TABLES FOR TESTING THE GOODNESS OF FIT OF THEORY TO OBSERVATION.

By W. PALIN ELDERTON, *Actuary*.

[Received October 18, 1901.]

On the Test for Random Sampling.

ANY theoretical description by means of curve or series is *ceteris paribus* admissible as a graduation of a given set of frequency observations, provided the observed values do not differ from the values provided by this theory by more than the reasonable deviations due to random sampling. There may be utilitarian reasons (e.g. relative fewness of descriptive constants, or their easy calculation) or philosophical reasons (e.g. general theories as to the nature and distribution of causes producing frequency phenomena) why we should adopt one theoretical description rather than another, but apart from such reasons that theoretical description is best, which describes the observed frequencies with the "greatest probability." By "describing the observed frequencies with the greatest probability" we understand a good although conventional test of fitness. Suppose the theoretical description of the frequencies to be the actual distribution of the whole population; we ask in how many cases per 100 in a series of random samplings should we differ from the theoretical distribution by as wide a system of deviations as that observed, or by a still wider system? In other words we want to find out the probability P that in random sampling deviation-systems as great as or greater than that actually observed will arise. This point has been dealt with in a paper by Professor K. Pearson published in the *Philosophical Magazine**, and it is there shown that if there be $n' = n + 1$ frequency groups in the series, and m_r and m_r' be the theoretical and observed frequencies in any group, it is necessary to find

$$\chi^2 = S\left\{\frac{(m_r - m_r')^2}{m_r}\right\} = \text{sum}\left(\frac{\left\{\begin{array}{c}\text{squares of differences of theoretical}\\ \text{and observed frequencies}\end{array}\right\}}{\text{theoretical frequency}}\right),$$

* On the Criterion that a given System of Deviations from the Probable in the case of a Correlated System of Variables is such that it can be reasonably supposed to have arisen from Random Sampling, Vol. L. pp. 157—175, July, 1900.

and that P will then be calculated from:

$$P = \sqrt{\frac{2}{\pi}}\int_\chi^\infty e^{-\frac{1}{2}\chi^2}d\chi + \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}\chi^2}\left(\frac{\chi}{1} + \frac{\chi^3}{1.3} + \frac{\chi^5}{1.3.5} + \dots + \frac{\chi^{n'-3}}{1.3.5\dots(n'-3)}\right)$$

if n' be even, and from:

$$P = e^{-\frac{1}{2}\chi^2}\left(1 + \frac{\chi^2}{2} + \frac{\chi^4}{2.4} + \frac{\chi^6}{2.4.6} + \dots + \frac{\chi^{n'-3}}{2.4.6\dots(n'-3)}\right)$$

if n' be odd.

Now although χ^2 can be found quite easily without any special mathematical knowledge, the calculation of P from the above formulæ is very troublesome. But it is quite clear that some test of the above kind is absolutely needful in all biometric enquiries in which we wish to test theory against observation. In the paper referred to a small table for P in terms of n' and χ^2 was given, but this table beside being far from extensive enough for actual practice, was based in some entries on values of the probability integral which had not been calculated by the use of higher differences. The present Table I. is an attempt to provide a more extensive and accurate system of values for P. It gives the values of P for $n' = 3$ to 30 and from $\chi^2 = 1$ to 30 by units and from $\chi^2 = 30$ to 70 by tens.

Method of Calculating Tables.

In order to simplify the work of calculating P for values lying outside the range of this table, or in cases where interpolation would not give sufficiently accurate results a series of additional tables are given which were used in the calculation of Table I. Thus Table II. gives the values of $\log\left(\chi\sqrt{\frac{2}{\pi}}\,e^{-\frac{1}{2}\chi^2}\right)$ and $\log(e^{-\frac{1}{2}\chi^2})$ to eight figures. Table V. gives $\log e^{-\frac{1}{2}}$ and $\log\sqrt{\frac{2}{\pi}}$ to ten figures*. Table III. gives the cologarithms of $n(n-2)(n-4)\dots 1$ (or 2) needed for the coefficients of the powers of χ to eight figures. Table IV. gives the values of $\sqrt{\frac{2}{\pi}}\int_\chi^\infty e^{-\frac{1}{2}\chi^2}d\chi$ for $\chi^2 = 1$ to 30 to eight figures, i.e. as long as it is practically sensible. Further values of this integral may be deduced from the tables for $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$, which are given for $t = 0$ to 4·80 to eleven places of decimals for the higher values in Czuber's *Theorie der Beobachtungsfehler*, Leipzig, 1891.

