

Radiative Equilibrium in the Outer Layers of a Star: the Temperature Distribution and the Law of Darkening. By E. A. Milne, B.A., Fellow of Trinity College, Cambridge.

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§ 1. *Object of the Investigation.*

A mass of material is said to be in radiative equilibrium when the temperature at each point is steady and when the only agency effecting the transfer of heat is thermal radiation. Heat may be being generated internally by the liberation of energy, as in the interior of a star; or there may be no generation of heat, in which case the amount of radiant energy emitted by any portion is equal to that absorbed. In either case there is no accumulation of heat, the steady state being preserved by the existence of net currents of radiation in different directions; and the amounts of these currents have been shown by Eddington* and Jeans† to depend on the gradients of energy-density in different directions.

The important case in practice is that in which the medium has a plane boundary and the surfaces of equal temperature are parallel planes. (For a star the effect of curvature is usually negligible.) When this is the case, and when there is no internal generation of energy, the problem of the distribution of temperature is a perfectly definite one; there is at each point a net flow of energy perpendicular to the boundary and the same for all points. In the interior at great distances from the boundary the temperature distribution is quite a simple one, and the complete solution has been given by Eddington and Jeans; but near the boundary the question is more difficult, since the distribution of radiation must satisfy the condition that at the boundary the entrant radiation is zero in all directions. The general features of radiative equilibrium near the boundary of a star were first considered by Schwarzschild, in a classical paper,‡ and applied to the case of the sun. Schwarzschild showed that if T_1 is the effective temperature, deduced from the total radiation from the whole disc with the aid of Stefan's constant, then the temperature at the boundary tends to a definite limit which is approximately $T_1/\sqrt{2}$ ($=T_1/1.414$); and he deduced from the temperature gradients an approximation to the law of darkening of the disc towards the limb. Schwarzschild, in evaluating the temperature distribution, ignored the dependence of the radiation on direction, and considered the radiation as consisting simply of an inward and outward stream; Jeans, making a further approximation, arrived at a different boundary temperature ($=T_1/1.278$) but the

* *M.N., R.A.S.*, 77, 16 (1916); 77, 598 (1916); 79, 22 (1918).

† *M.N., R.A.S.*, 78, 28 (1917).

‡ "Über das Gleichgewicht der Sonnenatmosphäre," *Gott. Nach.*, 1906, p. 41.

same law of darkening. Schwarzschild, however, had already* made a very complete investigation of the analogous problem for scattering, and we shall show (§ 7) that his solution can be applied to the absorption case.

Since the light emitted by a star is radiation derived almost entirely from the outer layers, the spectral distribution of this light is governed by the distribution of temperature near the boundary, and the theoretical distribution for radiative equilibrium is of some importance. The present paper shows the relation of the earlier solutions to one another, and carries them to a higher approximation; and it obtains a revised formula for the darkening towards the limb.

§ 2. *The Fundamental Equations.*

Consider a medium stratified in planes perpendicular to the axis of x , extending indefinitely on the positive side of its boundary $x=0$. Let I be the intensity of radiation of all wave-lengths at any point in a direction making an angle θ with the *negative* direction of the axis of x ; I is a function of x and θ which will sometimes be written $I(x, \theta)$. Let ρ , T be the density and temperature at any point; and let k be the coefficient of mass-absorption, supposed independent of wave-length. Lastly, let B (a function of x) be the intensity of black-body radiation corresponding to the temperature of the matter at any point x . By Kirchoff's law the emission per unit volume in all directions at any point x is $4\pi k\rho B$. By considering in the usual way the gains and losses of a narrow pencil of radiation during a short stretch of its paths, it is found that I must satisfy the equation

$$\cos \theta \frac{dI}{dx} = k\rho(I - B). \quad (1)$$

Setting

$$\tau = \int_0^x k\rho dx,$$

equation (1) becomes

$$\cos \theta \frac{dI}{d\tau} = I - B \quad (2)$$

The rate of absorption of energy per unit volume is

$$k\rho \int I d\omega;$$

equating this to the emission, we have as the condition for radiative equilibrium

$$2B = \int_0^\pi I \sin \theta d\theta \quad (3)$$

* "Über Diffusion in der Sonnenatmosphäre," *Berlin Sitz.*, 1914, p. 1183.

Multiplying (2) by $d\omega$ and integrating over the complete solid angle, we have on using (3)

$$\frac{d}{d\tau} \iint I \cos \theta d\omega = 0. \quad (4)$$

But $\iint I \cos \theta d\omega$ is the net flow of radiation per unit area crossing from the positive side to the negative side of a plane perpendicular to Ox ; and by (4) this is constant. Let us denote by πF the value of this constant net flow which characterises any given state of radiative equilibrium. Then we have

$$F = 2 \int_0^\pi I \cos \theta \sin \theta d\theta \quad (5)$$

The problem is to obtain the solution of (1) subject to (3) or its equivalent (5), and subject to the boundary condition of incident radiation zero,

