}) = \boldsymbol{\xi} + \sqrt{\frac{2}{\pi}}\boldsymbol{\omega}\boldsymbol{\delta}$; see [6].

3.6. Móri–Rohatgi–Székely measure

Móri et al. [11] introduced a vectorial measure of skewness. If $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{Y} - \boldsymbol{\mu})$ is the standardized variable, this measure can be written in terms of coordinates of $\mathbf{Z}$ as (see [7])

$$\begin{aligned}
s(\mathbf{Y}) &= E(\|\mathbf{Z}\|^2\mathbf{Z}) = E\left\{(\mathbf{Z}^T\mathbf{Z})\mathbf{Z}\right\} = \sum_{i=1}^{p} E(\mathbf{Z}_i^2\mathbf{Z}) = E\left(\sum_{i=1}^{p}\mathbf{Z}_i^2\mathbf{Z}_1, \ldots, \sum_{i=1}^{p}\mathbf{Z}_i^2\mathbf{Z}_p\right)\\
&= \left(\sum_{i=1}^{p} E(\mathbf{Z}_i^2\mathbf{Z}_1), \ldots, \sum_{i=1}^{p} E(\mathbf{Z}_i^2\mathbf{Z}_p)\right).
\end{aligned}$$

All the quantities involved in the above expression are specific non-central moments of third order of $\mathbf{Z}$. When $\mathbf{Y}$ has a multivariate SN distribution, $\mathbf{Z}$ is still SN, and so we can use once again the expression in (8).

3.7. Kollo measure

Kollo [7] observed that the vectorial measure of Móri et al. [11] involves only some of the moments of third order of $\mathbf{Z}$. He, instead, proposed to sum all these moments

$$\begin{aligned} b(\mathbf{X}) &= E\left\{\sum_{i,j}(\mathbf{Z}_i\mathbf{Z}_j)\mathbf{Z}\right\} = E\left\{\sum_{i,j}(\mathbf{Z}_i\mathbf{Z}_j)\mathbf{Z}_1,\ldots,\sum_{i,j}(\mathbf{Z}_i\mathbf{Z}_j)\mathbf{Z}_p\right\} \\ &= \left\{\sum_{i,j}E[(\mathbf{Z}_i\mathbf{Z}_j)\mathbf{Z}_1],\ldots,\sum_{i,j}E[(\mathbf{Z}_i\mathbf{Z}_j)\mathbf{Z}_p]\right\} \end{aligned}$$

and use it as a skewness measure.

For the SN distribution, all the needed moments are easily obtained from (8).

3.8. Song measure of shape

Song [14] defined a completely different measure of the shape of a distribution based on Rényi's entropy of order λ (see [12]) defined as

$$\mathcal{I}_R(\lambda) = \mathcal{I}_R(\lambda, f) = \frac{1}{1-\lambda}\log\int f^{\lambda}\mathrm{d}x.$$

The Song measure is then given by $\mathcal{S}(f) = -2\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{I}_R(1)$, which leads to the measure in (6). This measure is location and scale invariant as well.

For obtaining the variance of the logarithm of the density for the SN case, we may use the delta method, $\mathrm{var}\,(G(\mathbf{Y})) \approx G(\boldsymbol{\mu}) + [G'(\boldsymbol{\mu})]\mathrm{var}(\mathbf{Y})[G'(\boldsymbol{\mu})]^T$, as an approximation. For the multivariate SN case, we readily have

$$G(\mathbf{y}) = \log\{f(\mathbf{y})\} = -\frac{1}{2}\log(|\boldsymbol{\Omega}|) - \frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Omega}^{-1}(\mathbf{y}-\boldsymbol{\xi})(\mathbf{y}-\boldsymbol{\xi})^T\right\} + \log\left\{2\Phi\left(\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\mathbf{y}-\boldsymbol{\xi})\right)\right\}$$

and its derivative as

$$G'(\mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{y}}G(\mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{y}}\log f(\mathbf{y}) = -\boldsymbol{\Omega}^{-1}(\mathbf{y}-\boldsymbol{\xi}) + \frac{\phi\left\{\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\mathbf{y}-\boldsymbol{\xi})\right\}}{\Phi\left\{\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\mathbf{y}-\boldsymbol{\xi})\right\}}\boldsymbol{\omega}^{-1}\boldsymbol{\alpha}.$$

From these, we then obtain

$$\begin{aligned} \mathcal{S}(f) \approx &\left[-\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{\xi}) + \frac{\phi\left\{\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{\xi})\right\}}{\Phi\left\{\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{\xi})\right\}}\boldsymbol{\omega}^{-1}\boldsymbol{\alpha}\right] \times \left[\boldsymbol{\Omega} - \frac{2}{\pi}\boldsymbol{\omega}\boldsymbol{\delta}\boldsymbol{\delta}^T\boldsymbol{\omega}\right] \\ &\times \left[-\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{\xi}) + \frac{\phi\left\{\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{\xi})\right\}}{\Phi\left\{\boldsymbol{\alpha}^T\boldsymbol{\omega}^{-1}(\boldsymbol{\mu}-\boldsymbol{\xi})\right\}}\boldsymbol{\omega}^{-1}\boldsymbol{\alpha}\right]^T. \end{aligned}$$