In calculating the tables Erskine Scott's 10-Figure Logarithms and Filipowski's 7-Figure Antilogarithms were used. The method of calculation was, briefly, as follows. Tables were first made of $\log\left(\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}\chi^2}\right)$ and $\log e^{-\frac{1}{2}\chi^2}$ by continuous

* Thus incidentally the ordinates of the normal probability curve, $y = \frac{1}{2}\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}$, are given for the squares of the abscissæ.

addition and after adding $\log \chi$ to the former, the resulting figures were reduced from ten to eight places of decimals so as to avoid the error that would arise from the accumulation of small differences in the eleventh place in the value of $\log e^{-\frac{1}{2}}$. These tables were carefully checked by addition and by examining every tenth value in the continuous work. The table of $\text{colog}\,[n] \left(= \log \frac{1}{n(n-2)(n-4)\ldots}\right)$ was originally calculated to ten places. The only other auxiliary table required was for $\sqrt{\frac{2}{\pi}} \int_\chi^\infty e^{-\frac{1}{2}\chi^2} d\chi$, and these values were calculated to seven places of decimals by second differences from a table of values of $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$*. It was quite safe to omit third differences. The values of P were then calculated from formulæ (i) and (ii) given above. In making the table, to find χ^{2s}, a column of $s \log \chi^2$ was first set up, and then by means of four moveable slips of paper (two for n' even and two for n' odd) a second column calculated giving the sum of $s \log \chi^2$, $\text{colog}\,(2s+1)$ and $\log \left(\sqrt{\frac{2}{\pi}} \chi e^{-\frac{1}{2}\chi^2}\right)$. These figures were checked by addition. The use of slips with $\text{colog}\,(n)$ written on them saved a very large amount of copying. The antilogarithms of the items of the second column were then put in a third column and the values of $\sqrt{\frac{2}{\pi}} \int_\chi^\infty e^{-\frac{1}{2}\chi^2} d\chi$ having been written at the top of it, the figures given in Table I. were found by continuous addition. The values for n' even were calculated in like manner. The numbers obtained were tested when possible against those originally published in the *Phil. Mag.* and against a few additional values calculated by Miss M. A. Lewenz. The work was of course checked at every stage, but when the table was completed the second differences in each column were examined and found to run smoothly. The like method of differences was appealed to in the case of discrepancies between the short table and the present table, which were not due to the approximate value taken for the probability integral. It is hoped that the table as it now stands is substantially free from error.

In using the present method of testing goodness of fit it is essential to bear in mind a warning given in the paper in the *Phil. Mag.* referred to above (footnote, p. 164): "A theoretical probability curve without limited range will never at the extreme tails exactly fit observation. The difficulty is obvious where the observations go by units and the theory by fractions. We ought to take our final theoretical groups to cover as much of the tail area as amounts to at least a unit of frequency in such cases."

Further we ought to be careful to read the corresponding *areas* of the frequency curve and not merely the mid-ordinates, when we have not a great number of groups, or when, although the groups are numerous, the frequency is very skew.

* The Table in Galloway's *Treatise on Probabilities* was the one actually used.

Illustration of use of Tables.

In the table below we have the distribution of the cephalic index in 900 skulls of modern Bavarian peasants. The frequency is given in the second column. In the third column we have the distribution as indicated by the normal curve of errors. Is this a reasonable description of the series of measurements? In the fourth column are given the values of $m_r - m_r'$ and in the fifth those of $(m_r - m_r')^2/m_r$. The resulting value of χ^2 is 18·36 and $n' = 24$. Table I. gives us: $n' = 24$, $\chi^2 = 18$, $P = ·757489$, and $\chi^2 = 19$, $P = ·701224$. Hence the required probability is nearly ·737, or roughly in every three cases out of four a random sampling would lead to a system of deviations diverging more widely from theory. Thus the fit may be considered a good one.

