That $V(x, y)$ thus defined is harmonic follows at once from (3) since $\frac{\partial \psi}{\partial s}$ is easily seen, either by direct computation or from its value:

$$\frac{\partial}{\partial n}[\log \rho_1 - \log \rho]$$

to be a harmonic function of (x, y).

For the proof of the second part of the theorem formula (2) is particularly adapted. We have here to prove that if (x, y) approaches a point P on the circumference $V(x, y)$ approaches as its limit the value of V_c at P. The idea upon which this proof rests is that when (x, y) is near to P a small arc including P corresponds to a large range of values of ψ and, therefore, when we take the arithmetic mean as indicated in (2) the value of V_c at P will predominate.* The exact proof based upon the idea just stated merely requires the writing down of a few inequalities.

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ON THE POLYNOMIALS OF STIELTJES.

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BY a Stieltjes polynomial will here be understood any polynomial satisfying a regular linear differential equation of the second order

$$\begin{aligned}&\frac{d^2y}{dx^2} + \left(\frac{1-\lambda_1}{x-e_1} + \cdots + \frac{1-\lambda_r}{x-e_r}\right)\frac{dy}{dx}\\ &+ \frac{\varphi(x) = A_0x^{r-2} + A_1x^{r-3} + \cdots + A_{r-2}}{(x-e_1)\cdots(x-e_r)}y = 0\end{aligned} \tag{I}$$

in which the singular points $e_1, \cdots, e_r, \infty$ are real and in which also r exponent-differences $\lambda_1, \cdots, \lambda_r$ are (algebraically) less than unity. We shall here for the most part confine our

* It will be seen that this idea is similar to that suggested by Schwarz. (Ges. Werke, vol. 2, p. 360. See also Klein-Fricke: Modulfunctionen, vol. 1, p. 512.) We avoid, however, the artificiality of Schwarz's method.

attention to the case in which the number of singular points is equal to 4. The differential equation may then be written as follows:

$$\frac{d^2y}{dx^2}+\left(\frac{1-\lambda_1}{x-e_1}+\frac{1-\lambda_2}{x-e_2}+\frac{1-\lambda_3}{x-e_3}\right)\frac{dy}{dx} - \frac{Ax-B}{(x-e_1)(x-e_2)(x-e_3)}y=0. \tag{II}$$

The three singular points in the finite plane we will suppose to succeed each other in the order $e_1 < e_2 < e_3$. If their positions and exponent-differences are given, the accessory parameters A and B are completely determined by the requirement that one solution of the equation shall be a polynomial of the nth degree.* The former parameter is the product of the two exponents for ∞ which are $-n$ and

$$n-\sum_{i=1}^{i=3}\lambda_i+2;$$

the value of the latter is found from an equation of the $(n+1)$th degree. We have accordingly a group of $n+1$ polynomials which belong to the same set of exponent differences. If the exponent differences for the three singular points in the finite plane are each equal to $\frac{1}{2}$, the group consists of Lamé polynomials. These, as is well known, have the common property that their roots are all real and included between e_1 and e_3. In 1881 Klein † showed, however, that they could be distinguished, each from every other, by the manner in which its n roots are divided between the two segments e_1e_2 and e_2e_3. To each of the $n+1$ ways of assigning n roots to the two segments there corresponds, therefore, one and only one polynomial. ‡ This result was subsequently proved by Stieltjes § to hold also when the exponent-differences $\lambda_1, \cdots, \lambda_3$ have any values algebraically less than one. More recently Klein in his lectures upon linear differential equations || has remarked that the $n+1$ values of $B'=\frac{B}{A}$ belonging to a group of $n+1$ Lamé poly-

* Heine's Kugelfunctionen, vol. 1, p. 474.

† *Math. Ann.*, vol. 18, p. 237.

‡ The theorem as here stated is a special case of a corresponding theorem for a differential equation with $r+1$ singular points.