4. Comparison of the measures

Based on the above skewness measures for the multivariate SN distribution, it is possible to compare all of them and also investigate what features of the distribution are captured by which measure. For similar work on kurtosis comparisons of elliptical distributions, one may refer to Zografos [16]. It should be noted that all the measures are location and scale invariant, a desirable property indeed for any measure of skewness. Tables 1 and 2 present the values of all the skewness measures for different choices of the parameters of the bivariate SN distribution, along with the picture of the corresponding contour plot. The directions of the vectorial measures can be seen in these plots as well.

Mardia's measure appears to be always greater than the Balakrishnan–Brito–Quiroz measure and the ranking of the measures is very similar for $\gamma_{1,p}$, Q and Q^*, with an exception for Case 1, wherein Mardia's measure is the highest among the considered cases, indicating an outstanding skewness, while Q and Q^* show a regular asymmetry. However, the scale of these three measures appears to be quite different, since cases such as 5, 6 and 7 that have almost the same value for the Q^* modified Balakrishnan–Brito–Quiroz overall measure (between 0.024 and 0.030), but have quite different values for Mardia's measure, varying from 0.805 to 0.493, and for the simulated Q measure (ranging between 2.022 and 5.689).

Table 1
Skewness measures for some bivariate skew-normal distributions.

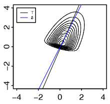
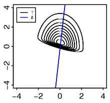
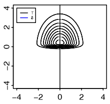
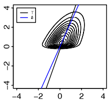
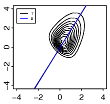
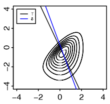
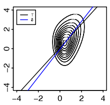
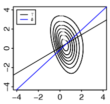
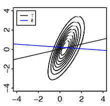
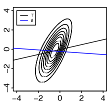
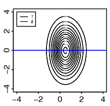
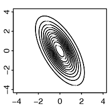
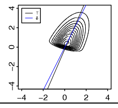
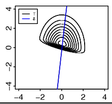
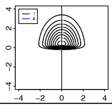
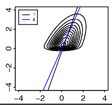
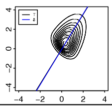
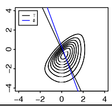
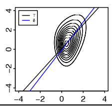
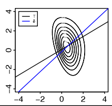
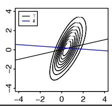
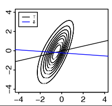
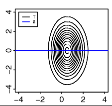
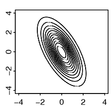
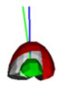
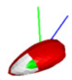
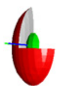
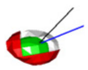
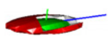
#	Contour plot	Parameters	Mardia	Balakrishnan, Brito and Quiroz			Isogai
			$\gamma_{1,p}$	Q^*	Q	T	δ
1		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 10\end{bmatrix}$	0.963	0.094	29.275	$\begin{bmatrix}0.019\\ 0.288\end{bmatrix}$	$\begin{bmatrix}0.699\\ 1.572\end{bmatrix}$
2		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 10\end{bmatrix}$	0.959	0.118	44.745	$\begin{bmatrix}0.003\\ 0.334\end{bmatrix}$	$\begin{bmatrix}0.125\\ 1.565\end{bmatrix}$
3		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}0\\ 10\end{bmatrix}$	0.958	0.139	87.641	$\begin{bmatrix}0.000\\ 0.358\end{bmatrix}$	$\begin{bmatrix}0.000\\ 1.578\end{bmatrix}$
4		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}0\\ 10\end{bmatrix}$	0.958	0.148	74.824	$\begin{bmatrix}0.004\\ 0.333\end{bmatrix}$	$\begin{bmatrix}0.631\\ 1.578\end{bmatrix}$
5		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 3\end{bmatrix}$	0.805	0.024	5.689	$\begin{bmatrix}0.066\\ 0.159\end{bmatrix}$	$\begin{bmatrix}0.795\\ 1.512\end{bmatrix}$
6		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ -3\end{bmatrix}$	0.736	0.029	2.022	$\begin{bmatrix}0.041\\ 0.118\end{bmatrix}$	$\begin{bmatrix}0.381\\ -1.430\end{bmatrix}$
7		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 1\end{bmatrix}$	0.493	0.030	2.257	$\begin{bmatrix}0.120\\ 0.050\end{bmatrix}$	$\begin{bmatrix}0.885\\ 1.327\end{bmatrix}$
8		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 1\end{bmatrix}$	0.343	0.058	1.817	$\begin{bmatrix}0.145\\ 0.004\end{bmatrix}$	$\begin{bmatrix}0.730\\ 0.913\end{bmatrix}$
9		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ -1\end{bmatrix}$	0.107	0.042	1.272	$\begin{bmatrix}0.118\\ 0.000\end{bmatrix}$	$\begin{bmatrix}0.534\\ -0.267\end{bmatrix}$
10		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}-2\\ 1\end{bmatrix}$	0.107	0.042	1.347	$\begin{bmatrix}-0.118\\ 0.000\end{bmatrix}$	$\begin{bmatrix}-0.534\\ 0.267\end{bmatrix}$
11		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}1\\ 0\end{bmatrix}$	0.019	0.004	0.302	$\begin{bmatrix}0.051\\ 0.000\end{bmatrix}$	$\begin{bmatrix}0.707\\ 0.000\end{bmatrix}$
12		Ω $\begin{bmatrix}1 & -1\\ -1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}0\\ 0\end{bmatrix}$	0.000	0.000	0.000	$\begin{bmatrix}0.000\\ 0.000\end{bmatrix}$	$\begin{bmatrix}0.000\\ 0.000\end{bmatrix}$