Cephalic Index of Bavarian Skulls.

Index	Observed	Calculated*	$m_r - m_r'$	$\frac{(m_r - m_r')^2}{m_r}$†
Under 71·5	2	1	− 1	1
72	0	1	+ 1	1
73	2·5	1·5	− 1	·67
74	1·5	3·5	+ 2	1·14
75	3·5	7·5	+ 4	2·13
76	12·5	13·5	+ 1	·07
77	17	23	+ 6	1·57
78	37	35·5	− 1·5	·06
79	55	52·5	− 2·5	·12
80	71·5	69·5	− 2	·06
81	82	86	+ 4	·19
82	116	98·5	−17·5	3·11
83	98	103	+ 5	·24
84	107	99·5	− 7·5	·57
85	82	88·5	+ 6·5	·48
86	74	72	− 2	·06
87	58	54	− 4	·30
88	34·5	37·5	+ 3	·24
89	19	23·5	+ 4·5	·86
90	10	14	+ 4	1·14
91	8	7·5	− 0·5	·03
92	3	3·5	+ 0·5	·07
93	1·5	2	+ 0·5	·125
Over 93·5	4·5	2	− 2·5	3·125
Totals	900	900	0	$\chi^2 = 18·36$

* The calculated values are given to the nearest half skull because the observed values only run to this unit.

† The numbers in the fifth column were obtained from the squares of those in the fourth by dividing them by the corresponding numbers in the third. The squaring is at once done from *Barlow's Tables* and the division to the accuracy required by *Crelle's Rechentafeln*. Both these books are indispensable to biometricians.

TABLE I.

χ^2	$n'=3$	$n'=4$	$n'=5$	$n'=6$	$n'=7$	$n'=8$	$n'=9$	$n'=10$	$n'=11$
1	·606531	·801253	·909796	·962566	·985612	·994829	·998249	·999438	·999828
2	·367879	·572407	·735759	·849146	·919699	·959840	·981012	·991468	·996340
3	·223130	·391625	·557825	·699986	·808847	·885002	·934357	·964295	·981424
4	·135335	·261464	·406006	·549416	·676676	·779778	·857123	·911413	·947347
5	·082085	·171797	·287298	·415880	·543813	·659963	·757576	·834308	·891178
6	·049787	·111610	·199148	·306219	·423190	·539750	·647232	·739919	·815263
7	·030197	·071897	·135888	·220640	·320847	·428880	·536632	·637119	·725444
8	·018316	·046012	·091578	·156236	·238103	·332594	·433470	·534146	·628837
9	·011109	·029291	·061099	·109064	·173578	·252656	·342296	·437274	·532104
10	·006738	·018566	·040428	·075235	·124652	·188573	·265026	·350485	·440493
11	·004087	·011726	·026564	·051380	·088376	·138619	·201699	·275709	·357518
12	·002479	·007383	·017351	·034787	·061969	·100558	·151204	·213308	·285057
13	·001503	·004637	·011276	·023379	·043036	·072109	·111850	·162607	·223672
14	·000912	·002905	·007295	·015609	·029636	·051181	·081765	·122325	·172992
15	·000553	·001817	·004701	·010363	·020256	·036000	·059145	·090937	·132061
16	·000335	·001134	·003019	·006844	·013754	·025116	·042380	·066881	·099632
17	·000203	·000707	·001933	·004500	·009283	·017396	·030109	·048716	·074364
18	·000123	·000440	·001234	·002947	·006232	·011970	·021226	·035174	·054964
19	·000075	·000273	·000786	·001922	·004164	·008187	·014860	·025193	·040263
20	·000045	·000170	·000499	·001250	·002769	·005570	·010336	·017913	·029253
21	·000028	·000105	·000317	·000810	·001835	·003770	·007147	·012650	·021093
22	·000017	·000065	·000200	·000524	·001211	·002541	·004916	·008880	·015105
23	·000010	·000040	·000127	·000338	·000796	·001705	·003364	·006197	·010747
24	·000006	·000025	·000080	·000217	·000522	·001139	·002292	·004301	·007600
25	·000004	·000016	·000050	·000139	·000341	·000759	·001554	·002971	·005345
26	·000002	·000010	·000032	·000090	·000223	·000504	·001050	·002043	·003740
27	·000001	·000006	·000020	·000057	·000145	·000333	·000707	·001399	·002604
28	·000001	·000004	·000012	·000037	·000094	·000220	·000474	·000954	·001805
29	·000001	·000002	·000008	·000023	·000061	·000145	·000317	·000648	·001246
30	·000000	·000001	·000005	·000015	·000039	·000095	·000211	·000439	·000857
40	·000000	·000000	·000000	·000000	·000001	·000001	·000003	·000008	·000017
50	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
60	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
70	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000