§ *Acta Mathematica*, vol. 6.

|| Cf. "Lineare Differentialgleichungen der zweiten Ordnung," pp. 341–346.

nomials are all included between e_1 and e_3. If, moreover, these are arranged in the order $B_1' < B_2' \cdots < B'_{n+1}$, the number of roots of the corresponding polynomial contained in $e_1 e_2$ is successively $0, 1, 2, \cdots, n$ and in $e_2 e_3$ successively $n, n-1, \cdots, 0$. The purpose of this paper is to prove first that these conclusions hold not only when the three exponent differences are equal to $\frac{1}{2}$ but for any values which are less than 1. It will be shown further that between two consecutive roots of any polynomial in the series will be included one and only one root of the preceding or following polynomial. These propositions are based upon a number of new theorems which relate to the roots of the fundamental integrals P^{λ_i}, P_i^0 belonging to the singular points e_i of a differential equation of the form (II). In conclusion, it will be shown that the roots of the accessory polynomial φ in the general differential equation (I) for a Stieltjes polynomial are all real, and, like the roots of the latter polynomial, are included between the two outermost singular points e_1 and e_r.

§ 1. In the differential equation here to be considered

$$\frac{d^2y}{dx^2} + \left(\frac{1-\lambda_1}{x-e_1} + \cdots + \frac{1-\lambda_3}{x-e_3}\right)\frac{dy}{dx} - \frac{Ax-B}{(x-e_1)\cdots(x-e_3)}\,y = 0 \tag{1}$$

it will be assumed that all the constants, whether singular points, exponent-differences, or accessory parameters, are real. We will also suppose that their values are given with the single exception of that of B. The axis of x is divided by the singular points into four segments, which, by a proper assignment of the subscripts to the singular points, may be taken to be ∞e_1, $e_1 e_2$, $e_2 e_3$, $e_3 \infty$. To each singular point in the finite plane there belong two fundamental integrals with the exponents λ_i, 0. These have the form

$$\begin{aligned} P^{\lambda_i} &= (x-e_i)^{\lambda_i}[1 + L_1(x-e_i) + L_2(x-e_i)^2 + \cdots], \\ P_i^0 &= 1 + M_1(x-e_i) + M_2(x-e_i)^2 + \cdots, \end{aligned} \tag{2}$$

in which the coefficients are necessarily real.* In general,

* When λ_i is a negative integer, the first integral must, of course, be modified by the introduction of a logarithmic term; and when it is a positive integer, the second integral. The terms of the theorems to be subsequently given are, however, such as to exclude fundamental integrals with logarithmic terms.

these series converget hrough the whole of one segment ending in e_i, but only in a part of the other. We shall, however, extend the meaning of the symbols P^{λ_i}, P_0^i so as to include the analytical continuation of these series throughout the remainder of the latter segment. The two exponents for the singular point ∞ are determined by the relations

$$\begin{gathered} \lambda_\infty' \, \lambda_\infty'' = -A, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_\infty' + \lambda_\infty'' = 2, \end{gathered} \tag{3}$$

and the corresponding expansions are

$$P^{\lambda_\infty'} = \frac{1}{x^{\lambda_\infty'}}\left[1 + \frac{L_1'}{x} + \frac{L_2'}{x^2} + \cdots\right]$$

and a similar series for $P^{\lambda_\infty''}$.

§2. When the differential equation possesses a polynomial solution, this solution is a fundamental integral for each singular point. If, as will hereafter be assumed unless the contrary is expressly stated, the exponents $\lambda_i (i = 1, 2, 3)$ are restricted to values less than one, it will coincide, except for a constant factor, at each singular point in the finite plane with the integral P_i^0. Since also its expansion for the singular point ∞ begins with x^n, one of the exponents for this point is equal to $-n$, and the polynomial is the corresponding fundamental integral. The second exponent and the value of A are determined by the equations (3).