$$I(0, \theta) = 0, \quad (\frac{1}{2}\pi < \theta \leq \pi) \quad (6)$$

§ 3. *Schwarzschild's and Jeans' Solutions.*

Schwarzschild's early method is equivalent to the following. When $\frac{1}{2}\pi < \theta \leq \pi$, put $\psi = \pi - \theta$, $I(x, \theta) = I'(x, \psi)$, and retain the symbol I for $0 \leq \theta < \frac{1}{2}\pi$. Now make the assumption that it will be sufficient, in finding the temperature distribution, to suppose that I and I' are independent of direction. In this case, multiplying (2) by $d\omega$ and integrating over the regions $0 \leq \theta < \frac{1}{2}\pi$, $0 \leq \psi < \frac{1}{2}\pi$ in turn, we find

$$\frac{1}{2} \frac{dI}{d\tau} = I - B, \quad \frac{1}{2} \frac{dI'}{d\tau} = B - I' \quad (7)$$

and from (3) and (5)

$$2B = I + I', \quad F = I - I' \quad (8)$$

It is seen that equations (7) are the same as we should have found if we had assumed the radiation to be confined to an inward and outward stream parallel to Ox with a coefficient of absorption twice the true value; it is often sufficient, in fact, to use these equations of linear flow for rough investigations of the effects of scattering, selective absorption,* etc. From (7) and (8) with the boundary condition $I_0' = 0$ it is found that

$$I = F(1 + \tau), \quad I' = F\tau, \quad B = \frac{1}{2}F(1 + 2\tau) \quad (9)$$

If T_0 is the boundary temperature, T_1 the effective temperature deduced from the total radiation, then

$$T_1^4/T_0^4 = \pi F/\pi B_0 = 2 \quad (10)$$

* *E.g.* Schuster, *Astrophys. Jour.*, 21, 1, 1905.

The temperature distribution thus found may now be introduced into the exact equation (2). Integrating this equation and putting $\tau = 0$, it is found that

$$I(0, \theta) = \frac{1}{2}F(1 + 2 \cos \theta) \quad . \quad . \quad . \quad (11)$$

This gives the law of darkening of the apparent disc, as θ varies from 0 at the centre to $\frac{1}{2}\pi$ at the limb; it corresponds to a coefficient of darkening of $\frac{2}{3}$. But this law of darkening as found by Schwarzschild is inconsistent with the fundamental property of a net flux of πF . Instead, it gives for the total radiation escaping at the surface

$$2\pi \int_0^{\frac{1}{2}\pi} I(0, \theta) \cos \theta \sin \theta d\theta = \pi F \left(\frac{1}{2} + \frac{2}{3} \right),$$

or $\frac{7}{6}$ of its required amount.

The same applies to Jeans' solution. He obtains a next approximation by assuming that for the determination of the temperature distribution I may be taken to be a linear function* of $\cos \theta$, instead of Schwarzschild's assumption. It is then found that the solution analogous to (9) is

$$I = B + \frac{3}{4}F \cos \theta, \quad B = a + \frac{3}{4}F\tau \quad . \quad . \quad . \quad (12)$$

where a is a constant. This is, in fact, the solution appropriate to the interior, as found by Eddington and Jeans. From somewhat complicated considerations Jeans adopts for a the value $\frac{3}{8}F$ (in our notation). Inserting now the value

$$B = \frac{3}{8}F(1 + 2\tau) \quad . \quad . \quad . \quad (13)$$

in (2) and integrating, we have

$$I(0, \theta) = \frac{3}{8}F(1 + 2 \cos \theta) \quad . \quad . \quad . \quad (14)$$

Thus the temperature distribution and the law of darkening are identical with Schwarzschild's save as to a numerical factor, but the total escaping radiation is now

$$2\pi \int_0^{\frac{1}{2}\pi} I(0, \theta) \cos \theta \sin \theta d\theta = \frac{7}{8}\pi F,$$

instead of πF . The approximation is thus only a slightly better one. The boundary temperature is now given by

$$T_1^4/T_0^4 = \pi F/\pi B_0 = \frac{8}{3} \quad . \quad . \quad . \quad (15)$$

We shall now investigate closer approximations, and in particular approximations (which it is very convenient to have) which give the correct flux at the boundary.

* *I.e.*, the first two terms of an expansion in spherical harmonics.

§ 4. *An Integral Equation for the Temperature Distribution.*

Solving equation (2) for I in terms of B, we have

$$I = -e^{\tau \sec \theta} \int_{\tau}^{\infty} B \sec \theta e^{-\tau \sec \theta} d\tau \quad . \quad . \quad (16)$$

As before, write $I'(\tau, \psi)$ for $I(\tau, \theta)$ when $\frac{1}{2}\pi < \theta \leq \pi$, where $\psi = \pi - \theta$. On adjusting the constant of integration in (16) so that $I'(0, \psi) = 0$ for all ψ , and so that $I/I' \rightarrow 1$ as $\tau \rightarrow \infty$, we find

$$I(\tau, \theta) = e^{\tau \sec \theta} \int_{\tau}^{\infty} B \sec \theta e^{-\tau \sec \theta} d\tau \quad . \quad . \quad (17)$$

$$I'(\tau, \psi) = e^{-\tau \sec \psi} \int_0^{\tau} B \sec \psi e^{\tau \sec \psi} d\tau \quad . \quad . \quad (18)$$

