Self-dual vector spaces

Pierre-Yves Gaillard[1]

Let K be a field, V a K-vector space, $\mathrm{GL}(V)$ the automorphism group of V, and $V^* := \mathrm{Hom}_K(V,K)$ the dual of V, equipped with its natural $\mathrm{GL}(V)$-module structure. Say that V is **self-dual** if there is a $\mathrm{GL}(V)$-module isomorphism $V \simeq V^*$.

The purpose of this text is to answer the question: which vector spaces are self-dual?

A necessary condition is that V be finite dimensional (this follows from the Erdős–Kaplansky Theorem), and a sufficient condition is that V be the zero vector space. So, from now on we assume that V is nonzero and finite dimensional. We denote the dimension of V by n (note $n \geq 1$), and the cardinality of K by κ.

Theorem. *The vector space V is self-dual if and only if $\kappa+n \leq 4$. Moreover* $\dim \mathrm{Hom}_{\mathrm{GL}(V)}(V,V^*)$ *depends only on κ and n. Denoting this integer by $d(\kappa,n)$, we have $d(\kappa,n)=1$ if $\kappa+n \leq 4$, and $d(\kappa,n)=0$ otherwise. (Recall that we assume $V \neq 0$.)*

Loosely speaking: the only nonzero self-dual vector spaces are $\mathbb{F}_2, \mathbb{F}_3$ and $(\mathbb{F}_2)^2$. The unique $\mathrm{GL}((\mathbb{F}_2)^2)$-isomorphism $\phi : (\mathbb{F}_2)^2 \to ((\mathbb{F}_2)^2)^*$ is given by

$$\left(\phi \begin{pmatrix} a \\ c \end{pmatrix}\right) \begin{pmatrix} b \\ d \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad+bc.$$

Let us add for completeness sake that, for κ equal to 2 or 3, a $\mathrm{GL}(\mathbb{F}_\kappa)$-isomorphism $\psi : \mathbb{F}_\kappa \to (\mathbb{F}_\kappa)^*$ is given by $(\psi(x))(y) = xy$.

Note that $\mathrm{Hom}_{\mathrm{GL}(V)}(V,V^*)$ is isomorphic to the space $B(V)$ of $\mathrm{GL}(V)$-invariant bilinear forms on V.

To prove the theorem, it suffices to verify the statement about $d(\kappa,n)$. The case $n=1$ is left to the reader as an easy exercise. So we suppose from now on $n \geq 2$. Suppose there is a nonzero f in $B(V)$. We must show $\kappa = n = 2$ and $f = \det$.

Let $e_1,\dots,e_n$ be a basis of V, and set $a := f(e_1,e_1), b := f(e_1,e_2)$. By invariance we have $f(v,v) = a$ for all $v \neq 0$, and $f(v,w) = b$ for all linearly independent v and w; in particular $(a,b) \neq (0,0)$. We have

$$b = f(e_1, e_1+e_2) = f(e_1,e_1) + f(e_1,e_2) = a+b \implies a = 0 \neq b.$$

For $0 \neq \lambda \in K$ we have $b = f(e_1, \lambda e_2) = \lambda b$, hence $\lambda = 1$, hence $\kappa = 2$, hence $b = 1$, hence $f = \det$ if $n = 2$. If $n \geq 3$ we have

$$1 = f(e_1, e_2+e_3) = f(e_1,e_2) + f(e_1,e_3) = 1+1 = 0,$$

contradiction. This completes the proof.

[1]ORCID https://orcid.org/0000-0002-7960-1698

selfdual-bf, 2024/12/08, 13:56:12 -05'00', 1733671098