Table 2
Skewness measures for some bivariate skew-normal distributions.

#	Contour plot	Parameters	Srivastava β^2_{1p}	Móri, Rohatgi, Székely s	Kollo b	Song $\mathcal{S}(f)$
1		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 10\end{bmatrix}$	0.430	$\begin{bmatrix}0.361\\ 0.894\end{bmatrix}$	$\begin{bmatrix}0.613\\ 1.515\end{bmatrix}$	0.2326
2		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 10\end{bmatrix}$	0.171	$\begin{bmatrix}0.208\\ 0.934\end{bmatrix}$	$\begin{bmatrix}0.297\\ 1.331\end{bmatrix}$	0.2329
3		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}0\\ 10\end{bmatrix}$	0.456	$\begin{bmatrix}0.000\\ 0.955\end{bmatrix}$	$\begin{bmatrix}0.000\\ 0.955\end{bmatrix}$	0.2330
4		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}0\\ 10\end{bmatrix}$	0.246	$\begin{bmatrix}0.207\\ 0.933\end{bmatrix}$	$\begin{bmatrix}0.294\\ 1.327\end{bmatrix}$	0.2330
5		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 3\end{bmatrix}$	0.247	$\begin{bmatrix}0.493\\ 0.661\end{bmatrix}$	$\begin{bmatrix}0.966\\ 1.294\end{bmatrix}$	0.2364
6		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ -3\end{bmatrix}$	0.139	$\begin{bmatrix}0.442\\ -0.60\end{bmatrix}$	$\begin{bmatrix}0.010\\ -0.027\end{bmatrix}$	0.2226
7		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 1\end{bmatrix}$	0.056	$\begin{bmatrix}0.499\\ 0.374\end{bmatrix}$	$\begin{bmatrix}0.978\\ 0.734\end{bmatrix}$	0.1657
8		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 1\end{bmatrix}$	0.112	$\begin{bmatrix}0.468\\ 0.215\end{bmatrix}$	$\begin{bmatrix}0.824\\ 0.379\end{bmatrix}$	0.1076
9		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ -1\end{bmatrix}$	0.043	$\begin{bmatrix}0.321\\ -0.043\end{bmatrix}$	$\begin{bmatrix}0.236\\ -0.032\end{bmatrix}$	0.0352
10		Ω $\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}-2\\ 1\end{bmatrix}$	0.043	$\begin{bmatrix}-0.321\\ 0.043\end{bmatrix}$	$\begin{bmatrix}-0.236\\ 0.032\end{bmatrix}$	0.0352
11		Ω $\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}1\\ 0\end{bmatrix}$	0.009	$\begin{bmatrix}0.137\\ 0.000\end{bmatrix}$	$\begin{bmatrix}0.137\\ 0.000\end{bmatrix}$	0.0052
12		Ω $\begin{bmatrix}1 & -1\\ -1 & 2.5\end{bmatrix}$ α $\begin{bmatrix}0\\ 0\end{bmatrix}$	0.000	$\begin{bmatrix}0.000\\ 0.000\end{bmatrix}$	$\begin{bmatrix}0.000\\ 0.000\end{bmatrix}$	1.000

On the other hand, Song's measure presents a completely different ranking among the considered cases, emphasizing different behaviors: cases 1–5 have very similar values of $\mathcal{S}(f)$, while Srivastava measure as well as both Balakrishnan–Brito–Quiroz measures show quite different skewness among them.

Table 3
Skewness measures for some trivariate skew-normal distributions.

