Advances in Applied Mathematics 45 (2010) 505–517

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Catalan pairs: A relational-theoretic approach to Catalan numbers ☆

Filippo Disanto [a], Luca Ferrari [b], Renzo Pinzani [b,*], Simone Rinaldi [a]

[a] Dipartimento di Scienze Matematiche ed Informatiche, Pian dei Mantellini 44, 53100 Siena, Italy
[b] Dipartimento di Sistemi e Informatica, viale Morgagni 65, 50134 Firenze, Italy

ARTICLE INFO

Article history:
Received 27 September 2009
Accepted 14 February 2010

MSC:
06A07
05A19
05A05

Keywords:
Relation
Catalan numbers
Poset
(Pattern avoiding) permutation

ABSTRACT

We define the notion of a *Catalan pair* (which is a pair of binary relations (S, R) satisfying certain axioms) with the aim of giving a common language to several combinatorial interpretations of Catalan numbers. We show, in particular, that the second component R uniquely determines the pair, and we give a characterization of R in terms of forbidden configurations. We also propose some generalizations of Catalan pairs arising from some slight modifications of (some of the) axioms.

1. Introduction

A famous exercise of [26] proposes to the reader to show that every item of a long list of combinatorial structures provides a possible interpretation of the well-known sequence of Catalan numbers, where the n-th Catalan number is given by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the first terms of the considered sequence are $1, 1, 2, 5, 14, 42, 132, 429, \ldots$. In addition, since its appearance, many new combinatorial instances of Catalan numbers (in part due to Stanley as well [24]) have been presented by several authors ([4,5,17,15,16], to cite only a few). What makes Stanley's exercise even more scary

☆ This work has been partially supported by MIUR project: *Aspetti matematici e applicazioni emergenti degli automi e dei linguaggi formali*.
* Corresponding author.
E-mail addresses: disafili@yahoo.it (F. Disanto), ferrari@dsi.unifi.it (L. Ferrari), pinzani@dsi.unifi.it (R. Pinzani), rinaldi@unisi.it (S. Rinaldi).

0196-8858/$ – see front matter
doi:10.1016/j.aam.2010.02.004

is the request for an explicit bijection for each couple of structures: even the more skillful and bold student will eventually give up, frightened by such a long effort.

The motivation of the present work lies in the attempt of making the above job as easy as possible. We propose yet another instance of Catalan numbers, by showing that they count pairs of binary relations satisfying certain axioms. Of course this is not the first interpretation of Catalan numbers in terms of binary relations. For instance, a well-known appearance of Catalan numbers comes from considering the so-called *similarity relations*; these have been introduced by Fine [9] and further studied by several authors [12,18,22]. However, what we claim to be interesting in our setting is that many known Catalan structures can be obtained by suitably interpreting our relations in the considered framework. From the point of view of our student, this approach should result in a quicker way to find bijections: indeed, it will be enough to guess the correct translation of any two Catalan structures in terms of our binary relations to get, as a bonus, the desired bijection. We hope to make this statement much clearer in Section 3, where, after the definition of a *Catalan pair* and the proofs of some of its properties (pursued on Section 2), we explicitly describe some representations of Catalan pairs in terms of well-known combinatorial objects.

The rest of the paper is devoted to show that Catalan pairs are indeed a concept that deserves to be better investigated. In Section 4 we show that any Catalan pair is uniquely determined by its second component, and we also provide a characterization of such a component in terms of forbidden configurations (which, in our case, are forbidden posets). We also observe that the first component of a Catalan pair does not uniquely determine the pair itself, and we give a description of Catalan pairs having the same first component. Finally, we propose some generalizations of Catalan pairs by considering a slight, and very natural, modification of the crucial axiom in the definition of a Catalan pair and giving an account of what this fact leads to.

Throughout the paper the reader will find a (not at all exhaustive) series of open problems. We hope they can serve to stimulate future research on these topics.

2. Catalan pairs

In what follows, given any set X, we denote by $\mathcal{D} = \mathcal{D}(X)$ the *diagonal* of X, that is the relation $\mathcal{D} = \{(x, x) \mid x \in X\}$. Moreover, if θ is any binary relation on X, we denote by $\bar{\theta}$ the *symmetrization* of θ, i.e. the relation $\bar{\theta} = \theta \cup \theta^{-1}$.

2.1. Basic definitions

Given a set X of cardinality n, let $\mathcal{O}(X)$ be the set of strict order relations on X. By definition, this means that $\theta \in \mathcal{O}(X)$ when θ is an irreflexive and transitive binary relation on X. In symbols, this means that $\theta \cap \mathcal{D} = \emptyset$ and $\theta \circ \theta \subseteq \theta$ (for the definition of the composition operator $\circ$ on relations see, for instance, [14]).

Now let (S, R) be an ordered pair of binary relations on X. We say that (S, R) is a *Catalan pair* on X when the following axioms are satisfied:

(i) $S \in \mathcal{O}(X)$; (**ord S**)
(ii) $R \in \mathcal{O}(X)$; (**ord R**)
(iii) $\overline{R} \cup \overline{S} = X^2 \setminus \mathcal{D}$; (**tot**)
(iv) $\overline{R} \cap \overline{S} = \emptyset$; (**inters**)
(v) $S \circ R \subseteq R$; (**comp**)

Remarks.

1. Observe that, since S and R are both strict order relations, the two axioms (**tot**) and (**inters**) can be explicitly described by saying that, given $x, y \in X$, with $x \neq y$, exactly one of the following holds: xSy, xRy, ySx, yRx.

2. Axiom (**comp**) could be reformulated by using strict containment, i.e. $S \circ R \subset R$. In fact, it is not difficult to realize that equality cannot hold since X is finite. However we prefer to keep our notation, thus allowing to extend the definition of a Catalan pair to the infinite case.