§3. Without, however, as yet restricting the value of A in (1), we now proceed to examine the changes in the position of the real roots of P_1^0 when the parameter B is continuously varied. For this purpose let the differential equation be put into the form

$$\frac{d^2\,(Ky)}{dx^2} + Gy = 0, \tag{4}$$

where

$$K = \Pi(x - e_i)^{\frac{1-\lambda_i}{2}}, \quad G = \frac{-K(Ax - B)}{(x - e_1)\cdots(x - e_3)} - \frac{d^2K}{dx^2}. \tag{5}$$

By multiplying the differential equation through by an appropriate constant, the values of K and G may be made real for any interval of the axis of x which is included between two consecutive singular points, and at the same time the sign of K may be made positive. It will hereafter be assumed that K and G have been thus modified for the interval which may be under consideration. The equation is then of the same nature and form as that considered by Sturm in a highly important memoir in the first volume of *Liouville's Journal* (p. 106), K and G being used by him to represent any functions of x and of a parameter m which are real, one valued, and continuous for the interval considered. The present article is based upon some of the results there obtained. From these results we select now the following: If at every point of an interval ab ($a < b$) of the axis of x the value of K is positive; if, moreover, throughout this interval G increases with the parameter m while K either remains unaltered or decreases; then any root of a real solution y of the differential equation which is included between a and b will

(1) decrease if $\frac{d}{dm}\left(\frac{K}{y}\frac{dy}{dx}\right)$ is negative for $x = a$.

(2) increase if $\frac{d}{dm}\left(\frac{K}{y}\frac{dy}{dx}\right)$ is positive for $x = b$.

We may also remark parenthetically that in the proof of this theorem Sturm shows that under these conditions $\frac{d}{dm}\left(\frac{K}{y}\frac{dy}{dx}\right)$ retains the same sign throughout the entire interval, a result which we shall later use in § 6. To apply now the theorem itself we will begin by placing $m = B$, $y = P_1^0$. The values of K and G given by (5) meet the requirement of being continuous except at the singular points. Moreover, when B increases, K remains unaltered; G, on the other hand, increases only within the two segments e_1e_2 and $e_3\infty$. Since in the consideration of P_1^0 the latter segment is to be excluded, we will take for the limits of the interval $a = e_1 + \varepsilon_1$, $b = e_2 - \varepsilon_2$, where ε_1, ε_2 are infinitesimal positive quantities. To determine the sign of

$$\frac{d}{dB}\left(\frac{K}{P_1^0}\frac{dP_1^0}{dx}\right)$$

for the lower limit we observe that the value of

$$\frac{1}{P_1^0}\frac{dP_1^0}{dx}$$

for $x = e_1$ is equal to the coefficient M_1 in the series (2) for P_1^0. The substitution of this series in (1) gives

$$M_1 = \frac{Ae_1 - B}{(1 - \lambda_1)(e_1 - e_2)(e_1 - e_3)}, \tag{6}$$

whence follows

$$\frac{d}{dB}\left(\frac{1}{P_1^0}\frac{dP_1^0}{dx}\right)\Bigg|_{x=e_1} = \frac{-1}{(1-\lambda_1)(e_1-e_2)(e_1-e_3)}.$$

When λ_1 is less than 1, the last expression has a negative sign. Since the sign is also the same for $a = e_1 + \varepsilon_1$, all the conditions of the above theorem are fulfilled. Hence when B increases, the roots of P_1^0 which are situated in e_1e_2 decrease and move toward e_1. Furthermore, since P_1^0 cannot vanish for $x = e_1$, the number of its roots in the segment must either remain constant or increase. The changes in the position of its roots in $\infty\, e_1$ can be determined from the same theorem by placing $m = -B$, $a = -\infty$, $b = e_1 - \varepsilon_1$. The second of the two alternatives in the theorem must then be selected. Hence as B decreases, the roots increase and move toward e_1, and the number of roots in the segment must either remain constant or increase. The theorem can be applied in similar manner to P_2^0 and P_3^0. The conclusions thus reached can be united with the preceding in the following theorem.

I. *If when the exponent difference λ_i of one of the singular points e_i in the finite plane is less than unity B is increased, the roots of P_i^0 will move toward e_i, if contained in the segment e_1e_2 or $e_3\infty$, and will recede from e_i, if contained in the segment $\infty\, e_1$ or e_2e_3. In the former case the number of roots in the segment cannot diminish; in the latter case it cannot increase.*

It will be easily seen that

$$y = (x - e_i)^{\lambda_i}\bar{y}$$

in (1) replaces the two exponents λ_i, 0 by 0, $-\lambda_i$. The value of B is at the same time changed only by the addition of a constant. *Theorem I will, therefore, hold for the fundamental integrals $P_0^{\lambda_i}$ when the exponent λ_i is greater than -1. Subsequent theorems can in like manner be converted into theorems relating to the integrals P^{λ_i} and holding for exponent differences which are greater than -1.* Due account must, of course, be taken of the change in the values of A and B resulting from the substitution.