TABLE I.—*continued.*

χ^2	$n'=12$	$n'=13$	$n'=14$	$n'=15$	$n'=16$	$n'=17$	$n'=18$	$n'=19$	$n'=20$
1	·999950	·999986	·999997	·999999	1·	1·	1·	1·	1·
2	·998496	·999406	·999774	·999917	·999970	·999990	·999997	·999999	1·
3	·990726	·995544	·997934	·999074	·999598	·999830	·999931	·999972	·999989
4	·969917	·983436	·991191	·995466	·997737	·998903	·999483	·999763	·999894
5	·931167	·957979	·975193	·985813	·992127	·995754	·997771	·998860	·999431
6	·873365	·916082	·946153	·966491	·979749	·988095	·993187	·996197	·997929
7	·799073	·857613	·902151	·934711	·957650	·973260	·983549	·990125	·994213
8	·713304	·785131	·843601	·889327	·923783	·948867	·966547	·978637	·986671
9	·621892	·702931	·772943	·831051	·877517	·913414	·940261	·959743	·973479
10	·530387	·615960	·693934	·762183	·819739	·866628	·903610	·931906	·952946
11	·443263	·528919	·610817	·686036	·752594	·809485	·856564	·894357	·923839
12	·362642	·445680	·527643	·606303	·679028	·743980	·800136	·847237	·885624
13	·293326	·369041	·447812	·526524	·602298	·672758	·736186	·791573	·838571
14	·232993	·300708	·373844	·449711	·525529	·598714	·667102	·729091	·783691
15	·182498	·241436	·307354	·378154	·451418	·524638	·595482	·661967	·722598
16	·141130	·191236	·249129	·313374	·382051	·452961	·523834	·592547	·657277
17	·107876	·149597	·199304	·256178	·318864	·385597	·454366	·523105	·589868
18	·081581	·115691	·157520	·206781	·262666	·323897	·388841	·455653	·522438
19	·061094	·088529	·123104	·164949	·213734	·268663	·328532	·391823	·456836
20	·045341	·067086	·095210	·130141	·171932	·220220	·274229	·332819	·394578
21	·033371	·050380	·072929	·101632	·136830	·178510	·226291	·279413	·336801
22	·024374	·037520	·055362	·078614	·107804	·143191	·184719	·231985	·284256
23	·017676	·027726	·041677	·060270	·084140	·113735	·149251	·190590	·237342
24	·012733	·020341	·031130	·045822	·065093	·089504	·119435	·155028	·196152
25	·009117	·014822	·023084	·034566	·049943	·069824	·094710	·124915	·160542
26	·006490	·010734	·017001	·025887	·038023	·054028	·074461	·099758	·130189
27	·004595	·007727	·012441	·019254	·028736	·041483	·058068	·078995	·104653
28	·003238	·005532	·009050	·014228	·021569	·031620	·044938	·062055	·083428
29	·002270	·003940	·006546	·010450	·016085	·023936	·034526	·048379	·065985
30	·001585	·002792	·004710	·007632	·011921	·018002	·026345	·037446	·051798
40	·000036	·000072	·000138	·000255	·000453	·000778	·001294	·002087	·003272
50	·000001	·000001	·000003	·000006	·000012	·000023	·000042	·000075	·000131
60	·000000	·000000	·000000	·000000	·000000	·000001	·000001	·000002	·000004
70	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000