In a Catalan pair (S, R), S (resp. R) will be referred to as the *first* (resp. *second*) *component*. Two Catalan pairs (S_1, R_1) and (S_2, R_2) on the (not necessarily distinct) sets X_1 and X_2, respectively, are said to be *isomorphic* when there exists a bijection ξ from X_1 to X_2 such that xS_1y if and only if $\xi(x)S_2\xi(y)$ and xR_1y if and only if $\xi(x)R_2\xi(y)$. We say that a Catalan pair has *size* n when it is defined on a set X of cardinality n and the set of isomorphism classes of Catalan pairs of size n will be denoted by $\mathcal{C}(n)$. We will be mainly interested in the set $\mathcal{C}(n)$, even if, in several specific cases, we will deal with "concrete" Catalan pairs. However, in order not to make our paper dull reading, we will use the term "Catalan pair" when referring both to a specific Catalan pair and to an element of $\mathcal{C}(n)$. In the same spirit, to mean that a Catalan pair has size n, we will frequently write "$(S, R) \in \mathcal{C}(n)$", even if $\mathcal{C}(n)$ is a set of isomorphism classes. In each situation, the context will clarify which is the exact meaning of what we have written down.

As an immediate consequence of the definition of a Catalan pair (specifically, from the fact that all the axioms are universal propositions), the following property holds.

Proposition 2.1. *Let (S, R) be a Catalan pair on X. For any $\widetilde{X} \subseteq X$, denote by $\widetilde{S}$ and $\widetilde{R}$ the restrictions of S and R to $\widetilde{X}$, respectively. Then $(\widetilde{S}, \widetilde{R})$ is a Catalan pair on $\widetilde{X}$.*

2.2. First properties of Catalan pairs

In order to get trained with the above definition, we start by giving some elementary properties of Catalan pairs.

Proposition 2.2. *Given a Catalan pair (S, R), the following properties hold*:

1. $S \circ R^{-1} \subseteq R^{-1}$;
2. $R \circ S \subseteq R \cup S$.

Proof.

1. If $xSyR^{-1}z$, then xSy and zRy. Since x and z are necessarily distinct (this follows from axiom (**inters**)), we must have exactly one of zRx, xRz, zSx and xSz. It is then easy to check that the three cases xRz, zSx, xSz cannot hold. For instance, if xRz, then $xRzRy$, whence xRy, against (**inters**) (since, by hypothesis, xSy). Using also the property (**comp**) the reader can prove that both zSx and xSz lead to a contradiction. Thus zRx, i.e. $xR^{-1}z$.
2. Suppose that $xRySz$. Once again, observe that the elements x and z are necessarily distinct, thus we must have exactly one of xRz, xSz, zRx and zSx. Similarly as above, it can be shown that neither zRx nor zSx can hold. □

As a consequence of Proposition 2.2 we have the following result, whose proof easily follows from the above remark 1 by straightforward computations.

Proposition 2.3. *Let (S, R) be a pair of binary relations on X satisfying axioms* (**ord S**), (**ord R**), (**tot**) *and* (**inters**). *Then axiom* (**comp**) *is equivalent to*:

$$S \circ \overline{R} \subseteq \overline{R}. \tag{1}$$

The above property will be useful in the sequel, when we will investigate the properties of the relation R.

Proposition 2.4. *Let (S, R) be a pair of binary relations on X satisfying axioms* (**ord S**), (**ord R**), (**tot**) *and* (**inters**). *Then axiom* (**comp**) *is equivalent to*:

$$\bar{S} \circ R \subseteq R \cup S^{-1}. \qquad (\textbf{comp}^*)$$

Proof. Assume that axiom (**comp**) holds and let $x\bar{S}yRz$. Since $x\bar{S}y$, we have two possibilities: if xSy, then $xSyRz$ and xRz. Instead, if ySx, then, being also yRz, we get that both the cases xSz and zRx cannot occur. Therefore it must be either zSx or xRz, which means that $(x, z) \in R \cup S^{-1}$.

Conversely, assume that condition (**comp***) holds, and suppose that $xSyRz$. We obviously deduce $x\bar{S}yRz$, and so we have either xRz or zSx. If zSx, then $zSxSy$, whence zSy, against the hypothesis yRz. Therefore it must be xRz. □

We conclude this section by observing that the choice of the name "Catalan pair" is motivated by the fact that Catalan pairs are indeed counted by Catalan numbers. More precisely, the set $\mathcal{C}(n)$ of nonisomorphic Catalan pairs on a set having n elements has cardinality C_n, the n-th Catalan number. A formal proof of this result can be found in [7], and is not reported here in order to keep the length of the paper to a minimum.

3. Combinatorial interpretations of Catalan pairs

In this section we wish to convince the reader that many combinatorial structures counted by Catalan numbers can be interpreted in terms of Catalan pairs. More precisely, we deem that most of the main Catalan structures can be described using a suitable Catalan pair (S, R), where S and R are somehow naturally defined on the objects of the class. To support this statement, we will take into consideration here four examples, involving rather different combinatorial objects, such as matchings, permutations, trees and partitions. For each of them, we will provide a combinatorial interpretation in terms of Catalan pairs.

3.1. Perfect noncrossing matchings (and Dyck paths)

Our first example will be frequently used throughout all the paper. Given a set A of even cardinality, a *perfect noncrossing matching* of A is a noncrossing partition of A having all the blocks of cardinality 2. There is an obvious bijection between perfect noncrossing matchings and well formed strings of parentheses.

A graphical device to represent a perfect noncrossing matching of A consists of drawing the elements of A as points on a straight line and join with an arc each couple of corresponding points in the matching. Using this representation, we can define the following relations on the set X of arcs of a given perfect noncrossing matching:

- for any $x, y \in X$, we say that xSy when x is included in y;
- for any $x, y \in X$, we say that xRy when x is on the left of y.

The reader is invited to check that the above definition yields a Catalan pair (S, R) on the set X.

Example. Let $X = \{a, b, c, d, e, f, g\}$, and let S and R be defined as follows:

$$S = \{(b,a), (f,e), (f,d), (e,d), (g,d)\},$$
$$R = \{(a,c), (a,d), (a,e), (a,f), (a,g), (b,c), (b,d), (b,e), (b,f), (b,g), (c,d), (c,e), (c,f), (c,g), (e,g), (f,g)\}.$$

It is easy to check that (S, R) is indeed a Catalan pair on X of size 7, which can be represented as in Fig. 1(a).