#	Contour plot	Parameters	Mardia $\gamma_{1,p}$	Balakrishnan, Brito and Quiroz Q^*	Balakrishnan, Brito and Quiroz Q	Balakrishnan, Brito and Quiroz T	Isogai δ
1		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$ α $\begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$	0.974	0.0095	2.847	$\begin{bmatrix} 0.0007 \\ 0.0664 \\ 0.0664 \end{bmatrix}$	$\begin{bmatrix} 0.089 \\ 1.112 \\ 1.112 \end{bmatrix}$
2		Ω $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$ α $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$	0.941	0.0024	2.374	$\begin{bmatrix} 0.022 \\ 0.051 \\ 0.032 \end{bmatrix}$	$\begin{bmatrix} 0.633 \\ 0.990 \\ 2.771 \end{bmatrix}$
3		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ α $\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$	0.913	0.0988	7.195	$\begin{bmatrix} 0.191 \\ 0.0000 \\ 0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.995 \\ 0.000 \\ 0.000 \end{bmatrix}$
4		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$ α $\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}$	0.493	0.0074	1.598	$\begin{bmatrix} 0.015 \\ -0.049 \\ -0.036 \end{bmatrix}$	$\begin{bmatrix} 0.331 \\ -1.494 \\ -1.241 \end{bmatrix}$
5		Ω $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$ α $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$	0.492	0.0086	1.410	$\begin{bmatrix} 0.052 \\ 0.000 \\ -0.003 \end{bmatrix}$	$\begin{bmatrix} 0.000 \\ -0.442 \\ -2.654 \end{bmatrix}$
6		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ α $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$	0.019	0.0081	1.435	$\begin{bmatrix} 0.064 \\ 0.006 \\ 0.008 \end{bmatrix}$	$\begin{bmatrix} 0.685 \\ 0.857 \\ 0.343 \end{bmatrix}$

The vectorial measures yield very similar results in terms of skewness directions, especially when the distribution is quite asymmetric. Some differences are remarkable in cases for distributions that are not extremely skewed such as Cases 8 and 9. It is useful to note that Cases 9 and 10 deal with the same reflected distribution, and in these cases, all the measures are the same (the vectorial ones are reflected as well).

Tables 3 and 4 present the values of all the measures for the case of trivariate SN distribution. Differences among the measures are much more pronounced in this situation.

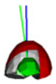
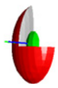
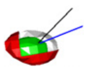
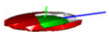
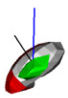
The Mardia and Balakrishnan–Brito–Quiroz measures are not in agreement overall, not only in the scale of measures, but also in the ranking of skewness. The directions of the vectorial measures are also quite different from the bivariate case, with agreement being present when skewness is present only in one direction (such as in Case 3).

5. Performance of measures and comparisons

The measures considered in the preceding sections are not directly comparable with each other. So, for comparing them, we should have measures obtained on the same scale. For the purpose of obtaining such a set of comparable indices, we consider the sample version for each of the skewness measures considered as test statistics for the hypothesis of Normal against SN distribution.

Thus, we obtain a number of different statistics for testing the same null hypothesis and p-values and powers for each of these tests quantify the ability of each skewness measure to identify the specific asymmetry present in the skew-normal

Table 4
Skewness measures for some trivariate skew-normal distributions.

#	Contour plot	Parameters	Srivastava β_{1p}^2	Móri, Rohatgi, Székely s	Kollo b	Song $\mathcal{S}(f)$
1		Ω $\begin{bmatrix}1 & 0 & 0\\ 0 & 2.5 & 0\\ 0 & 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ 10\\ 10\end{bmatrix}$	0.126	$\begin{bmatrix}0.152\\ 0.681\\ 0.681\end{bmatrix}$	$\begin{bmatrix}0.367\\ 1.643\\ 1.643\end{bmatrix}$	0.2322
2		Ω $\begin{bmatrix}1 & 1 & 1\\ 1 & 2.5 & 1\\ 1 & 1 & 10\end{bmatrix}$ α $\begin{bmatrix}2\\ 3\\ 3\end{bmatrix}$	0.112	$\begin{bmatrix}0.443\\ 0.590\\ 0.504\end{bmatrix}$	$\begin{bmatrix}1.310\\ 1.746\\ 1.492\end{bmatrix}$	0.2356
3		Ω $\begin{bmatrix}1 & 0 & 0\\ 0 & 2.5 & 0\\ 0 & 0 & 5\end{bmatrix}$ α $\begin{bmatrix}10\\ 0\\ 0\end{bmatrix}$	0.304	$\begin{bmatrix}0.955\\ 0.0000\\ 0.0000\end{bmatrix}$	$\begin{bmatrix}0.955\\ 0.000\\ 0.000\end{bmatrix}$	0.2330
4		Ω $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2.5\end{bmatrix}$ α $\begin{bmatrix}2\\ -3\\ -3\end{bmatrix}$	0.171	$\begin{bmatrix}0.370\\ -0.555\\ -0.500\end{bmatrix}$	$\begin{bmatrix}0.250\\ -0.375\\ -0.338\end{bmatrix}$	0.2367
5		Ω $\begin{bmatrix}1 & 1 & 1\\ 1 & 2.5 & 1\\ 1 & 1 & 10\end{bmatrix}$ α $\begin{bmatrix}2\\ -1\\ -1\end{bmatrix}$	0.027	$\begin{bmatrix}0.310\\ -0.057\\ -0.120\end{bmatrix}$	$\begin{bmatrix}0.047\\ -0.009\\ -0.018\end{bmatrix}$	0.0389
6		Ω $\begin{bmatrix}1 & 0 & 0\\ 0 & 2.5 & 0\\ 0 & 0 & 1\end{bmatrix}$ α $\begin{bmatrix}2\\ 1\\ 1\end{bmatrix}$	0.093	$\begin{bmatrix}0.467\\ 0.214\\ 0.234\end{bmatrix}$	$\begin{bmatrix}1.228\\ 0.562\\ 0.614\end{bmatrix}$	0.1333