TABLE I.—*continued.*

χ^2	$n'=21$	$n'=22$	$n'=23$	$n'=24$	$n'=25$	$n'=26$	$n'=27$	$n'=28$	$n'=29$	$n'=30$
1	1·	1·	1·	1·	1·	1·	1·	1·	1·	1·
2	1·	1·	1·	1·	1·	1·	1·	1·	1·	1·
3	·999996	·999998	·999999	1·	1·	1·	1·	1·	1·	1·
4	·999954	·999980	·999992	·999997	·999999	1·	1·	1·	1·	1·
5	·999722	·999868	·999939	·999972	·999987	·999994	·999998	·999999	1·	1·
6	·998898	·999427	·999708	·999855	·999929	·999966	·999984	·999993	·999997	·999999
7	·996685	·998142	·998980	·999452	·999711	·999851	·999924	·999962	·999981	·999991
8	·991868	·995143	·997160	·998371	·999085	·999494	·999726	·999853	·999924	·999960
9	·982907	·989214	·993331	·995957	·997595	·998596	·999194	·999546	·999748	·999863
10	·968171	·978912	·986304	·991277	·994547	·996653	·997981	·998803	·999302	·999599
11	·946223	·962787	·974749	·983189	·989012	·992946	·995549	·997239	·998315	·998988
12	·916076	·939617	·957379	·970470	·979908	·986567	·991173	·994294	·996372	·997728
13	·877384	·908624	·933161	·951990	·966121	·976501	·983974	·989247	·992900	·995384
14	·830496	·869599	·901479	·926871	·946650	·961732	·973000	·981254	·987189	·991377
15	·776408	·822952	·862238	·894634	·920759	·941383	·957334	·969432	·978436	·985015
16	·716624	·769650	·815886	·855268	·888076	·914828	·936203	·952947	·965819	·975536
17	·652974	·711106	·763362	·809251	·848662	·881793	·909083	·931122	·948589	·962181
18	·587408	·649004	·705988	·757489	·803008	·842390	·875773	·903519	·928149	·944272
19	·521826	·585140	·645328	·701224	·751990	·797120	·836430	·870001	·898136	·921288
20	·457930	·521261	·583040	·641912	·696776	·746825	·791556	·830756	·864464	·892927
21	·397132	·458944	·520738	·581087	·638725	·692609	·741964	·786288	·825349	·859149
22	·340511	·399510	·459889	·520252	·579267	·635744	·688697	·737377	·781291	·820189
23	·288795	·343979	·401730	·460771	·519798	·577564	·632947	·685013	·733041	·776543
24	·242392	·293058	·347229	·403808	·461597	·519373	·575965	·630316	·681535	·728932
25	·201431	·247164	·297075	·350285	·405760	·462373	·518975	·574462	·627835	·678248
26	·165812	·206449	·251682	·300866	·353165	·407598	·463105	·518600	·573045	·625491
27	·135264	·170853	·211226	·255967	·304453	·355884	·409333	·463794	·518247	·571705
28	·109399	·140151	·175681	·215781	·260040	·307853	·358458	·410973	·464447	·517913
29	·087759	·114002	·144861	·180310	·220131	·263916	·311082	·360899	·412528	·465066
30	·069854	·091988	·118464	·149402	·184752	·224289	·267611	·314154	·363218	·414004
40	·004995	·007437	·010812	·015369	·021387	·029164	·039012	·051237	·066128	·083937
50	·000221	·000365	·000586	·000921	·001416	·002131	·003144	·004551	·006467	·009032
60	·000007	·000013	·000022	·000038	·000064	·000104	·000168	·000264	·000407	·000618
70	·000000	·000000	·000001	·000001	·000002	·000004	·000007	·000011	·000019	·000030

TABLE II.