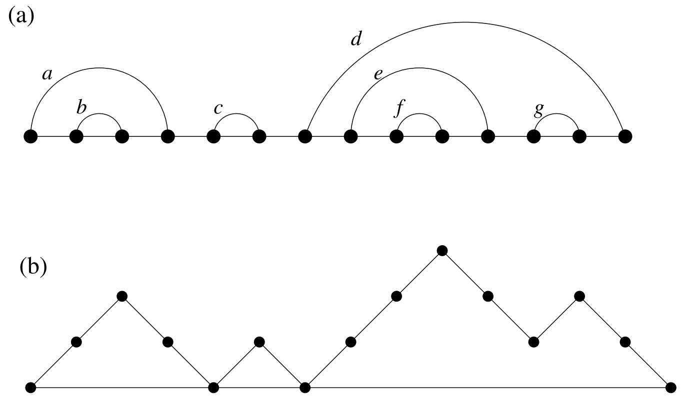


Fig. 1. The graphical representation of a Catalan pair in terms of a noncrossing matching, and the associated Dyck path.

An equivalent way to represent perfect noncrossing matchings is to use Dyck paths: just interpret the leftmost element of an arc as an up step and the rightmost one as a down step. For instance, the matching represented in Fig. 1(a) corresponds to the Dyck path depicted in Fig. 1(b). Coming back to Catalan pairs, the relations S and R are suitably interpreted using the notion of tunnel. A *tunnel* in a Dyck path [8] is a horizontal segment joining the midpoints of an up step and a down step, remaining below the path and not intersecting the path anywhere else. Now define S and R on the set X of the tunnels of a Dyck paths by declaring, for any $x, y \in X$:

- xSy when x lies above y;
- xRy when x is completely on the left of y.

3.2. *Pattern avoiding permutations*

Let n, m be two positive integers with $m \leqslant n$, and let $\pi = \pi(1)\cdots\pi(n) \in S_n$ and $\nu = \nu(1)\cdots\nu(m) \in S_m$. We say that π *contains* the pattern ν if there exist indices $i_1 < i_2 < \cdots < i_m$ such that $(\pi(i_1), \pi(i_2), \ldots, \pi(i_m))$ is in the same relative order as $(\nu(1), \ldots, \nu(m))$. If π does not contain ν, we say that π is ν-*avoiding*. See [2] for plenty of information on pattern avoiding permutations. For instance, if $\nu = 123$, then $\pi = 524316$ contains ν, while $\pi = 632541$ is ν-avoiding. We denote by $S_n(\nu)$ the set of ν-avoiding permutations of S_n. It is known that, for each pattern $\nu \in S_3$, $|S_n(\nu)| = C_n$ (see, for instance, [2]).

It is possible to give a description of the class of 312-avoiding permutations by means of a very natural set of Catalan pairs. More precisely, let $[n] = \{1, 2, \ldots, n\}$; for every permutation $\pi \in S_n$, define the following relations S and R on $[n]$:

- iSj when $i < j$ and (j, i) is an inversion in π, i.e. $\pi(i) > \pi(j)$;
- iRj when $i < j$ and (i, j) is a noninversion in π, i.e. $\pi(i) < \pi(j)$.

Proposition 3.1. *The permutation $\pi \in S_n$ is* 312*-avoiding if and only if (S, R) is a Catalan pair of size n.*

Proof. The axioms (i) to (iv) in the definition of a Catalan pair are satisfied by (S, R) for any permutation π, as the reader can easily check. Moreover, π is 312-avoiding if and only if, given any three positive integers $i < j < k$, it can never happen that both (j, i) and (k, i) are inversions and (j, k) is a noninversion. This happens if and only if $S \circ R$ and S are disjoint. But, from the above definitions of S and R, it must be $S \circ R \subseteq R \cup S$, whence $S \circ R \subseteq R$. $\square$

The present interpretation in terms of 312-avoiding permutations can be connected with the previous ones using Dyck paths and perfect noncrossing matchings, giving rise to a very well-known

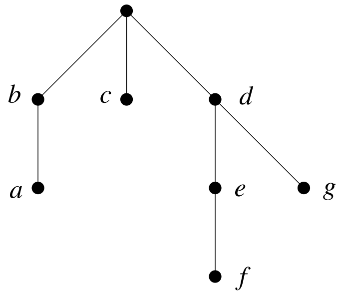


Fig. 2. The plane tree corresponding to the Catalan pair represented in Fig. 1.

bijection, whose origin is very hard to be traced back (see, for instance, [20]). We leave all the details to the interested reader.

3.3. *Plane trees*

Let $\mathcal{T}_n$ be the set of plane trees having n edges. We say that a node b is a *descendant* of a node a when b belongs to the subtree generated by a. In this situation, we also say that a is an *ancestor* of b. For any two nodes b and c, we define their *minimum common ancestor* to be the root a of the minimum subtree containing both b and c. Finally, we will say that b lies *on the left of* c when, called a the minimum common ancestor of b and c, b belongs to a subtree of a which is on the left of the subtree of a containing c.

Given $t \in \mathcal{T}_n$, let X denote the set of nodes of t other than the root. Define two relations S and R on X as follows:

- xSy when y is a descendant of x;
- xRy when x lies on the left of y.

Then the pair (S, R) is indeed a Catalan pair on X, and it induces the well-known bijection between plane trees and Dyck paths. Fig. 2 depicts the plane tree corresponding to the Catalan pair (S, R) represented in Fig. 1.

3.4. *Noncrossing partitions*

Let $\mathcal{P}_n$ be the set of noncrossing partitions on the linearly ordered set $X_n = \{x_1, x_2, \dots, x_n\}$. The elements of each $p \in \mathcal{P}_n$ are subsets of X_n and, for any $x \in X_n$, we will denote the subset in p containing x by $[x]_p$.