§4. The theorem just given leaves it uncertain whether the number of roots of the fundamental integrals in e_1e_2 and $e_3\,\infty$ increases indefinitely with B. A like doubt exists concerning the number of roots in the other two segments when B is indefinitely diminished. To decide this point let the equation (1) be deprived of its second term by the transformation

$$y = (x - e_1)^{\frac{\lambda_1 - 1}{2}} \cdots (x - e_3)^{\frac{\lambda_3 - 1}{2}} \bar{y} .$$

It then becomes

$$\frac{d^2y}{dx^2} + \left(R(x) + \frac{B}{(x - e_1) \cdots (x - e_3)}\right) y = 0,$$

in which $R(x)$ denotes a rational function which is finite except at the singular points. By decreasing B the coefficient of y can be made greater than any given positive constant c^2 for all values of x between ∞ and e_1 and by increasing B for all values between e_3 and ∞. Now the equation $y'' + c^2y = 0$ has for its general integral $A \cos cx + B \sin cx$, and any real solution has therefore an infinite number of positive, as also of negative roots, which cumulate in the vicinity of the point ∞. But a second theorem of Sturm's states that if φ and ψ are one-valued functions which are finite and continuous for any interval of the axis of x, and if for this interval $\varphi < \psi$, then between two successive roots of a real solution of $y'' + \varphi y = 0$ which are situated in this interval there must lie at least one root of every real solution of $y'' + \psi y = 0$. It follows that when a solution of (1) is real in $e_3 \infty$, the number of roots contained in this segment can be increased indefinitely by increasing B; on the other hand, when it is real in $\infty\, e_1$, the number of roots in the segment is increased indefinitely by decreasing B. These conclusions also hold for the segment e_1e_2, and e_2e_3 respectively, as will be seen by making the substitutions

$$\bar{x} - e_2 = \frac{1}{x - e_2}, \quad \cdot \quad y = (\bar{x} - e_2)^{\lambda'_\infty} \bar{y}.$$

For the first substitution exchanges e_2 and ∞, the second restores the equation to its original form, and the two together alter the value of B only by the introduction of a negative factor

$$\frac{1}{(e_2 - e_1)\,(e_2 - e_3)}$$

and by the addition of a constant. Hence

II. *If a solution of the differential equation* (1) *is real in either of the segments* e_1e_2 *or* $\infty\, e_3$, *the number of its roots in the segment increases indefinitely when* B *is increased without limit*; *if it is real in* $\infty\, e_1$ *or* e_2e_3, *the number of roots in the segment increases indefinitely when* B *is diminished without limit.*

This theorem, it will be noticed, is independent of the values of the exponent-differences.

§5. We have thus shown that the number of roots of P_i^0 in one of the two segments ending in e_i increases without limit when B is indefinitely increased and in the other when it is indefinitely decreased. When A is positive, it is possible to reduce the number of roots to zero by varying B in the opposite direction. Take, for instance, the integral P_1^0. This was so defined in (2) as to have for $x = e_1$ a positive sign. The value of its derivative is at the same time equal to M_1, an expression for which was given in (6). If we take $\frac{B}{A} > e_1$, it will have a negative sign. When, therefore, x decreases from e_1 to ∞, P_1^0 begins by increasing, and it will remain positive unless its derivative changes sign. But its derivative being, like itself, holomorphic throughout the interval considered, can only change signs by passing through zero. We then have

$$\frac{d^2P_1^0}{dx^2} = \frac{(Ax - B)P_1^0}{(x - e_1) \cdots (x - e_3)}, \tag{7}$$