χ^2	$\log\left\{\chi\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}\chi^2}\right\}$	$\log e^{-\frac{1}{2}\chi^2}$	χ^2	$\log\left\{\chi\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}\chi^2}\right\}$	$\log e^{-\frac{1}{2}\chi^2}$
1	$\bar{1}$·68479282	$\bar{1}$·78285276	51	$\overline{11}$·68121586	$\overline{12}$·92549071
2	$\bar{1}$·61816058	$\bar{1}$·56570552	52	$\overline{11}$·46828520	$\overline{12}$·70834347
3	$\bar{1}$·48905897	$\bar{1}$·34855828	53	$\overline{11}$·25527432	$\overline{12}$·49119623
4	$\bar{1}$·33438109	$\bar{1}$·13141104	54	$\overline{11}$·04218593	$\overline{12}$·27404899
5	$\bar{1}$·16568886	$\bar{2}$·91426380	55	$\overline{12}$·82902315	$\overline{12}$·05690175
6	$\bar{2}$·98813224	$\bar{2}$·69711655	56	$\overline{12}$·61578858	$\overline{13}$·83975450
7	$\bar{2}$·80445839	$\bar{2}$·47996931	57	$\overline{12}$·40248475	$\overline{13}$·62260726
8	$\bar{2}$·61630713	$\bar{2}$·26282207	58	$\overline{12}$·18911408	$\overline{13}$·40546002
9	$\bar{2}$·42473615	$\bar{2}$·04567483	59	$\overline{13}$·97567885	$\overline{13}$·18831278
10	$\bar{2}$·23046765	$\bar{3}$·82852759	60	$\overline{13}$·76218123	$\overline{14}$·97116554
11	$\bar{2}$·03401675	$\bar{3}$·61138035	61	$\overline{13}$·54862328	$\overline{14}$·75401830
12	$\bar{3}$·83576379	$\bar{3}$·39423311	62	$\overline{13}$·33500696	$\overline{14}$·53687106
13	$\bar{3}$·63599760	$\bar{3}$·17708587	63	$\overline{13}$·12133415	$\overline{14}$·31972382
14	$\bar{3}$·43494271	$\bar{4}$·95993863	64	$\overline{14}$·90760662	$\overline{14}$·10257658
15	$\bar{3}$·23277708	$\bar{4}$·74279139	65	$\overline{14}$·69382607	$\overline{15}$·88542934
16	$\bar{3}$·02964420	$\bar{4}$·52564414	66	$\overline{14}$·48099412	$\overline{15}$·66828209
17	$\bar{4}$·82566143	$\bar{4}$·30849690	67	$\overline{14}$·26611232	$\overline{15}$·45113485
18	$\bar{4}$·62092598	$\bar{4}$·09134966	68	$\overline{14}$·05218213	$\overline{15}$·23398761
19	$\bar{4}$·41551928	$\bar{5}$·87420242	69	$\overline{15}$·83820498	$\overline{15}$·01684037
20	$\bar{4}$·20951024	$\bar{5}$·65705518	70	$\overline{15}$·62418221	$\overline{16}$·79969313
21	$\bar{4}$·00295765	$\bar{5}$·43990794	71	$\overline{15}$·41011512	$\overline{16}$·58254589
22	$\bar{5}$·79591210	$\bar{5}$·22276070	72	$\overline{15}$·19600496	$\overline{16}$·36539865
23	$\bar{5}$·58841744	$\bar{5}$·00561346	73	$\overline{16}$·98185290	$\overline{16}$·14825141
24	$\bar{5}$·38051190	$\bar{6}$·78846622	74	$\overline{16}$·76766009	$\overline{17}$·93110417