Given $x \in X_n$, we set $u(x) = \max\{y \mid (\forall t \in [x]_p) y < t\}$. Thus $u(x)$ is given by the predecessor of the minimum of $[x]_p$. Observe that $u(x)$ is not defined on the elements belonging to $[x_1]_p$, x_1 being the minimum of X_n.

Given $p \in \mathcal{P}_n$, define S and R on X_n as follows:

- S is the transitive closure of the relation $\{(x, u(x)) \mid x \in X_n\}$;
- xRy when $x < y$ and (y, x) is not in S.

It can be proved that all the axioms of a Catalan pair are satisfied. As an example, we provide a proof of axiom (**comp**).

Suppose that $xSyRz$, thus being, by the definition of S, $x = y_0 Su(y_0) = y_1 Su(y_1) = y_2 S \dots Su(y_{k-1}) = y_k = yRz$. If we had zSx, this would imply, by transitivity, zSy which is not, since yRz. Moreover we also have that $x < z$. Indeed, supposing $x > z$, since $y < z$ there exists an i such that $y_{i+1} < z < y_i$ (z cannot be one of the y_j's since yRz); therefore we have two distinct possibilities for z: either $z \in [y_i]_p$ or z lies entirely below an arc connecting two elements of $[y_i]_p$. In both cases

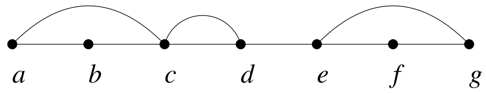


Fig. 3. The noncrossing partition corresponding to the Catalan pair represented in Fig. 1.

we have that zSy_{i+1} (in the first case this is obvious, whereas in the second case it follows from the fact that the block containing z is on the right of $y_{i+1} = u(y_i)$). From here we get zSy, which is a contradiction. Thus we have shown that xRz.

This Catalan pair (S, R) induces a bijection between noncrossing partitions and plane trees which we have not been able to find in the literature. Fig. 3 depicts the noncrossing partition corresponding to the Catalan pair (S, R) represented in Fig. 1.

4. Properties of the posets defined by *S* and *R*

In the present section we investigate some features of the posets associated with the (strict) order relations S and R. An immediate observation which follows directly from the definition of a Catalan pair is the following, which we state without proof. In the sequel $[X, S]$ denotes the poset having support X and (strict) order relation S.

Proposition 4.1. *Given a finite set X, consider the graphs X_1 and X_2 determined by the relations R and S. Then X_1 and X_2 are edge-disjoint subgraphs of the complete graph $K(X)$ on X whose union gives the whole $K(X)$.*

4.1. The poset defined by S

As a consequence of the interpretation of Catalan pairs in the class of plane trees given in Section 3.3 we obtain a bijection between the class $\mathcal{C}(n)$ and that of plane trees with $n+1$ nodes. Such a bijection suggests the following lemma.

Lemma 4.1. *Given a Catalan pair (S, R) on X, let x, y be two incomparable elements (if any) of S. Then, there exists no element t such that tSx and tSy.*

Proof. Just observe that the statement of the proposition is equivalent to a known property of plane trees, namely that two nodes x, y of a plane tree whose minimum common ancestor is neither x or y cannot have a common descendant. $\square$

The interpretation in terms of plane trees allows us to give a characterization of the class of posets defined by the first component S of a Catalan pair $(S, R) \in \mathcal{C}(n)$.

More precisely define the set $\mathbf{S}(n) = \{[X, S] \mid (\exists R)(S, R) \in \mathcal{C}(n)\}$. A combinatorial characterization of the posets in $\mathbf{S}(n)$ is a consequence of the next proposition.

Proposition 4.2. *If $[X, S] \in \mathbf{S}(n)$, then the Hasse diagram of $[X, S]$ is a forest of rooted trees, where the roots of the trees are the maximal elements of S.*

Proof. First observe that, thanks to Lemma 4.1, the poset $[X, S]$ has k connected components, where k is the number of its maximal elements. Now take x, y belonging to the same connected component and suppose that x and y are incomparable in S. By Lemma 4.1 the set of all lower bounds of $\{x, y\}$ is empty. Thus, the Hasse diagram of each connected component of $[X, S]$ is a direct acyclic graph, that is a tree, rooted at its maximum element, and this concludes our proof. $\square$

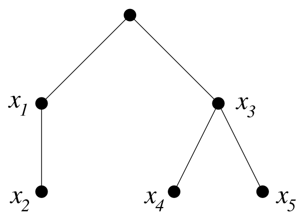
Obviously, each forest of rooted trees having n nodes can be represented, using the same concept of descendant as in the case of plane trees, by an element of $\mathbf{S}(n)$. This fact, together with the previous proposition, gives the desired characterization of S in terms of forests of rooted trees.

As a consequence of the previous statements we have the following enumerative property.

Corollary 4.1. *There is a bijection between* $\mathbf{S}(n)$ *and the set of rooted trees with* $n+1$ *nodes.*

Proof. Just add to the Hasse diagram of each element $[X,S]$ of $\mathbf{S}(n)$ a new root, linking such a root to the maximum of each connected component. □

Below the rooted tree on 6 nodes associated with $[X,S] \in \mathbf{S}(5)$ is shown, where $S = \{(x_2,x_1), (x_4,x_3), (x_5,x_3)\}$.



The above corollary implies that $|\mathbf{S}(n)|$ is given by the number of rooted trees having $n+1$ nodes, which is sequence A000081 in [23].

Recall that a rooted tree can be seen as a graph-isomorphism class of plane rooted trees. Since we have shown that Catalan pairs can be interpreted by using plane rooted trees, it easily follows that, given $S \in \mathbf{S}(n)$, the set of Catalan pairs (S,R) having S as the first component can be interpreted as the set of all plane rooted trees which are isomorphic (as graphs) to the Hasse diagram of $[X,S]$.

Thus the first component of a Catalan pair does not uniquely determine the pair. This should be also clear by examining the following two perfect noncrossing matchings, which are associated with the same S, but determine a different R, whence a different Catalan pair.

4.2. The poset defined by R

From the point of view of Catalan pairs, it turns out that the strict order relation R completely defines a Catalan pair. To prove this, we first need a technical definition which will also be useful again later.