Inserting these in the equation of radiative equilibrium (3) we have

$$2B(\tau) = \int_0^{\frac{1}{2}\pi} e^{\tau \sec \theta} \sin \theta d\theta \int_{\tau}^{\infty} B(t) \sec \theta e^{-t \sec \theta} dt \\ + \int_0^{\frac{1}{2}\pi} e^{-\tau \sec \psi} \sin \psi d\psi \int_0^{\tau} B(t) e^{t \sec \psi} dt \quad . \quad (19)$$

Write $t = \tau + y \cos \theta$ in the first integral, $t = \tau - y \cos \theta$ in the second, and put $\cos \theta = \mu$. Then

$$2B(\tau) = \int_0^1 d\mu \int_0^{\infty} B(\tau + y\mu) e^{-y} dy + \int_0^1 d\mu \int_0^{\tau/\mu} B(\tau - y\mu) e^{-y} dy.$$

We now change the orders of integration. The second integral, which is the double integral of $B(\tau - y\mu) e^{-y}$ over the shaded area

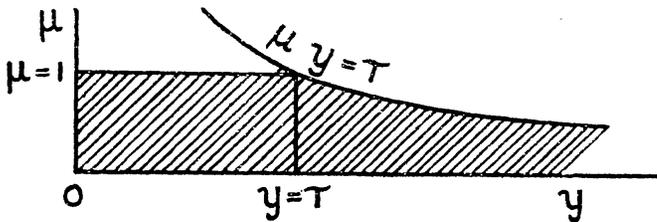


FIG. 1.

in fig. 1, must be split up into an integral over a rectangle and one over a curvilinear area. We have altogether .

$$2B(\tau) = \int_0^{\infty} e^{-y} dy \int_0^1 B(\tau + y\mu) d\mu + \int_0^{\tau} e^{-y} dy \int_0^1 B(\tau - y\mu) d\mu \\ + \int_{\tau}^{\infty} e^{-y} dy \int_0^{\tau/y} B(\tau - y\mu) d\mu \quad . \quad (20)$$

Now put

$$\int_{\tau}^{\infty} B(\tau) d\tau = C(\tau), \quad B(\tau) = C'(\tau).$$

Performing the integrations in (20) with respect to μ , we have

$$2C'(\tau) = \int_0^\infty \frac{C(\tau+y) - C(\tau)}{y} e^{-y} dy - \int_0^\tau \frac{C(\tau-y) - C(\tau)}{y} e^{-y} dy + C(\tau) \int_\tau^\infty \frac{e^{-y}}{y} dy,$$

or

$$C'(\tau) = \int_0^\tau \frac{C(\tau+y) - C(\tau-y)}{2y} e^{-y} dy + \int_\tau^\infty \frac{C(\tau+y)}{2y} e^{-y} dy \quad (21)$$

This gives the temperature distribution as the solution of a certain integral equation; this equation holds in the interior and right up to the boundary. It appears to be of simpler form than equations hitherto given.

It is interesting to consider for a moment the limiting form of this equation for large values of τ . It is seen to be

$$C'(\tau) = \int_0^\infty \frac{C(\tau+y) - C(\tau-y)}{2y} e^{-y} dy \quad (22)$$

A solution of this, as is readily verified, is given by

$$C(\tau) = a\tau + b\tau^2, \quad B(\bullet) = C'(\tau) = a + 2b\tau \quad (23)$$

where a, b are arbitrary constants. The corresponding solutions for I and I' , for τ large, are found to be

$$I(\tau, \theta) = B(\tau) + 2b \cos \theta, \quad I'(\tau, \psi) = B(\tau) - 2b \cos \psi \quad (24)$$

and insertion in the flux equation (5) gives

$$b = \frac{3}{8}F \quad (25)$$

We have again the solution for the interior given by Eddington and Jeans. It may be observed that the method they use, that of spherical harmonics, would be equivalent in the present analysis to expanding $C(\tau+y)$ and $C(\tau-y)$ in (22), formally, in Taylor's series, and integrating term by term. Thus

$$C'(\tau) = \int_0^\infty \left\{ yC'(\tau) + \frac{y^3}{3!}C'''(\tau) + \dots \right\} \frac{e^{-y}}{y} dy,$$

or

$$0 = \frac{1}{3}B''(\tau) + \frac{1}{5}B^{(4)}(\tau) + \frac{1}{7}B^{(6)}(\tau) + \dots \quad (25a)$$

a linear equation of infinite order with constant coefficients. Formal methods of solving this do not lead to a solution other than (23); and it appears probable,* though I have not succeeded in constructing a rigorous proof, that (23) is the general solution

* Note added April 2.—Mr. R. H. Fowler has kindly drawn my attention to a recent paper by Schürer, *Leipzig Berichte*, 70, ii, 185 (1918), from one of the theorems in which it follows that (23) is the only solution of equation (25a).

of (22). In that case there are no higher harmonics in I than the first.