distribution. Both p-value and power of the test are probabilities and their use allows us to compare different statistics, no matter what the original scales of them were.

It is known [16] that if the distribution is multivariate normal, the Song's measure is $p/2$, while all other considered measures are zero. A test based on the empirical version of the Song index will consider as rejection region the area exceeding the theoretical value, while for all other tests, the rejection region will cover the area not exceeding the theoretical value.

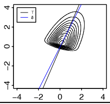
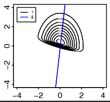
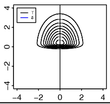
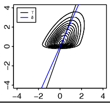
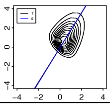
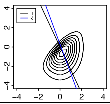
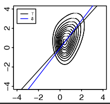
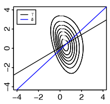
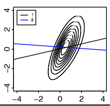
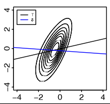
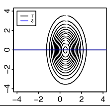
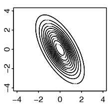
By proceeding via simulation, a sensitivity index (p-value) for these skewness measures can be provided by enumerating the number of samples from multivariate normal distribution having each index of skewness not exceeding the theoretical value obtained for the SN. Similarly, a specificity index (the power of the test) can be obtained by sampling from SN and enumerating the number of samples that have each measure of skewness not exceeding the expected theoretical value. For the Song's measure, sensitivity and specificity are obtained by considering the reverse rejection regions.

The implementation of these simulations needs the introduction of sample versions of the skewness measures we are considering here.

Let $X_1, \ldots, X_n$ denote a sample of $p \times 1$ observations from any p-dimensional distribution. A sample version of all the skewness measures described in the preceding sections can be obtained by replacing $\boldsymbol{\xi}$, $\boldsymbol{\Omega}$ and $\boldsymbol{\alpha}$ with the empirical estimates of these quantities ($\hat{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\Omega}}$ and $\hat{\boldsymbol{\alpha}}$), obtained, for example, by using the maximum likelihood method by hypothesizing SN distribution for observed data [1].

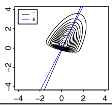
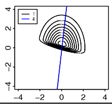
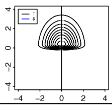
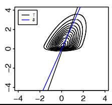
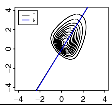
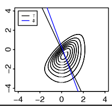
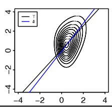
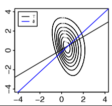
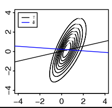
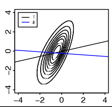
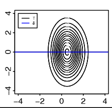
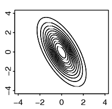
In order to obtain a single test statistic for the vectorial measures (Balakrishnan–Brito–Quiroz, Isogai, Móri–Rohatgi–Székely and Kollo), we propose two different metrics, namely, the *sum* and the *maximum*. For the Balakrishnan–Brito–Quiroz

Table 5
p-values and power for test for bivariate normality against skew-normal.