which for values of x less than e_1 and hence less than $\frac{B}{A}$ has the same sign as P_1^0. In passing, for the first time, through zero, $\frac{dP_1^0}{dx}$ should, therefore, decrease with x. This, however, is impossible since it is initially negative. It follows that P_1^0 remains positive as x decreases, and hence cannot vanish between e_1 and ∞. When, on the other hand, $\frac{B}{A} < e_1$, it cannot vanish between e_1 and e_2. For suppose x to increase from e_1 to e_2. Initially, P_1^0 and $\frac{dP_1^0}{dx}$ are both positive, and the latter must be the first to vanish. But $\frac{d^2P_1^0}{dx^2}$ would then be positive, since $x > e_1 > \frac{B}{A}$. In passing, for the first time, through zero $\frac{dP_1^0}{dx}$ must therefore increase with x. This obviously is impossible. We conclude, therefore, that P_1^0 has no root between e_1 and e_2. The same line of reasoning can be extended to P_2^0 and P_3^0. Since e_1 and e_3, the extreme singular points in the finite plane, have the same function in the differential equation, we see at once that when $\frac{B}{A} < e_3$, the integral P_3^0 will have no roots in $e_3\infty$ and when $\frac{B}{A} > e_3$, none in e_2e_3. As regards P_2^0, however, the condition $\frac{B}{A} > e_2$ is sufficient only to secure that P_2^0 and its derivative shall have a positive sign for $x = e_2$. To obtain a like sign for $\frac{d^2P_2^0}{dx^2}$ when for $e_2 < x < e_3$ we set $\frac{dP_2^0}{dx} = 0$, and thereby to insure that neither P_2^0 nor its derivative shall vanish between e_2 and e_3, it will be necessary to take $\frac{B}{A} > e_3$. For a similar reason we take $\frac{B}{A} < e_1$ to insure that no roots shall lie in e_1e_2. We have, therefore, the following results:

III. *Given a differential equation of the form* (1) *in which* $\lambda_1, \lambda_2, \lambda_3$ *are less than one and* A *is positive;*

(1) *If* $\frac{B}{A} < e_1$, *no root of* P_1^0 *or* P_2^0 *can lie in* e_1e_2 *and no root of* P_3^0 *in* $e_3\infty$.

(2) *If* $\frac{B}{A} > e_3$, *no root of* P_3^0 *or* P_2^0 *can lie in* e_2e_3 *and no root of* P_1^0 *in* ∞e_1.

(3) *If* $e_1 < \frac{B}{A} < e_3$, *no root of* P_1^0 *can lie in* ∞e_1 *and no root of* P_3^0 *in* $e_3\infty$.

§6. The number of roots of P_1^0 and of P_2^0 (and also of their derivatives) contained in e_1e_2 has thus been shown to increase from 0 to ∞ when $\frac{B}{A}$ increases from e_1 to $+\infty$. The roots cannot be drawn from the imaginary domain directly into the segment, because this would necessitate first the formation of a double root at a non-singular point, which is impossible. They must therefore first make their appearance at the two extremities of the segment. The roots of P_1^0, as we have seen, move toward e_1 and hence enter the segment at e_2, say in the order $a_1, a_2, \cdots$; the roots of P_2^0, on the other hand, move toward e_2 and enter at e_1, say in the order, $b_1, b_2, \cdots$. Since the two integrals are in general independent solutions of the equation, the roots of one alternate with those of the other. Hence the number of roots of neither integral can exceed that of other by more than a unit. Suppose now that i roots of each integral have entered the segment. Immediately after the last of these has entered, their order of succession will be $b_i, a_1, b_{i-1}, \cdots, b_1, a_i$.