Returning to the general equation (21), we notice that it defines the nature of the singularity at the origin. The value of the gradient $B'(\tau)$ must be logarithmically infinite as $\tau \rightarrow 0$; for, differentiating (21) with regard to τ and putting $\tau = 0$, we have

$$B'(0) = \int_0^{\infty} \frac{B(y)}{y} e^{-y} dy \quad . \quad . \quad . \quad (26)$$

and $B(0)$ is not zero.

§ 5. *Solutions by Successive Approximation.*

Equation (21) is suitable for solution by successive approximation; taking an approximate temperature distribution to start with, we can insert the corresponding $C(\tau)$ in the right-hand side of (21), and the result of the integration will be a new approximation for $C'(\tau)$ or $B(\tau)$. The accuracy can be checked by testing the constancy of flux, *e.g.* by comparing the flux at the boundary with the flux as $\tau \rightarrow \infty$; or by testing for radiative equilibrium, of which (21) is the direct expression. The process can be repeated if necessary. We can, however, go further. If we adopt as a first approximation the solution for the interior,

$$B(\tau) = C'(\tau) = a + 2b\tau \quad . \quad . \quad . \quad (23)$$

the second approximation will contain the constants a and b ; b is given by (25), in order that the flux may be correct in the interior, but a is still arbitrary, the boundary temperature being unknown. We can therefore choose a so that in this second approximation the total flux at the boundary is the required amount πF , and we can hope that the solution giving the correct flux for $\tau = 0$ and for $\tau \rightarrow \infty$ will be a good approximation throughout. Alternatively, we could choose a so that the condition of radiative equilibrium is satisfied at the boundary, which would be the same thing as choosing a so that the third approximation gave the same boundary temperature as the second. One would expect, however, that the former approximation was the better, the constancy of flux being so characteristic of radiative equilibrium, whilst mere consistency between successive approximations to the boundary temperature might be expected to be of secondary importance. The device of choosing a is perhaps arbitrary to this extent, that if a long series of successive approximations are carried out starting from solution (23), the limiting approximation is independent of a ; but it is convenient to adopt it.

Inserting (23) in the left-hand side of (21) and integrating, we find as a second approximation

$$B(\tau) = C'(\tau) = a + 2b\tau + \frac{1}{2}e^{-\tau}(b - a - b\tau) + \frac{1}{2}(a\tau + b\tau^2) \int_{\tau}^{\infty} \frac{e^{-y}}{y} dy \quad . \quad (27)$$

This has the same form as (23) for τ large, and exhibits the logarithmic singularity in $B'(\tau)$ for $\tau=0$. For the radiation escaping at the boundary we have from (17) on putting $\cos \theta = \mu$,

$$\begin{aligned} I(0, \theta) &= \int_0^\infty B(\tau) \mu^{-1} e^{-\tau/\mu} d\tau \\ &= (a + 2b\mu) + \frac{1}{2\mu} \int_0^\infty (b - a - b\tau) e^{-\tau(1+1/\mu)} d\tau \\ &\quad + \frac{1}{2\mu} \int_0^\infty (a\tau + b\tau^2) e^{-\tau/\mu} d\tau \int_\tau^\infty \frac{e^{-y}}{y} dy. \end{aligned}$$

The last term is easily evaluated on setting $y = \tau z$ and changing the order of integration, and we find finally

$$I(0, \theta) = \frac{1}{2}(a + b) + b\mu + \mu \left(\frac{1}{2}a + b\mu \right) \log \frac{\mu + 1}{\mu} \quad (28)$$

The corresponding flux at the boundary is found to be

$$2\pi \int_0^{\frac{1}{2}\pi} I(0, \theta) \sin \theta \cos \theta d\theta = \pi \left[a \left(\frac{1}{3} + \frac{2}{3} \log 2 \right) + \frac{1}{2} b \right] \quad (29)$$

and this should be equal to πF or $\frac{8}{3}\pi b$. Again, we find

$$\frac{1}{2} \int_0^{\pi/2} I(0, \theta) \sin \theta d\theta = \frac{3}{8}a + b \left(\frac{5}{12} + \frac{1}{3} \log 2 \right) \quad (30)$$

and by the condition of radiative equilibrium (3) this should be equal to $B(0)$ or $\frac{1}{2}(a + b)$.

First let us select a so that the flux condition at the boundary is satisfied. This gives

$$\begin{aligned} a \left(\frac{1}{3} + \frac{2}{3} \log 2 \right) + \frac{1}{2} b &= \frac{8}{3} b, \\ \frac{a}{b} &= \frac{13}{4 + 8 \log 2} = 1.362 \quad (31) \end{aligned}$$

With this value $B(0) = \frac{1}{2}(a + b) = 1.181b$, whilst the right-hand side of (30) is $1.158b$, an error of 2.0 per cent. The boundary temperature is given by

$$T_1^4/T_0^4 = F/B(0) = \frac{8}{3}b/1.181b = 2.258, \quad T_1/T_0 = 1.226 \quad (32)$$

Alternatively, if we select a so that the radiative equilibrium condition is satisfied at the boundary, we have

$$\begin{aligned} \frac{3}{8}a + b \left(\frac{5}{12} + \log 2 \right) &= \frac{1}{2}(a + b), \\ \frac{a}{b} &= \frac{8 \log 2 - 2}{3} = 1.182 \quad (33) \end{aligned}$$