25	$\bar{5}$·17222904	$\bar{6}$·57131898	75	$\overline{16}$·55342762	$\overline{17}$·71395693
26	$\bar{6}$·96359847	$\bar{6}$·35417173	76	$\overline{16}$·33915654	$\overline{17}$·49680968
27	$\bar{6}$·75464644	$\bar{6}$·13702449	77	$\overline{16}$·12484787	$\overline{17}$·27966244
28	$\bar{6}$·54539633	$\bar{7}$·91987725	78	$\overline{17}$·91050256	$\overline{17}$·06251520
29	$\bar{6}$·33586907	$\bar{7}$·70273001	79	$\overline{17}$·69612157	$\overline{18}$·84536796
30	$\bar{6}$·12608346	$\bar{7}$·48558277	80	$\overline{17}$·48170578	$\overline{18}$·62822072
31	$\bar{7}$·91605644	$\bar{7}$·26843553	81	$\overline{17}$·26725605	$\overline{18}$·41107348
32	$\bar{7}$·70580334	$\bar{7}$·05128829	82	$\overline{17}$·05277323	$\overline{18}$·19392624
33	$\bar{7}$·49533808	$\bar{8}$·83414105	83	$\overline{18}$·83825810	$\overline{19}$·97677900
34	$\bar{7}$·28467333	$\bar{8}$·61699381	84	$\overline{18}$·62371146	$\overline{19}$·75963176
35	$\bar{7}$·07382065	$\bar{8}$·39984657	85	$\overline{18}$·40913404	$\overline{19}$·54248452
36	$\bar{8}$·86279064	$\bar{8}$·18269932	86	$\overline{18}$·19452656	$\overline{19}$·32533727
37	$\bar{8}$·65159301	$\bar{9}$·96555208	87	$\overline{19}$·97988972	$\overline{19}$·10819003
38	$\bar{8}$·44023670	$\bar{9}$·74840484	88	$\overline{19}$·76522419	$\overline{20}$·89104279
39	$\bar{8}$·22872997	$\bar{9}$·53125760	89	$\overline{19}$·55053062	$\overline{20}$·67389555
40	$\bar{8}$·01708042	$\bar{9}$·31411036	90	$\overline{19}$·33580963	$\overline{20}$·45674831
41	$\bar{9}$·80529511	$\bar{9}$·09696312	91	$\overline{19}$·12106183	$\overline{20}$·23960107
42	$\bar{9}$·59338058	$\overline{10}$·87981588	92	$\overline{20}$·90628780	$\overline{20}$·02245383
43	$\bar{9}$·38134293	$\overline{10}$·66266864	93	$\overline{20}$·69148812	$\overline{21}$·80530659
44	$\bar{9}$·16918780	$\overline{10}$·44552140	94	$\overline{20}$·47666333	$\overline{21}$·58815935
45	$\overline{10}$·95692047	$\overline{10}$·22837416	95	$\overline{20}$·26181397	$\overline{21}$·37101211
46	$\overline{10}$·74454589	$\overline{10}$·01122691	96	$\overline{20}$·04694054	$\overline{21}$·15386486
47	$\overline{10}$·53206866	$\overline{11}$·79407967	97	$\overline{21}$·83204355	$\overline{22}$·93671762
48	$\overline{10}$·31949311	$\overline{11}$·57693243	98	$\overline{21}$·61712348	$\overline{22}$·71957038
49	$\overline{10}$·10682329	$\overline{11}$·35978519	99	$\overline{21}$·40218080	$\overline{22}$·50242314
50	$\overline{11}$·89406301	$\overline{11}$·14263795	100	$\overline{21}$·18721596	$\overline{22}$·28527590