But if the roots are to alternate also after b_{i+1} or a_{i+1} enters the segment, $a_1, a_2, \cdots, a_i$ must cross over $b_i, b_{i-1}, \cdots, b_1$ respectively. The two integrals must then for a moment have i roots common and therefore coincide save as to a constant factor. The next coincidence will take place when they each have $i+1$ roots in the segment and so on. The first coincidence will occur when neither integral has a root in the segment. It cannot be when $\frac{B}{A} < e_1$, for the integrals themselves then have the same sign throughout the segment and their derivatives opposite signs. If, on the other hand, $\frac{B}{A} > e_1$, the same conditions hold as in §3. The sign of $\frac{d}{dB}\left(\frac{K}{P_1^0}\frac{dP_1^0}{dx}\right)$ is, namely, negative for $a = e_1 + \varepsilon_1$ and hence also as we there noted, throughout the entire segment. Furthermore, as long as $\frac{B}{A} < e_2$, the sign of $\frac{d}{dB}\left(\frac{K}{P_2^0}\cdot\frac{dP_2^0}{dx}\right)$ will be positive for $b = e_2 - \varepsilon_2$ and hence also throughout the entire segment. At any point of the segment $\frac{1}{P_1^0}\frac{dP_1^0}{dx}$ therefore diminishes as B increases, while $\frac{1}{P_2^0}\frac{dP_2^0}{dx}$ increases. Since for $\frac{B}{A} \gtreqless e_1$, the former is positive and the latter negative, they must become equal before either integral vanishes, and they clearly become equal but once. When this happens, the two solutions must coincide save as to a constant factor. The following result is now evident.

IV. *When $\frac{B}{A}$ is increased from e_1 to ∞, the integrals P_1^0 and P_2^0 will coincide, except for a constant factor, an infinite number of times, and when $\frac{B}{A}$ is decreased from e_3 to ∞, the integrals P_2^0 and P_3^0 will coincide in like manner an infinite number of times. With each consecutive coincidence of the first two integrals the number of roots of each in e_1e_2 increases by a unit, and with each coincidence of the second two integrals the number of roots of each in e_2e_3 increases also by a unit, beginning in each case with zero.*

§7. We will next compare the relative position of the i and the $i+1$ common roots, say of P_1^0 and P_2^0, in two consecutive coincidences. Let the corresponding values of B be denoted by B_i and B_{i+1}. When, after coincidence, the roots of the two integrals separate, the roots of the one move in an opposite direction from those of the other. The i roots must therefore be included between the outermost of the $i+1$ roots. Moreover, since $B_{i+1} > B_i$, the value of G_{i+1} [see (5)] will be greater than that of G_i for every value of x included between e_1 and e_2. The differential equations corresponding to the $(i+1)$th and $(i+2)$th coincidences therefore meet the conditions of the following theorem of Sturm: If in two differential equations

$$\frac{d^2(K_1y)}{dx^2} + G_1y = 0, \quad \frac{d^2(K_2y)}{dx^2} + G_2y = 0$$

G_1, K_1, G_2, K_2 are one valued and continuous for $a < x < b$, and if, moreover, throughout this interval $G_1 \lesseqqgtr G_2$ and $K_1 \geqq K_2$, then between two consecutive roots of any real solution of the first equation which are included in this interval, there will lie at least one root of a real solution of the second equation. It follows that between any two of the i roots of our two fundamental integrals when coincident for the $(i+1)$th time will lie at least one of the $i+1$ roots when coincident the next time. Combining this with our preceding conclusion, we see that the $i+1$ roots must alternate with the i roots. A similar inference can of course be made for P_2^0 and P_3^0.

V. *In two consecutive coincidences of* P_1^0 *and* P_2^0, *the* i *roots of the one coincidence which lie in* e_1e_2 *will alternate with the* $i+1$ *roots of the other coincidence. So also in two consecutive coincidences of* P_2^0 *and* P_3^0, *the* i *roots of the one coincidence which lie in* e_2e_3 *will alternate with the* $i+1$ *of the other.*

§8. For special values of A and B all three fundamental integrals coincide except for certain constant factors. We have in consequence an integral which is holomorphic over the entire plane. Since the differential equation is regular, this integral can have only a pole at ∞ and hence is a polynomial P, say of the nth degree. As before noted, A is the negative of the product of the two exponents for ∞, namely, $-n$ and $n+2-\lambda_1-\lambda_2-\lambda_3$, and is accordingly positive. Limits for the value of $\frac{B}{A}$ can be obtained immediately from Theorem IV, or found by the following considerations which permit of extension to differential equations of higher order. First it can be shown that no root of the polynomial can lie in either of the two outermost segments.* Suppose, if possible, that the smallest root is less than e_1 or the largest one greater than e_3 and let x in (1) be placed equal to either

* This is also evident from Theorem III.

root. If placed equal to the smallest root, P' and P'' have opposite signs, and if placed equal to the largest root, the same sign.* In either case the first two terms of the equation have like signs and the third term vanishes, so that the equation can not be identically satisfied. All the roots must, therefore, lie between e_1 and e_3. Next, suppose $\frac{B}{A}$ less than e_1 and place x equal to any intermediate value. Since x is now smaller than the smallest root, P'' and P have the same sign and P' the opposite sign. All three terms of the equation have in consequence the same sign and the equation again involves a contradiction. In like manner it can be proved that $\frac{B}{A} > e_3$.