With this value the boundary flux is $2.524\pi b$ or $.946\pi F$ instead of πF , an error of 5.4 per cent. The boundary temperature is given by

$$T_1^4/T_0^4 = \frac{8}{3}b/1.091b = 2.444, \quad T_1/T_0 = 1.254 \quad (34)$$

For completeness we will now obtain briefly the corresponding *first* approximations. In this case the temperature distribution is taken to be

$$B(\tau) = a + 2b\tau \quad . \quad . \quad . \quad (35)$$

and the corresponding escaping radiation is given by

$$I(o, \theta) = a + 2b \cos \theta \quad . \quad . \quad . \quad (36)$$

The corresponding flux at the boundary is

$$2\pi \int_0^{\frac{1}{2}\pi} I(o, \theta) \sin \theta \cos \theta d\theta = \pi(a + \frac{4}{3}b) \quad . \quad . \quad (37)$$

which should be equal to πF or $\frac{8}{3}\pi b$; and by the condition of radiative equilibrium at the boundary the expression

$$\frac{1}{2} \int_0^{\frac{1}{2}\pi} I(o, \theta) \sin \theta d\theta = \frac{1}{2}(a + b) \quad . \quad . \quad (38)$$

should be equal to $B(o)$ or a . If we select a so that the flux condition is satisfied, we find $a/b = \frac{4}{3}$, which makes the value of (38) $\frac{7}{8}b$ instead of $\frac{4}{3}b$, an error of 12.5 per cent.; and the boundary temperature is given by

$$T_1^4/T_0^4 = 2, \quad T_1/T_0 = 1.189. \quad . \quad . \quad (39)$$

If we select a so that the radiative equilibrium condition is satisfied, we find $a/b = 1$, which gives a flux of $\frac{7}{8}\pi F$ instead of πF , an error of again 12.5 per cent.; and the boundary temperature is given by

$$T_1^4/T_0^4 = \frac{8}{3}, \quad T_1/T_0 = 1.278 \quad . \quad . \quad (40)$$

This latter is identical with Jeans' approximation.

It appears that the second approximations are distinctly more accurate than the first, and that of them the former (adjusted to give the right boundary flux) is the better. It must be remembered that the tests of accuracy we have been applying are definite tests of convergency to a limit; the right-hand side of (30), for example, is precisely the value of $B(o)$ that would be obtained from a *third* approximation, so that with $a/b = 1.181$ a third approximation would give

$$T_1^4/T_0^4 = \frac{8}{3}b/1.158b = 2.303, \quad T_1/T_0 = 1.232.$$

It would appear that, as regards the first approximations, there is nothing to choose between Jeans' approximation ($a/b = 1$) and its analogue ($a/b = \frac{4}{3}$); however, the latter is much closer to the asymptote to the better of our second approximations

($a/b = 1.362$), and, as we shall see later, it gives a better law of darkening.

The function occurring in (27) is tabulated* in Table I. for the two values $a/b = 1.362$, $a/b = 1.182$, b being put equal to unity; the graph in the case $a/b = 1.362$ is shown in fig. 2, with the asymptote $B(\tau) = a + 2b\tau$ also. The curve has a vertical