#	Contour plot	Parameters			Mardia	Balakrishnan, Brito and Quiroz				Isogai	
					$\gamma_{1,p}$	Q^*	Q	T_{sup}	T_{sum}	δ_{sup}	δ_{sum}
1		Ω	$\begin{bmatrix}1&1\\1&2.5\end{bmatrix}$	p-value:	0.995	0.979	0.913	0.992	0.994	0.407	0.376
		α	$\begin{bmatrix}2\\10\end{bmatrix}$	Power:	0.334	0.447	0.461	0.470	0.445	0.543	0.553
2		Ω	$\begin{bmatrix}1&0\\0&2.5\end{bmatrix}$	p-value:	0.996	1.000	0.997	0.997	0.514	0.266	0.000
		α	$\begin{bmatrix}2\\10\end{bmatrix}$	Power:	0.341	0.454	0.455	0.439	0.537	0.385	0.583
3		Ω	$\begin{bmatrix}1&0\\0&2.5\end{bmatrix}$	p-value:	0.992	0.999	1.000	0.909	1.000	0.543	0.196
		α	$\begin{bmatrix}0\\10\end{bmatrix}$	Power:	0.312	0.468	0.434	0.439	0.426	0.545	0.214
4		Ω	$\begin{bmatrix}1&1\\1&2.5\end{bmatrix}$	p-value:	0.990	0.998	0.856	0.998	0.998	0.415	0.358
		α	$\begin{bmatrix}0\\10\end{bmatrix}$	Power:	0.329	0.499	0.485	0.484	0.462	0.528	0.491
5		Ω	$\begin{bmatrix}1&1\\1&2.5\end{bmatrix}$	p-value:	0.917	0.768	0.814	0.787	0.924	0.373	0.395
		α	$\begin{bmatrix}2\\3\end{bmatrix}$	Power:	0.405	0.249	0.350	0.481	0.365	0.549	0.566
6		Ω	$\begin{bmatrix}1&0\\0&2.5\end{bmatrix}$	p-value:	0.899	0.906	0.662	0.730	0.894	0.387	0.302
		α	$\begin{bmatrix}2\\-3\end{bmatrix}$	Power:	0.517	0.413	0.366	0.480	0.484	0.555	0.450
7		Ω	$\begin{bmatrix}1&1\\1&2.5\end{bmatrix}$	p-value:	0.581	0.814	0.661	0.625	0.786	0.267	0.357
		α	$\begin{bmatrix}2\\1\end{bmatrix}$	Power:	0.375	0.518	0.326	0.361	0.370	0.421	0.551
8		Ω	$\begin{bmatrix}1&0\\0&2.5\end{bmatrix}$	p-value:	0.435	0.947	0.591	0.735	0.740	0.045	0.241
		α	$\begin{bmatrix}2\\1\end{bmatrix}$	Power:	0.310	0.554	0.359	0.481	0.428	0.191	0.363
9		Ω	$\begin{bmatrix}1&1\\1&2.5\end{bmatrix}$	p-value:	0.113	0.869	0.528	0.596	0.509	0.005	0.008
		α	$\begin{bmatrix}2\\-1\end{bmatrix}$	Power:	0.102	0.618	0.392	0.449	0.379	0.025	0.047
10		Ω	$\begin{bmatrix}1&1\\1&2.5\end{bmatrix}$	p-value:	0.097	0.889	0.569	0.607	0.512	0.004	0.008
		α	$\begin{bmatrix}-2\\1\end{bmatrix}$	Power:	0.101	0.655	0.462	0.488	0.409	0.031	0.044
11		Ω	$\begin{bmatrix}1&0\\0&2.5\end{bmatrix}$	p-value:	0.031	0.309	0.144	0.150	0.116	0.014	0.006
		α	$\begin{bmatrix}1\\0\end{bmatrix}$	Power:	0.032	0.228	0.108	0.117	0.094	0.041	0.015
12		Ω	$\begin{bmatrix}1&-1\\-1&2.5\end{bmatrix}$	p-value:	0.000	0.000	0.000	0.000	0.000	0.000	0.000
		α	$\begin{bmatrix}0\\0\end{bmatrix}$	Power:	0.000	0.000	0.000	0.000	0.000	0.000	0.000

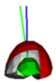
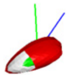
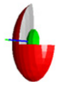
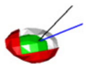
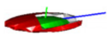
index, we compute $T_{\text{sum}} = \sum_{i=1}^{p} T_i$, for the Isogai's measure $\delta_{\text{sum}} = \sum_{i=1}^{p} \delta_i$, for the Móri–Rohatgi–Székely measure $s_{\text{sum}} = \sum_{i=1}^{p} s_i$ and for Kollo measure $b_{\text{sum}} = \sum_{i=1}^{p} b_i$. Similarly, for all these measures, we compute the maxima statistic as $T_{\text{sup}} = \sup_{i\in(1,\ldots,p)} T_i$, $\delta_{\text{sup}} = \sup_{i\in(1,\ldots,p)} \delta_i$, $s_{\text{sup}} = \sup_{i\in(1,\ldots,p)} s_i$, $b_{\text{sup}} = \sup_{i\in(1,\ldots,p)} b_i$, respectively.

Table 6
p-values and power for test for bivariate normality against skew-normal.