VI. *If the three exponent differences are less than unity and the differential equation admits a polynomial solution, the value of* $\frac{B}{A}$ *must lie between* e_1 *and* e_3.

Theorem IV shows that no two polynomials which belong to the same exponent differences can have the same number of roots in e_1e_2 or in e_2e_3. As there are only $n+1$ polynomials in all, one and only one polynomial can be found for each different distribution of the n roots between these two segments. We have thus a new demonstration of this fact. From the same theorem it follows also :

VII. *If the* $n+1$ *Stieltjes polynomials which belong to the same set of exponent differences are so arranged that the corresponding values of* B *succeed each other in the order* $B_1 < B_2 < \cdots < B_{n+1}$, *the number of roots in* e_1e_2 *will successively be* $0, 1, 2, \cdots, n$ *and in* e_2e_3 *successively* $n, n-1, \cdots, 0$.

§9. The reasoning by which Theorem VI was established in §8 can be applied without trouble to a differential equation of the rth order

$$\frac{d^2y}{dx^2} + \left(\frac{1-\lambda_1}{x-e_1} + \cdots + \frac{1-\lambda_r}{x-e_r}\right)\frac{dy}{dx} \tag{8}$$

$$+ \frac{\varphi(x) = n(-n+r-1+\sum\lambda_i)(x-a_1)\cdots(x-a_{r-2})}{(x-e_1)\cdots(x-e_r)}\, y = 0$$

which admits a polynomial solution. If each exponent difference λ_i is less than 1, it can be proved first that the roots of the polynomial solution are included between e_1 and e_r, and then that the real roots of the accessory polynomial $\varphi(x)$ are included between the same limits. For the best

* Since the roots of P are all real.

demonstration* that all the roots of the polynomial solution are real, the reader is referred to an article by Bôcher in the April number of the BULLETIN. The method which he has there employed I shall make use of to prove that the roots of the accessory polynomial φ are likewise real. Let P denote the polynomial solution and $x_1, \cdots, x_{n-1}$ the roots of its derivative which are, of course, real. If P be substituted in the differential equation and x be placed equal to a root a of φ, we get

$$P''(a) + \left(\frac{1-\lambda_1}{a-e_1} + \cdots + \frac{1-\lambda_r}{a-e_r}\right)P'(a) = 0,$$

or dividing by $P'(a)$,

$$\frac{1}{a-x_1} + \cdots + \frac{1}{a-x_{n-1}} + \frac{1-\lambda_1}{a-e_1} + \cdots + \frac{1-\lambda_r}{a-e_r} = 0.$$

If now a is an imaginary root $p + qi$ for which q is positive, the pure imaginary part of each fraction will have a negative sign. The equation therefore involves a contradiction. Hence

VIII. *The roots of the accessory polynomial φ of the differential equation* (8) *for a Stieltjes polynomial are all real and included between the two extreme singular points, e_1 and e_r.*

WESLEYAN UNIVERSITY,
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NOTE ON STOKES'S THEOREM IN CURVILINEAR CO-ORDINATES.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THE expression for the curl of a vector point-function, when required in terms of orthogonal curvilinear coördinates, is usually obtained by direct transformation from their values in rectangular coördinates. The proof of Stokes's theorem given in my Lectures on electricity and magnetism, due to Helmholtz, can be easily adapted to curvilinear coördinates so as to prove the theorem independently of rectangular coördinates.

Let P_1, P_2, P_3 be the projections of a vector P on the

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* The proof given by Stieljes in the sixth volume of the Acta Mathematica is based upon mechanical considerations.