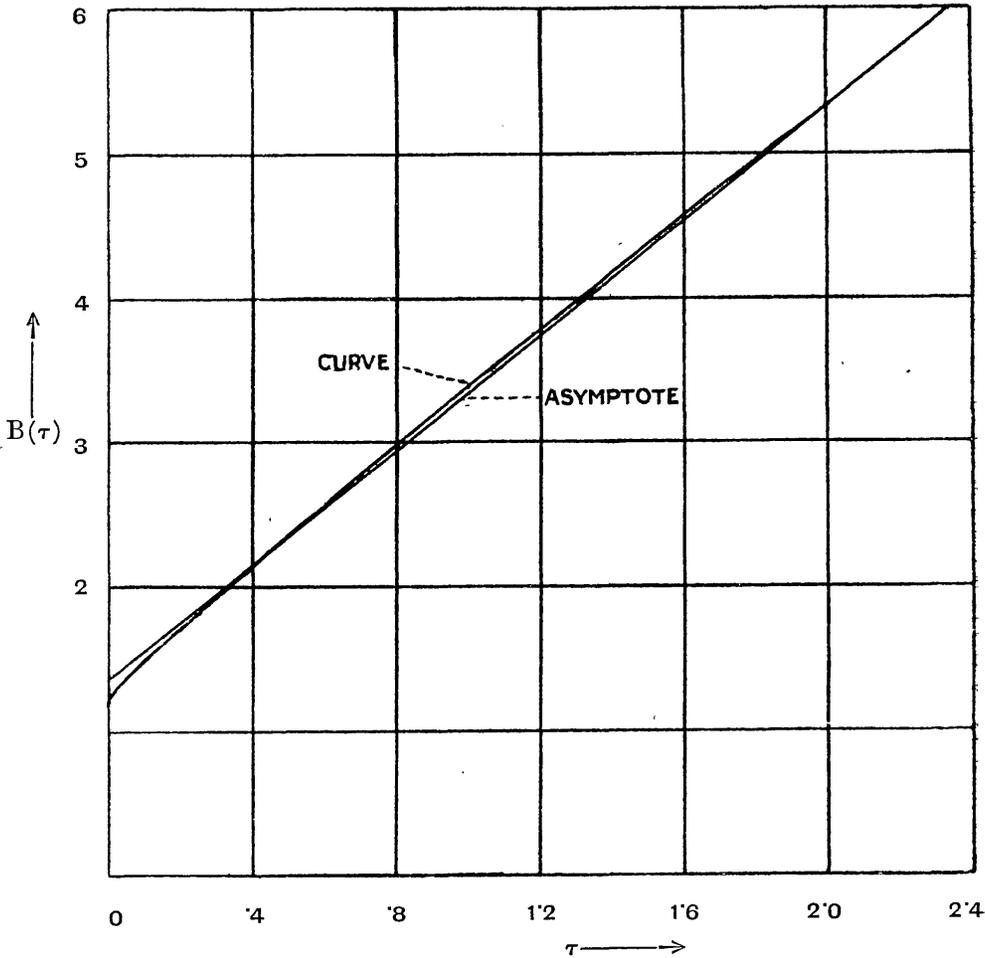


FIG. 2.—Energy-density Distribution (T^4) in Radiative Equilibrium.

tangent at $\tau = 0$, and crosses its asymptote near $\tau = 0.5$; beyond $\tau = 0.2$ it departs very little from its asymptote. We may assume that this curve represents the actual distribution of energy-density $B(\tau)$ in radiative equilibrium with fair accuracy. (The column (η) will be explained in § 7.)

* The exponential integral function $Ei(x) = \int_x^\infty \frac{e^{-x}}{x} dx$ is tabulated by Glaisher, *Phil. Trans.*, 160, 367 (1870), and by Jahnke and Emde, *Funktionentafeln* (Leipzig, 1909, p. 21). What we have denoted by $Ei(x)$ these writers call $-Ei(-x)$.

TABLE I.

Second Approximations to Energy-density Distribution.

r .	$a=1.362, b=1. (\delta).$		$a=1.182, b=1. (\epsilon).$		$(\eta).$
	$R(r).$	$a+2br.$	$R(r).$	$a+2br.$	
0.0	1.181	1.362	1.091	1.182	1.15
0.02	1.261	1.402	1.163	1.222	...
0.05	1.353	1.462	1.248	1.282	...
0.1	1.486	1.562	1.371	1.382	...
0.2	1.701	1.762	1.573	1.582	1.69
0.4	2.154	2.162	2.009	1.982	2.16
0.6	2.566	2.562	2.413	2.382	2.59
0.8	2.969	2.962	2.808	2.782	2.99
1.0	3.370	3.362	3.204	3.182	3.42
1.2	3.771	3.762	4.600	3.582	3.83
1.4	4.169	4.162	3.997	3.982	4.25
1.6	4.567	4.562	3.394	4.382	4.65
2.0	5.366	5.362	5.190	5.182	5.47
3.0	7.364	7.362	7.184	7.182	7.53
4.0	9.362	9.362	9.182	9.182	9.61

§ 6. *The Law of Darkening.*

Each of the approximations to the temperature distribution provides a definite law of darkening over the disc. If q denotes the ratio of the intensity of radiation escaping in direction θ to that escaping normally ($\theta=0$), we have for Schwarzschild's approximation, from (11)

$$q = \frac{1 + 2 \cos \theta}{3}; \quad (41)$$

for the analogue of Jeans' approximation ((36) with $a/b = \frac{4}{3}$) we have

$$q = \frac{2 + 3 \cos \theta}{5}, \quad (42)$$

and for Jeans' approximation itself ((36) with $a/b = 1$)

$$q = \frac{1 + 2 \cos \theta}{3}, \quad (43)$$

the same as Schwarzschild's; and for our second approximations we have from (28)

$$q = \frac{\frac{1}{2}(a+b) + b \cos \theta + \cos \theta (\frac{1}{2}a + b \cos \theta) \log \{(1 + \cos \theta) / \cos \theta\}}{\frac{1}{2}a + \frac{3}{2}b + (\frac{1}{2}a + b) \log 2} \quad (44)$$

with $a/b = 1.362, 1.182$ in turn. Let us denote these five laws of darkening by $(\alpha), (\beta), (\gamma), (\delta), (\epsilon)$; they are shown tabulated in

Table II. The coefficient of darkening u being defined by the equation

$$q = 1 - u + u \cos \theta \quad (45)$$

it is clear that (α) , (β) , (γ) correspond to coefficients of darkening of $\frac{2}{3}$, $\frac{3}{5}$, $\frac{2}{3}$ respectively. The equivalent *mean* coefficients of darkening \bar{u} for (δ) and (ϵ) are tabulated also; this is defined as that giving the same ratio of average intensity over the disc to intensity at the centre as the actual law, and is given by

$$\int_0^{\frac{1}{2}\pi} \frac{I(\circ, \theta)}{I(\circ, \circ)} \cos \theta \sin \theta d\theta = \int_0^{\frac{1}{2}\pi} (1 - \bar{u} + \bar{u} \cos \theta) \cos \theta \sin \theta d\theta = \frac{1}{2}(1 - \frac{1}{3}\bar{u}).$$

TABLE II.