#	Contour plot	Parameters			Srivastava	Móri, Rohatgi and Székely		Kollo		Song
					β_{1p}^2	$s_{\sup}$	s_{sum}	$b_{\sup}$	b_{sum}	$\mathcal{S}(f)$
1		Ω	$\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$	p-value:	0.998	0.997	0.995	0.998	0.996	0.981
		α	$\begin{bmatrix}2\\ 10\end{bmatrix}$	Power:	0.504	0.419	0.467	0.503	0.520	0.696
2		Ω	$\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$	p-value:	0.940	0.998	0.994	0.998	0.978	0.982
		α	$\begin{bmatrix}2\\ 10\end{bmatrix}$	Power:	0.458	0.398	0.487	0.514	0.525	0.692
3		Ω	$\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$	p-value:	0.999	0.998	0.944	0.966	0.843	0.974
		α	$\begin{bmatrix}0\\ 10\end{bmatrix}$	Power:	0.858	0.336	0.097	0.490	0.438	0.724
4		Ω	$\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$	p-value:	0.979	0.999	0.988	0.995	0.957	0.980
		α	$\begin{bmatrix}0\\ 10\end{bmatrix}$	Power:	0.453	0.400	0.445	0.467	0.472	0.700
5		Ω	$\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$	p-value:	0.978	0.941	0.988	0.994	0.991	0.989
		α	$\begin{bmatrix}2\\ 3\end{bmatrix}$	Power:	0.493	0.417	0.439	0.462	0.501	0.827
6		Ω	$\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$	p-value:	0.896	0.913	0.983	0.244	0.112	0.972
		α	$\begin{bmatrix}2\\ -3\end{bmatrix}$	Power:	0.669	0.503	0.586	0.223	0.225	0.545
7		Ω	$\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$	p-value:	0.661	0.811	0.923	0.972	0.963	0.877
		α	$\begin{bmatrix}2\\ 1\end{bmatrix}$	Power:	0.281	0.402	0.467	0.499	0.537	0.429
8		Ω	$\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$	p-value:	0.842	0.771	0.828	0.954	0.919	0.784
		α	$\begin{bmatrix}2\\ 1\end{bmatrix}$	Power:	0.549	0.458	0.499	0.598	0.639	0.459
9		Ω	$\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$	p-value:	0.571	0.481	0.368	0.541	0.272	0.426
		α	$\begin{bmatrix}2\\ -1\end{bmatrix}$	Power:	0.422	0.360	0.260	0.265	0.241	0.307
10		Ω	$\begin{bmatrix}1 & 1\\ 1 & 2.5\end{bmatrix}$	p-value:	0.563	0.486	0.353	0.556	0.310	0.416
		α	$\begin{bmatrix}-2\\ 1\end{bmatrix}$	Power:	0.463	0.379	0.306	0.279	0.253	0.336
11		Ω	$\begin{bmatrix}1 & 0\\ 0 & 2.5\end{bmatrix}$	p-value:	0.140	0.102	0.076	0.544	0.231	0.089
		α	$\begin{bmatrix}1\\ 0\end{bmatrix}$	Power:	0.109	0.083	0.053	0.265	0.209	0.069
12		Ω	$\begin{bmatrix}1 & -1\\ -1 & 2.5\end{bmatrix}$	p-value:	0.000	0.000	0.000	0.000	0.000	0.000
		α	$\begin{bmatrix}0\\ 0\end{bmatrix}$	Power:	0.000	0.000	0.000	0.000	0.000	0.000

For sensitivity analysis, we simulated 1000 samples of size 100 from the multivariate normal model with parameter settings as listed in Tables 1–4. For each sample, we computed every empirical index of skewness and counted the proportion of samples for which the skewness index fell in the rejection region.

Table 7
p-values and power for test for trivariate normality against skew-normal.

#	Contour plot	Parameters		Mardia	Balakrishnan, Brito and Quiroz				Isogai	
				$\gamma_{1,p}$	Q^*	Q	T_{sup}	T_{sum}	δ_{sup}	δ_{sum}
1		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$	p-value:	0.969	0.930	0.873	0.524	0.926	0.048	0.170
		α $\begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$	Power:	0.464	0.511	0.275	0.238	0.519	0.101	0.374
2		Ω $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$	p-value:	0.886	0.612	0.814	0.376	0.773	0.419	0.425
		α $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$	Power:	0.424	0.330	0.164	0.275	0.317	0.466	0.552
3		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	p-value:	0.900	1.000	0.909	0.999	0.999	0.012	0.000
		α $\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$	Power:	0.334	0.634	0.590	0.693	0.674	0.402	0.005
4		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$	p-value:	0.830	0.805	0.662	0.325	0.727	0.267	0.236
		α $\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}$	Power:	0.511	0.464	0.174	0.246	0.416	0.461	0.397
5		Ω $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$	p-value:	0.132	0.866	0.528	0.388	0.234	0.386	0.124
		α $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$	Power:	0.116	0.734	0.331	0.340	0.196	0.390	0.146
6		Ω $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	p-value:	0.270	0.793	0.591	0.463	0.458	0.020	0.160
		α $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$	Power:	0.244	0.540	0.218	0.411	0.300	0.112	0.269

Also, the specificity is obtained by simulating 1000 samples of size 100 from the SN distribution with different parameter settings listed in Tables 1–4, and then by counting the proportion of samples falling in the same rejection region.