Given a strict order relation R on X, define the relation $\sim_R$ on the set X by declaring $x \sim_R y$ when, for all z, it is $z\overline{R}x$ if and only if $z\overline{R}y$. It is trivial to show that $\sim_R$ is an equivalence relation. In what follows, the equivalence classes of $\sim_R$ will be denoted using square brackets.

Lemma 4.2.

(i) *If* $x \sim_R y$, *then* $x \not R y$.
(ii) $x \sim_R y$ *if and only if, for all* z, zRx *iff* zRy *and* xRz *iff* yRz.
(iii) *If* (S,R) *is a Catalan pair, then, for all* $x, y \in [z]_{\sim_R}$, xSy *or* ySx, *i.e.* S *is a total order on each equivalence class of* $\sim_R$.
(iv) *Suppose* (S,R) *is a Catalan pair. If* xSy *and* $x \nsim_R y$, *then there exists* $a \in X$ *such that* $a\overline{R}x$ *and* aSy.
(v) *For all* $x, y \in X$, xRy *iff* $[x]_{\sim_R} R [y]_{\sim_R}$ *(that is, for every* $x' \in [x]_{\sim_R}$ *and* $y' \in [y]_{\sim_R}$, $x'Ry'$*).*

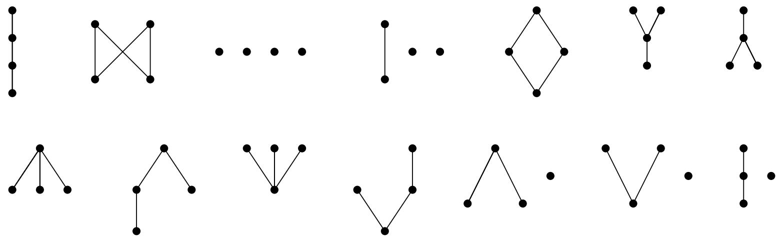


Fig. 4. The 14 posets of $\mathbf{R}(4)$.

Proof.

(i) Just observe that, if $x \sim_R y$, then $x\overline{R}y$ would imply $x\overline{R}x$, which is false.

(ii) Notice that, given that $x \sim_R y$, if zRx, then obviously $z\overline{R}x$, whence $z\overline{R}y$. If we had yRz, then, since zRx, it would also be yRx, which is impossible by (i). The fact that xRz implies yRz can be dealt with analogously.

(iii) It follows from (i) and Proposition 4.1.

(iv) From $x \nsim_R y$ it follows, by definition, that either there exists $a \in X$ such that $a\overline{R}x$ and $a\not\overline{R}y$, or there exists $b \in X$ such that $b\not\overline{R}x$ and $b\overline{R}y$. The second possibility cannot occur since, if such an element b existed, then, from the hypothesis xSy and from (1), we would have $x\overline{R}b$, a contradiction. Thus an element $a \in X$ with the above listed properties exists. In particular, since $a\not\overline{R}y$, it must be $a\overline{S}y$. If we had ySa, then, from xSy, it would follow xSa, a contradiction. Therefore it must be aSy, as desired.

(v) Suppose that xRy. If $a \sim_R x$, applying (ii) it follows that aRy. Now, if it is also $b \sim_R y$, applying (ii) once more yields aRb, which implies our thesis. $\square$

Theorem 4.1. *If (S_1, R), (S_2, R) are two Catalan pairs on X, then they are isomorphic.*

Proof. From Lemma 4.2(iii), each equivalence class of the relation $\sim_R$ is linearly ordered by the order relations S_1 and S_2.

Define a function F mapping X into itself such that, if $x \in X$ and there are exactly $k \geqslant 0$ elements in $[x]_{\sim_R}$ less than x with respect to the total order S_1, then $F(x)$ is that element in $[x]_{\sim_R}$ having exactly k elements less than it in the total order given by S_2. It is trivial to see that F is a bijection. Since $x \sim_R F(x)$, using Lemma 4.2(v), we get that xRy iff $F(x)RF(y)$.

To prove that xS_1y implies $F(x)S_2F(y)$ it is convenient to consider two different cases. First suppose that $x \sim_R y$; in this case our thesis directly follows from the definition of F. On the other hand, if $x \nsim_R y$, using Lemma 4.2(iv), there exists an element $a \in X$ such that $a\overline{R}x$ and aS_1y. Thus, considering the Catalan pair (S_2, R), it cannot be $F(x)\overline{R}F(y)$, since this would imply (by Lemma 4.2(v)) that $x\overline{R}y$, against xS_1y. Therefore it must be $F(x)\overline{S_2}F(y)$. More precisely, we get $F(x)S_2F(y)$, since, from $F(y)S_2F(x)\overline{R}a$, we would derive $F(y)\overline{R}a$ and so $y\overline{R}a$, which is impossible. With an analogous argument, we can also prove that $F(x)S_2F(y)$ implies xS_1y, which concludes the proof that F is an isomorphism between (S_1, R) and (S_2, R). $\square$

For the rest of the paper, we set $\mathbf{R}(n) = \{[X, R] \mid (\exists S)(S, R) \in \mathcal{C}(n)\}$.

The posets $[X, R] \in \mathbf{R}(4)$ are depicted in Fig. 4.

Among the possible 16 nonisomorphic posets on 4 elements, the two missing posets are shown in Fig. 5. They are respectively the poset $\mathbf{2} + \mathbf{2}$ (i.e. the direct sum of two copies of the 2-element chain) and the poset Z_4, called *fence of order* 4 (see, for instance, [6,19,26]).

The rest of this section is devoted to proving that the absence of the two posets $\mathbf{2} + \mathbf{2}$ and Z_4 is not an accident.

Proposition 4.3. *If $[X, R] \in \mathbf{R}(n)$, then $[X, R]$ does not contain any subposet isomorphic to $\mathbf{2} + \mathbf{2}$ or Z_4.*

2+2 Z_4

Fig. 5. The two posets not belonging to $\mathbf{R}(4)$.