Darkening towards the Limit.

Cos θ .	Sin θ .	(α).	(β).	(γ).	(δ).	(ϵ).	(η).
1·0	0	1·000	1·000	1·000	1·000	1·000	1·00
0·9	0·436	·933	·940	·933	·940	·938	...
0·8	0·600	·867	·880	·867	·879	·875	·875
0·6	0·800	·733	·760	·733	·758	·750	·755
0·4	0·915	·600	·640	·600	·635	·623	·625
0·2	0·978	·467	·520	·467	·507	·494	·495
0·1	0·995	·400	·460	·400	·439	·418	...
0·0	1·000	·333	·400	·333	·353	·342	·34
\bar{u}		·667	·600	·667	·609	·628	·627

The column headed sin θ gives the fraction of the radius of the disc measured from the centre.

Averaged over the disc, it is seen that the simple coefficient of darkening $\frac{2}{3}$ is much better than the Schwarzschild-Jeans value $\frac{2}{3}$; the true fall of intensity, which may be taken to be something between columns (δ) and (ϵ) , lies between $u = \frac{3}{5}$ and $u = \frac{2}{3}$; it agrees closely with $u = \frac{3}{5}$ up to about 98 per cent. of the radius, thereafter falling rapidly to a value little different from that given by $u = \frac{2}{3}$; this sudden fall close to the limb corresponds to the kink on the temperature curve near $\tau = 0$.

The observed intensity-distribution over the solar disc corresponds much better to $u = \frac{3}{5}$ than to $u = \frac{2}{3}$. Table III. gives the average of the determinations by Secchi, Vogel, Langley, and

TABLE III.

Sin θ .	0·0.	0·2.	0·4.	0·6.	0·7.	0·8.	0·9.	0·96.	0·98.	1·00.
Observed	1·00	·99	·97	·92	·87	·81	·70	·59	·49	(·40)
$u = \frac{2}{3}$	1·000	·987	·949	·880	·828	·760	·662	·568	·519	·400
$u = \frac{3}{5}$	1·000	·986	·943	·867	·809	·733	·624	·520	·466	·333

Frost.* The more recent results of Abbot, Fowle, and Aldrich † give the distribution of intensity for various wave-lengths separately; a few of the corresponding figures for integrated light are given by Abbot in his book *The Sun* (1912), p. 110, but it is convenient to avail ourselves of Lindblad's ‡ integrations of Abbot's observations; these are quoted in Table IV. (Abbot's observations, it

TABLE IV.

Sin θ .	0°00.	0°40.	0°55.	0°65.	0°75.	0°825.	0°875.	0°92.	0°95.
Observed	1·00	0·95	0·91	0·87	0·82	0·78	0·71	0·66	0·60
$u = \frac{2}{3}$	1·000	·949	·901	·856	·797	·739	·690	·635	·587
$u = \frac{2}{3}$	1·000	·943	·890	·840	·774	·710	·656	·595	·541

should be remembered, are corrected for the loss in transmission through the earth's atmosphere for each wave-length separately.) Abbot's results show somewhat less darkening even than corresponds to $u = \frac{2}{3}$, the average value of u as given by them being about 0·56.

§ 7. Comparison with Schwarzschild's Solution for Scattering.

A different form of the integral equation for the temperature distribution can be obtained. Equation (19) may be written, by an obvious transformation,

$$2B(\tau) = \int_{\tau}^{\infty} B(t) dt \int_{t-\tau}^{\infty} \frac{e^{-y}}{y} dy + \int_0^{\tau} B(t) dt \int_{\tau-t}^{\infty} \frac{e^{-y}}{y} dy,$$

or

$$B(\tau) = \frac{1}{2} \int_{\tau}^{\infty} B(t) Ei(t - \tau) dt + \frac{1}{2} \int_0^{\tau} B(t) Ei(\tau - t) dt \quad . \quad (46)$$

where

$$Ei(z) = \int_z^{\infty} \frac{e^{-y}}{y} dy = \int_1^{\infty} \frac{e^{-zy}}{y} dy.$$

We may write this more compactly in the form

$$B(\tau) = \frac{1}{2} \int_0^{\infty} B(t) Ei(|t - \tau|) dt \quad . \quad . \quad (47)$$

This form suggests the integral equation used by Schwarzschild in the second paper cited, in his treatment of scattering. Assuming that a scattering particle scatters uniformly in all directions, the equations for the distribution of radiation in the case of pure scattering are formally similar to those for radiative equilibrium, the function $B(\tau)$ being replaced by a function $J(\tau)$ which though,

* As quoted by Müller, *Photometrie der Gestirne*; by Pringsheim, *Physik der Sonne* (p. 394); and by Schwarzschild, *Gott. Nach.*, 1906, p. 52.