Tables 5–8 report the p-values and the power of tests when testing for the normal distribution against the SN distribution in the bivariate and trivariate cases, respectively.

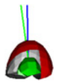
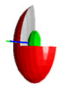
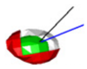
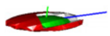
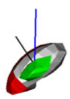
6. Discussion

Based on our empirical study, by considering different cases of the SN distributions in two and three dimensions, we observe the following points.

Song's measure, while being a good overall measure of the general shape of the distribution, does not seem to be a fair index of skewness; this is observed in the numerical results of Tables 2 and 4, and it is based on the fact that two different distributions, one symmetric and the other one skew, can have the same Song's measure in (6), since this index depends only on the levels of the density function and not on the values that the random variable assumes for each level of the density. Consequently, no knowledge is gained by Song's measure about the varying shape with respect to the center of the distribution.

A second aspect, remarkable from Tables 1 and 3, is that diverse elements of the skew-normal family, asymmetric in different directions, share the same Mardia and Balakrishnan–Brito–Quiroz overall measures, but they have different

Table 8
p-values and power for test for trivariate normality against skew-normal.

#	Contour plot	Parameters			Srivastava	Móri, Rohatgi and Székely		Kollo		Song
					β_{1p}^2	S_{sup}	S_{sum}	b_{sup}	b_{sum}	$S(f)$
1		Ω	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$	p-value:	0.923	0.879	0.982	0.978	0.991	0.903
		α	$\begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$	Power:	0.601	0.338	0.625	0.485	0.680	0.838
2		Ω	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$	p-value:	0.904	0.784	0.992	0.991	1.000	0.946
		α	$\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$	Power:	0.619	0.354	0.496	0.453	0.635	0.852
3		Ω	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	p-value:	1.000	0.998	0.732	0.864	0.709	0.933
		α	$\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$	Power:	0.771	0.495	0.124	0.568	0.432	0.757
4		Ω	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$	p-value:	0.964	0.731	0.961	0.552	0.711	0.992
		α	$\begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}$	Power:	0.773	0.408	0.623	0.454	0.563	0.956
5		Ω	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 10 \end{bmatrix}$	p-value:	0.433	0.239	0.205	0.124	0.115	0.195
		α	$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$	Power:	0.365	0.204	0.159	0.143	0.136	0.160
6		Ω	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	p-value:	0.833	0.539	0.709	0.949	0.959	0.564
		α	$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$	Power:	0.662	0.373	0.458	0.684	0.716	0.375

vectorial measures T and δ, thereby suggesting that a vectorial measure is relevant in describing this characteristic of a distribution.

We also remark that the equivalence of the Mardia and Malkovich–Afifi measures for the SN distribution is based on the existence of the canonical form for this family of distributions, and will hold for any family of distributions that possesses this property.

As already mentioned, from Tables 1–4, we note that some skew shape of the distribution are characterized by high values of all the measures (such as Cases 2, 3 and 4 of Table 1 and Cases 2 and 3 of Tables 3 and 4), while other cases show only a moderate value for some indices (such as Cases 1, 5 and 6 of Table 1 show a high Mardia index, high values for at least one dimension of the Isogai measure, but relatively small Balakrishnan–Brito–Quiroz overall measure). This allows us to conclude that even if the Q index assigns a relatively small value of skewness to distributions that are still asymmetric, it seems also to be characterized by a more detailed range of values indicating different types of skewness.

The comparison between measures may be directly performed by considering Tables 5–8. These results show clearly which are the poorer indices of skewness among those considered. In addition to the already mentioned Song's measure, these tables also reveal that the Isogai measure too does not seem to present high sensitivity and specificity, in both metrics considered.

In many cases, sensitivity is very high for all the remaining measures, while specificity displays more variability. By considering bivariate distributions (Table 5), Mardia's index seems to be the best choice if we are looking for types of skewness described by Cases 1– 4, while, for other cases, one of the summary metrics based on Balakrishnan–Brito–Quiroz measure seems to better describe asymmetry: the Q index seems to be the best one for Cases 6 and 10, and may also be chosen for Cases 8 and 11, while the sum of the vectorial values (T_{sum}) is the good choice for Cases 5, 7, 9 and could also be considered for Cases 8 and 11.

The results for the trivariate case are less clear in suggesting a choice, since in most cases sensitivity and specificity suggest different choices. Quite often, Mardia's index seems to perform well and moreover T_{sup} or T_{sum} show better specificity in several cases.

Taking into due consideration all the observations made above, in general, the measures of skewness of Mardia and Balakrishnan–Brito–Quiroz perform very well in most of the cases considered in terms of indication of skewness as well as in terms of sensitivity and specificity. Moreover, they both seem to agree in many cases. Of course, the measure of Balakrishnan–Brito–Quiroz also gives a direction of the skewness in addition to the magnitude. Therefore, these two measures are the ones to be recommended for practical use.

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