TABLE III.

Table of colog $[n]$:—$[n] = n(n-2)(n-4)\ldots\ldots$

n odd nos.	colog $[n]$	n even nos.	colog $[n]$
1	·00000000	2	$\overline{1}$·69897000
3	$\overline{1}$·52287875	4	$\overline{1}$·09691001
5	$\overline{2}$·82390874	6	$\overline{2}$·31875876
7	$\overline{3}$·97881070	8	$\overline{3}$·41566878
9	$\overline{3}$·02456819	10	$\overline{4}$·41566878
11	$\overline{5}$·98317551	12	$\overline{5}$·33648753
13	$\overline{6}$·86923215	14	$\overline{6}$·19035949
15	$\overline{7}$·69314089	16	$\overline{8}$·98623951
17	$\overline{8}$·46269197	18	$\overline{9}$·73096701
19	$\overline{9}$·18393837	20	$\overline{10}$·42993701
21	$\overline{11}$·86171908	22	$\overline{11}$·08751433
23	$\overline{12}$·49999124	24	$\overline{13}$·70730309
25	$\overline{13}$·10205123	26	$\overline{14}$·29232974
27	$\overline{15}$·67068747	28	$\overline{16}$·84517171
29	$\overline{16}$·20828947	30	$\overline{17}$·36805045
31	$\overline{18}$·71692778	32	$\overline{19}$·86290048
33	$\overline{19}$·19841384	34	$\overline{20}$·33142156
35	$\overline{21}$·65434579	36	$\overline{22}$·77511906
37	$\overline{22}$·08614407	38	$\overline{23}$·19533546
39	$\overline{24}$·49507946	40	$\overline{25}$·59327547
41	$\overline{26}$·88229561	42	$\overline{27}$·97002618
43	$\overline{27}$·24882715	44	$\overline{28}$·32657350
45	$\overline{29}$·59561464	46	$\overline{30}$·66381567
47	$\overline{31}$·92351678	48	$\overline{32}$·98257443
49	$\overline{32}$·23332070	50	$\overline{33}$·28360443
51	$\overline{34}$·52575052	52	$\overline{35}$·56760109
53	$\overline{36}$·80147465	54	$\overline{37}$·83520733
55	$\overline{37}$·06111196	56	$\overline{38}$·08701930
57	$\overline{39}$·30523711	58	$\overline{40}$·32359131
59	$\overline{41}$·53438510	60	$\overline{42}$·54544006
61	$\overline{43}$·74905526	62	$\overline{44}$·75304837
63	$\overline{45}$·94971471	64	$\overline{46}$·94686839
65	$\overline{46}$·13680135	66	$\overline{47}$·12732446
67	$\overline{48}$·31072655	68	$\overline{49}$·29481554
69	$\overline{50}$·47187746	70	$\overline{51}$·44971750
71	$\overline{52}$·62061911	72	$\overline{53}$·59238501
73	$\overline{54}$·75729625	74	$\overline{55}$·72315329
75	$\overline{56}$·88223499	76	$\overline{57}$·84233970
77	$\overline{58}$·99574426	78	$\overline{59}$·95024509
79	$\overline{59}$·09811717	80	$\overline{60}$·04715511
81	$\overline{61}$·18963215	82	$\overline{62}$·13334125
83	$\overline{63}$·27055406	84	$\overline{64}$·20906197
85	$\overline{65}$·34113514	86	$\overline{66}$·27456352
87	$\overline{67}$·40161588	88	$\overline{68}$·33008084
89	$\overline{69}$·45222588	90	$\overline{70}$·37583233
91	$\overline{71}$·49318448	92	$\overline{72}$·41205051
93	$\overline{73}$·52470154	94	$\overline{74}$·43892265
95	$\overline{75}$·54697793	96	$\overline{76}$·45665142
97	$\overline{77}$·56020620	98	$\overline{78}$·46542534
99	$\overline{79}$·56457100	100	$\overline{80}$·46542534

TABLE IV.

χ^2	$\sqrt{\frac{2}{\pi}}\int_{\chi}^{\infty} e^{-\frac{1}{2}\chi^2}\,d\chi$
1	·3173106
2	·1572992
3	·0832646
4	·0455003
5	·0253474
6	·0143060
7	·0081506
8	·0046776
9	·0026998
10	·0015654
11	·0009112
12	·0005321
13	·0003115
14	·0001828
15	·0001076
16	·0000634
17	·0000374
18	·0000221
19	·0000132
20	·0000078
21	·0000046
22	·0000027
23	·0000016
24	·0000011
25	·0000007
26	·0000004
27	·0000003
28	·0000002
29	·0000001
30	·0000000

TABLE V.

Function	Log. Function
$e^{-\frac{1}{2}}$	$\overline{1}$·7828527590
$\sqrt{\frac{2}{\pi}}$	$\overline{1}$·9019400615