Proof. Let $(S, R) \in \mathcal{C}(n)$ and suppose, ab absurdo, that $\mathbf{2}+\mathbf{2}$ is a subposet of $[X, R]$. Then, denoting by x, z and y, t the minimal and maximal elements of an occurrence of $\mathbf{2}+\mathbf{2}$ in $[X, R]$, respectively, and supposing that xRy and zRt, we would have, for instance, $t\bar{S}xRy$. By Proposition 2.4, since $t\not R y$, it is ySt. However, we also have $y\bar{S}zRt$ and $y\not R t$, whence tSy, which yields a contradiction with the previous derivation.

Similarly, suppose that Z_4 is a subposet of $[X, R]$. Then, supposing that xRy, xRt and zRt, we have $z\bar{S}xRy$, whence, by Proposition 2.4, ySz. However, it is also $ySzRt$, which implies yRt, and this is false. $\square$

We will now prove that the converse of the above proposition is also true, thus providing an order-theoretic necessary and sufficient condition for a strict order relation R to be the second component of a Catalan pair.

Proposition 4.4. *Let $R \in \mathcal{O}(X)$ such that $[X, R]$ does not contain subposets isomorphic to $\mathbf{2}+\mathbf{2}$ or Z_4. Then $[X, R] \in \mathbf{R}(n)$.*

Proof. Given $X = \{x_1, \ldots, x_n\}$, we define a binary relation $S = S(R)$ on X by making use of the equivalence relation $\sim_R$ defined at the beginning of the present section. More precisely:

- if $x_i \sim_R x_j$ and $i < j$, set $x_i S x_j$;
- if $x \nsim_R y$ and $x\overline{\not R}y$, set:
  i) xSy, when there exists $z \in X$ such that $z\overline{R}x$ and $z\overline{\not R}y$;
  ii) ySx, when there exists $z \in X$ such that $z\overline{\not R}x$ and $z\overline{R}y$.

We claim that $(S, R) \in \mathcal{C}(n)$.

It is trivial to show that axioms (**tot**) and (**inters**) in the definition of a Catalan pair are satisfied.

Next we show that axiom (**comp**) holds. Indeed, suppose that $xSyRq$ and $x\not R q$. From Lemma 4.2(ii), it would follow that $x \nsim_R y$. Thus, from xSy and the definition of S, we deduce that there is an element z such that $z\overline{R}x$ and $z\overline{\not R}y$. The reader can now check that the four elements x, y, q, z determine a subposet of $[X, R]$ isomorphic either to $\mathbf{2}+\mathbf{2}$ or Z_4, which is not allowed.

Using an analogous argument we can show that $S \circ R^{-1} \subseteq R^{-1}$. In fact, this will be useful below.

Finally, it remains to prove axiom (**ord S**), i.e. that $S \in \mathcal{O}(X)$. The fact that S is irreflexive is evident from its definition. To prove the transitivity of S, we first need to prove that, given $x, y \in X$, the two relations xSy and ySx cannot hold simultaneously. Indeed, if $x, y \in X$ were such that xSy and ySx, then it could not be $x \sim_R y$ and so, by definition, there would exist two elements $z, q \in X$ such that $z\overline{R}x$, $z\overline{\not R}y$, $q\overline{\not R}x$ and $q\overline{R}y$. It is not difficult to prove that the four elements x, y, z, q have to be all distinct (using the irreflexivity of R and S). Now, if we consider the poset determined by these four elements, in all possible cases a forbidden poset comes out, and we have reached a contradiction. Now suppose to have $xSySt$: we want to prove that necessarily xSt. In the case that $y \sim_R t$ holds we can easily exclude the following three possibilities: xRt, tRx and tSx with $t \sim_R x$. Moreover, it is impossible to have also tSx with $t \nsim_R x$ since, in this case, we should have $x \nsim_R y$ and so an element z, such that $z\overline{R}x$ and $z\overline{\not R}y$, should exist in X. The existence of such a z is forbidden because $tSx\overline{R}z$ would imply, thanks to the first part of this proof (namely axiom (**comp**) and the fact that $S \circ R^{-1} \subseteq R^{-1}$), $t\overline{R}z$ and so $ySt\overline{R}z$ which leads to $y\overline{R}z$ that is impossible. So, by properties (**tot**) and (**inters**), we have that, in the case $y \sim_R t$, it must be xSt as desired.

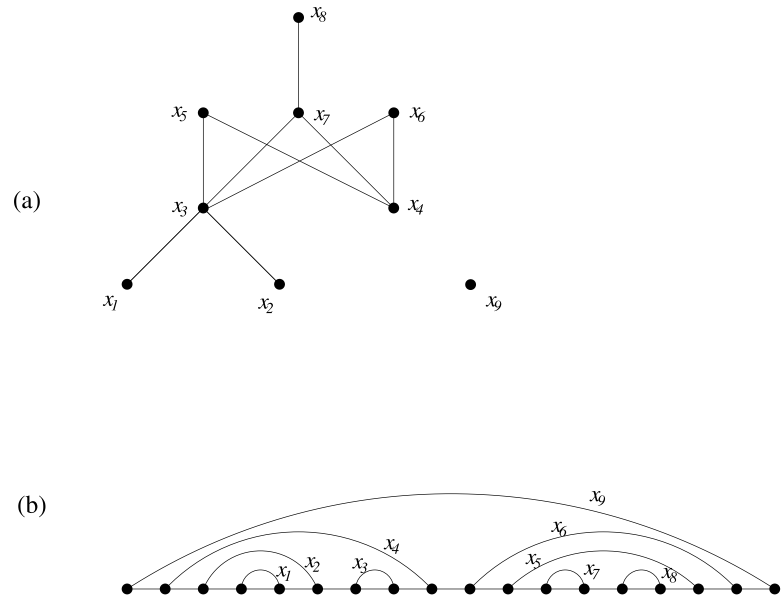


Fig. 6. A poset in $\mathbf{R}(9)$, together with the representation of the associated Catalan pair as a matching.