† *Annals Astrophys. Obs. Smithson. Inst.*, 3, 157 (1913).

‡ Lindblad, *Uppsala Universitets Årsskrift*, 1920, 1, 29.

like $B(\tau)$, measuring the energy-density distribution, has no longer any significance in connection with temperature; the boundary conditions are, however, different, for since there is no emission the radiation must be supposed provided by an external source. For a plane slab of scattering material of optical thickness τ_1 in front of a uniformly radiating surface (intensity I_1), Schwarzschild finds the equation

$$J(\tau) = \frac{1}{2} \int_0^{\tau_1} J(t) \text{Ei}(|t - \tau|) dt + \frac{1}{2} I_1 \text{Ei}_1(\tau_1 - t) \quad (48)$$

where $\text{Ei}_1(z)$ is defined by

$$\text{Ei}_1(z) = z \int_z^{\infty} \frac{e^{-y}}{y^2} dy = \int_1^{\infty} \frac{e^{-zy}}{y^2} dy.$$

Equation (48) tends to the form (47) as $\tau_1 \rightarrow \infty$. The radiation escaping from the front surface is given by

$$I(\circ, \theta) = I_1 e^{-\tau_1 \sec \theta} + \int_0^{\tau_1} J(t) e^{-\tau \sec \theta} \sec \theta d\theta \quad (49)$$

Schwarzschild proves that the solution of (48) is of the form

$$J(\tau) = I_1 \frac{\frac{1}{2} + \tau + L(\tau)}{\tau_1 + 1} \quad (50)$$

where $|L(\tau)| < \frac{1}{2}$, and that $L(\tau)$ tends to a definite limit function as $\tau_1 \rightarrow \infty$. He evaluates $L(\tau)$ approximately, for $\tau_1 = 1, 2, 4, 8$, by the usual method of writing down and solving a sufficient number of the ordinary simultaneous equations of which (48) is the limit; for $\tau_1 = 8$ there were twenty such equations required. The results for $\tau_1 = 8$, which may be taken as an approximation to those for $\tau_1 \rightarrow \infty$, are given in Tables I. and II. above, in the column headed (η). Those in Table II. are taken directly, but for those in Table I. it was necessary to multiply Schwarzschild's tabulated values by a certain factor (2.135), this factor (which was found by integrating his numerical values for $I(\circ, \theta)$) being chosen to give the appropriate net flow.*

It will be seen that these figures, both as regards the function $B(\tau)$ and the law of darkening, are in good agreement with our approximations (δ) and (ϵ); further, the boundary temperature corresponding to (η) is given by

$$T_1^4/T_0^4 = \frac{8}{3}/1.15 = 2.32, \quad T_1/T_0 = 1.232,$$

which lies between our determinations (32) and (34).

* This factor cannot be found directly, but it can be seen from (50) that its limiting value as $\tau_1 \rightarrow \infty$ is 2.

§ 8. *Summary.*

An integral equation is obtained for the density of radiant energy in radiative equilibrium and solved by successive approximation. The various approximations involve an arbitrary constant, which can be chosen in various ways so as to improve the accuracy of the solution. It is shown that the density-gradient taken with respect to the optical thickness ($dB(\tau)/d\tau$) has a logarithmic infinity at the boundary, and approximations are obtained for the boundary temperature T_0 in terms of the effective temperature T_1 . The temperature distribution thus found is applied to deduce the law of darkening of a stellar disc towards the limb; it is shown that the value $\frac{3}{5}$ for the coefficient of darkening is a better approximation than the Schwarzschild-Jeans value $\frac{2}{3}$; the distribution of intensity over the disc corresponds to a *mean* coefficient of about 0.61 or 0.62, but there is a more sudden drop in intensity very close to the limb.

Radiative Equilibrium and Spectral Distribution.

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(Communicated by the Director, Solar Physics Observatory, Cambridge.)

§ 1. *Introductory.*

In the investigation of radiative equilibrium it is necessary to exercise care in assuming that the radiation will possess certain of the properties of black-body radiation. For example, the density of radiant energy for isotropic black-body radiation is aT^4 , where a is Stefan's constant and T is the temperature; but it is not necessarily true that the energy-density in radiative equilibrium is equal to that of black-body radiation corresponding to the temperature of the matter at the point considered.

This point is of importance if one attempts to investigate the spectral distribution of the radiation.* The radiation differs from isotropic black-body radiation in that both the intensity and the spectral distribution vary from direction to direction, and the spectrum in any particular direction (for a given total intensity) will not in general be that of the corresponding black-body radia-

* When writing this paper the author was unaware that the question of the spectral distribution of the light in radiative equilibrium had been previously discussed, but since the paper was finished his attention has been drawn to a paper by Lindblad, "The Distribution of Intensity in the Continuous Spectra of the Sun and Fixed Stars," *Uppsala Universitets Årsskrift*, 1, 1920, which discusses very fully the theoretical spectrum for radiative equilibrium and the variation of colour over the disc, with detailed comparison with the observations of Abbot, Fowle, and Aldrich. However, it seems worth while making the present communication, as the method of calculation and the method of expressing the results are somewhat different.