In the other case, that is $y \nsim_R t$, let q be such that $q \not\overline{R} t$ and $q\overline{R}y$. Again, thanks to the first part of this proof, from $xSy\overline{R}q$ it follows that $q\overline{R}x$. On the other hand, if we had $x\overline{R}t$, since it is $x\not\overline{R}y$ and $t\not\overline{R}y$, it would be tSy (by the definition of S), which is impossible since, by hypothesis, ySt, and we have just shown that the last two relations lead to a contradiction. Therefore we must have $x\not\overline{R}t$, which, together with $q\not\overline{R}t$ and $q\overline{R}x$, implies that xSt, as desired. $\square$

In order to clarify the construction of S given in the proof of Proposition 4.4, consider the poset $R \in \mathbf{R}(9)$ shown in Fig. 6(a). It is $x_1 \sim_R x_2$, hence $x_1 S x_2$. Similarly we get $x_5 S x_6$. Moreover, for any fixed $i = 1, \ldots, 8$, we have $x_9 \nsim_R x_i$, and there exists x_j, $j \neq i$, such that $x_i \overline{R} x_j$, so we have $x_i S x_9$. Similarly we have $x_1 S x_4$, $x_2 S x_4$, $x_3 S x_4$, $x_7 S x_5$, $x_7 S x_6$, $x_8 S x_5$, $x_8 S x_6$, and we finally obtain the Catalan pair (S, R) represented by the matching depicted in Fig. 6(b).

Remark. Observe that, as a byproduct of the last proposition, we have found a presumably new combinatorial interpretation of Catalan numbers: C_n counts nonisomorphic posets of cardinality n which are simultaneously $(\mathbf{2}+\mathbf{2})$-free and Z_4-free. Such posets are called *series parallel interval orders*, see for example [1,21].

Open problem 1. We have shown that (S, R) is a Catalan pair if and only if $[X, R]$ does not contain neither $\mathbf{2}+\mathbf{2}$ nor Z_4. The class of $(\mathbf{2}+\mathbf{2})$-free posets have been deeply studied, see for example [10] or the more recent paper [3]. What about the enumeration of Z_4-free posets? A couple of interesting papers are [11], where some problems related with the avoidance of the poset Z_4 are considered for families of subsets, and [25], in which a functional equation for the generating function of the sequence counting Z_4-free posets is provided, as well as an asymptotic estimate for its coefficients.

Open problem 2. Can we define some interesting (and natural) partial order relation on the set $\mathbf{R}(n)$? Maybe some of the combinatorial interpretations of Catalan pairs can help in this task.

5. Generalizations of Catalan pairs

In this section we see how a slight modification of the axioms defining Catalan pairs determines some further combinatorial structures and number sequences, mostly related with permutations. In particular, we focus our attention on axiom (**comp**). If we relax such a condition, we are able to represent some classes of permutations which, in general, include 312-avoiding ones.

Consider all pairs of relations (S,R) on a set X satisfying axioms (**ord S**), (**ord R**), (**tot**) and (**inters**). In this situation, we call (S,R) a *factorial pair* on X. The set of all factorial pairs on X will be denoted $\mathcal{F}(X)$. As we did for Catalan pairs, we work up to isomorphism, and $\mathcal{F}(n)$ will denote the isomorphism class of factorial relations on a set X of n elements.

It is clear that, for any set X, $\mathcal{C}(X) \subseteq \mathcal{F}(X)$. Moreover, using an obvious extension of the bijection given in Section 3.2, it turns out that $|\mathcal{F}(n)| = n!$. More precisely, as it was in the case of 312-avoiding permutations and Catalan pairs, every permutation $\pi \in S_n$ can be uniquely represented as a factorial pair (S,R) of size n where $S = Inv(\pi)$ (i.e. the set of inversions of π) and $R = NInv(\pi)$ (i.e. the set of noninversions of π).

Given a factorial pair (S,R), we call the associated permutation its *permutation encoding*.

Example. Let $\pi = 53\,124$, π is the permutation encoding of the factorial pair $(S,R) \in \mathcal{F}(5)$ such that $S = Inv(\pi) = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4)\}$ and $R = NInv(\pi) = \{(2,5),(3,4),(3,5),(4,5)\}$.

Now we come to the main point of the present section, and show how relaxing axiom (**comp**) naturally leads to a family of interesting combinatorial structures which, in some sense, interpolates between the analogous combinatorial interpretations of Catalan pairs and factorial pairs.

Denote by $\mathcal{F}_{h,k}(X)$ the class of all pairs of relations (S,R) on the set X satisfying axioms (**ord S**), (**ord R**), (**tot**), (**inters**), and such that (**comp**) is replaced by the weaker axiom:

$$S^h \circ R^k \subseteq R. \qquad (\mathbf{comp\,(h,k)})$$

The next proposition (whose easy proof is left to the reader) illustrates how the sets $\mathcal{F}_{h,k}(X)$ are related to Catalan and factorial pairs.

Proposition 5.1.

(i) $\mathcal{C}(X) = \mathcal{F}_{1,1}(X)$.
(ii) *For all h and k we have that $\mathcal{F}_{h,k}(X) \subseteq \mathcal{F}(X)$.*
(iii) *If $a \leqslant b$, then $\mathcal{F}_{a,k}(X) \subseteq \mathcal{F}_{b,k}(X)$ and $\mathcal{F}_{h,a}(X) \subseteq \mathcal{F}_{h,b}(X)$.*

Each element of the family $\{\mathcal{F}_{h,k}(X)\colon\ h,k \geqslant 1\}$, where X is finite, can be characterized in terms of permutations avoiding a set of patterns. Indeed, for any fixed h and k, axiom (**comp (h, k)**) means that, for any $x_1, x_2, \dots, x_{h-1}, x_h$ and $y_1, y_2, \dots, y_k, y_{k+1}$ in X, if $x_1 S x_2 S \dots S x_{h-1} S x_h S y_1 R y_2 R \dots R y_k R y_{k+1}$, then $x_1 R y_{k+1}$. Now consider all (nonisomorphic) factorial pairs on a set having $h+k+1$ elements which do not satisfy the above condition: the permutation encoding of $\mathcal{F}_{h,k}(X)$ is given by the set of permutations avoiding all permutation encodings of such factorial pairs.

For example, consider the two families $\mathcal{F}_{h,1}(X)$ and $\mathcal{F}_{1,k}(X)$. The following two propositions completely characterize them in terms of pattern avoiding permutations. In both propositions (as well as in the subsequent corollary) X denotes the set $\{1,\dots,n\}$.

Proposition 5.2. *The permutation encoding of $\mathcal{F}_{1,k}(X)$ is given by $S_n((k+2)12\cdots k(k+1))$.*

Proof. Axiom (**comp (1, k)**) means that, for any $x, y_1, \dots, y_{k+1} \in X$, if $x S y_1 R y_2 \dots y_k R y_{k+1}$, then $x R y_{k+1}$. In terms of permutation encoding, the previous condition fails precisely for those permutations π of X where $(y_1,y_2),\dots,(y_k,y_{k+1})$ are noninversions, and $(x,y_1),(x,y_{k+1})$ are inversions. This is equivalent to say that π contains the pattern $(k+2)12\cdots k(k+1)$. $\square$

The next proposition can be proved using a similar argument.

Proposition 5.3. *The permutation encoding of $\mathcal{F}_{h,1}(X)$ is given by $S_n(\pi_2, \pi_3, \dots, \pi_{h+1})$, where $\pi_i \in S_{h+2}$, for every $2 \leqslant i \leqslant h+1$, and π_i is obtained from $(h+2)(h+1)\cdots 21$ by moving i to the rightmost position.*

Corollary 5.1. *The cardinality of $\mathcal{F}_{2,1}(X)$ is given by the n-th Schröder number.*

Proof. From the previous proposition we get that the permutation encoding of $\mathcal{F}_{2,1}(X)$ is given by $S_n(4312, 4213)$. In [13] it is shown that the above set of pattern avoiding permutations (or, more precisely, the one obtained by reversing both patterns) is counted by Schröder numbers. $\square$

Open problem 3. The enumeration of the sets $\mathcal{F}_{h,k}(X)$ has to be almost completely carried out, except for some specific cases. For instance, concerning $\mathcal{F}_{3,1}(X)$, Proposition 5.3 states that its permutation encoding is given by $S_n(53214, 54213, 54312)$. The first terms of its counting sequence are $1, 2, 6, 24, 117, 652, 3988, \dots$, which are not in [23].

References

[1] J. Berman, P. Dwinger, Lattices with series-parallel and interval orders and a generalization of Catalan numbers, J. Combin. Math. Combin. Comput. 16 (1994) 75–85.
[2] M. Bona, Combinatorics of Permutations, Chapman & Hall, CRC, Boca Raton, FL, 2004.
[3] M. Bousquet-Mélou, A. Claesson, M. Dukes, S. Kitaev, $(2+2)$-free posets, ascent sequences and pattern avoiding permutations, J. Combin. Theory Ser. A, in press.
[4] A. Burstein, S. Elizalde, T. Mansour, Restricted Dumont permutations, Dyck paths, and noncrossing partitions, Discrete Math. 306 (2006) 2851–2869.
[5] A. Claesson, Generalized pattern avoidance, European J. Combin. 22 (2001) 961–971.
[6] A. Conflitti, On Whitney numbers of the order ideals of generalized fences and crowns, Discrete Math. 309 (2009) 615–621.
[7] F. Disanto, L. Ferrari, R. Pinzani, S. Rinaldi, Catalan pairs and relations, (preliminary version) available at http://front.math.ucdavis.edu/0901.3509.
[8] S. Elizalde, Multiple pattern-avoidance with respect to fixed points and excedances, Electron. J. Combin. 11 (2004) #R51, 40 pp.
[9] T. Fine, Extrapolation when very little is known about the source, Inform. and Control 16 (1970) 331–359.
[10] P.C. Fishburn, Intransitive indifference with unequal indifference intervals, J. Math. Psych. 7 (1970) 144–149.
[11] J.R. Griggs, G.O.H. Katona, No four subsets forming an N, J. Combin. Theory Ser. A 115 (2008) 677–685.
[12] O. Guibert, S. Pelat-Alloin, Extending Fine sequences: a link with forbidden patterns, at http://arxiv.org/abs/math/0507408v1.
[13] D. Kremer, Permutations with forbidden subsequences and a generalized Schroeder number, Discrete Math. 218 (2000) 121–130.
[14] J.P.S. Kung, G.-C. Rota, C.H. Yan, Combinatorics: The Rota Way, Cambridge University Press, Cambridge, 2009.
[15] C.L. Mallows, L. Shapiro, Balls on the lawn, J. Integer Seq. 2 (1999), article 99.1.5.
[16] T. Mansour, S. Severini, Enumeration of $(k,2)$-noncrossing partitions, Discrete Math. 308 (2008) 4570–4577.
[17] R.J. Marsh, P. Martin, Pascal arrays: counting Catalan sets, available at http://arxiv.org/abs/math/0612572.
[18] J.W. Moon, Some enumeration problems for similarity relations, Discrete Math. 26 (1979) 251–260.
[19] E. Munarini, N. Zagaglia Salvi, On the rank polynomial of the lattice of order ideals of fences and crowns, Discrete Math. 259 (2002) 163–177.
[20] H. Prodinger, A correspondence between ordered trees and noncrossing partitions, Discrete Math. 46 (1983) 205–206.
[21] M.S. Rhee, J.G. Lee, A characterization theorem for families, J. Korean Math. Soc. 32 (1995) 1–5.
[22] L.W. Shapiro, A Catalan triangle, Discrete Math. 14 (1976) 83–90.
[23] N.J.A. Sloane, The on-line encyclopedia of integer sequences, at http://www.research.att.com/~njas/sequences/index.html.
[24] R.P. Stanley, Catalan addendum, available at http://www-math.mit.edu/~rstan/ec/catadd.pdf.
[25] R.P. Stanley, Enumeration of posets generated by disjoint unions and ordinal sums, Proc. Amer. Math. Soc. 45 (1974) 295–299.
[26] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999.