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LIII. On the resistance of fluids

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ment could be tried than the following. Take a piece of copper wire and cover with suboxide by gently heating over a Bunsen burner. Fuse the covered wire into a capillary glass tube. A scarlet glass is thus obtained which is black whilst hot, *i. e.* exhibits continuous absorption (1), and scarlet whilst cold, exhibiting partial absorption (2). 2 and 3 are bridged over by a host of bodies like ZnO. When hot they, as a rule, exhibit absorption at the blue end of the spectrum (2), but when cold they are white or colourless (3).

Respecting the merging of Class I. into Class II. nothing much at present can be said, since questions are involved which require further investigation.

LIII. *On the Resistance of Fluids.* *By* LORD RAYLEIGH, *F.R.S.**

[Plate V.]

THERE is no part of hydrodynamics more perplexing to the student than that which treats of the resistance of fluids. According to one school of writers, a body exposed to a stream of perfect fluid would experience no resultant force at all, any augmentation of pressure on its face due to the stream being compensated by equal and opposite pressures on its rear. And indeed it is a rigorous consequence of the usual hypotheses of perfect fluidity and of the continuity of the motion, that the resultant of the fluid pressures reduces to a *couple* tending to turn the broader face of the body towards the stream. On the other hand, it is well known that in practice an obstacle does experience a force tending to carry it down stream, and of magnitude too great to be the direct effect of friction; while in many of the treatises calculations of resistances are given leading to results depending on the inertia of the fluid without any reference to friction.

It was Helmholtz who first pointed out that there is nothing in the nature of a perfect fluid to forbid a finite slipping between contiguous layers, and that the possibility of such an occurrence is not taken into account in the common mathematical theory, which makes the fluid flow according to the same laws as determine the motion of electricity in uniform conductors. Moreover the electrical law of flow (as it may be called for brevity) would make the velocity infinite at every sharp edge encountered by the fluid; and this would require a negative pressure of infinite magnitude. It is no answer to

* Communicated by the Author, having been communicated in substance to the British Association at Glasgow.

this objection that a mathematically sharp edge is an impossibility, inasmuch as the electrical law of flow would require negative pressure in cases where the edge is not perfectly sharp, as may be readily proved from the theory of the simple circular vortex, in which the velocity varies inversely as the distance from the axis.

The application of these ideas to the problem of the resistance of a stream to a plane lamina immersed transversely amounts to a justification of the older theory as at least approximately correct. Behind the lamina the fluid is at rest under a pressure equal to that which prevails at a distance, the region of rest being bounded by a surface of separation or discontinuity which joins the lamina tangentially, and is determined mathematically by the condition of constant pressure. On the anterior surface of the lamina there is an augmentation of pressure corresponding to the loss of velocity.

The relation between the velocity and pressure in a steady stream of incompressible fluid may be obtained immediately by considering the transference of energy along an imaginary tube bounded by stream-lines. In consequence of the steadiness of the motion, there must be the same amount of energy transferred in a given time across any one section of the tube as across any other. Now if p and v be the pressure and velocity respectively at any point, and ρ be the density of the fluid, the energy corresponding to the passage of the unit of volume is $p+\frac{1}{2}\rho v^2$, of which the first term represents potential, and the second kinetic energy; and thus $p+\frac{1}{2}\rho v^2$ must retain the same value at all points of the same stream-line. It is further true, though not required for our present purpose or to be proved so simply, that $p+\frac{1}{2}\rho v^2$ retains a constant value not merely on the same stream-line, but also when we pass from one stream-line to another, provided that the fluid flows throughout the region considered in accordance with the electrical law.

If u be the velocity of the stream, the increment of pressure due to the loss of velocity is $\frac{1}{2}\rho u^2-\frac{1}{2}\rho v^2$, and can never exceed $\frac{1}{2}\rho u^2$, which value corresponds to a place of rest where the whole of the energy, originally kinetic, has become potential. The old theory of resistances went on the assumption that the velocity of the stream was destroyed over the whole of the anterior face of the lamina, and therefore led to the conclusion that the resistance amounted to $\frac{1}{2}\rho u^2$ for each unit of area exposed. It is evident at once that this is an overestimate, since it is only near the middle of the anterior face that the fluid is approximately at rest; towards the edge of the lamina the fluid moves outwards with no inconsiderable velocity, and at the edge itself retains the full velocity of the original stream. Nevertheless the amount of error involved in the theory referred to is not great, as appears from the result of Kirchhoff's calculation of the case of two dimensions, from which it follows that the resistance per unit of area is $\frac{\pi}{4+\pi}\rho u^2$ instead of $\frac{1}{2}\rho u^2$.

It is worthy of notice that by a slight modification of the conditions of the problem the estimate $\frac{1}{2}\rho u^2$ may be made accurate. For this purpose the lamina is replaced by the bottom of a box-shaped vessel, whose sides project in the direction from which the stream is flowing, and are sufficiently extended to cause approximate quiescence over the whole of the bottom (Plate V. fig. 1). In the absence of friction, the sides themselves do not contribute any thing to the resistance. It appears from this argument that the increase of resistance due to concavity can never exceed a very moderate value.

Although not very closely connected with the principal subject of this communication, it may be well to state the corresponding result in the case of a compressible fluid such as air. If p_0 be the normal pressure in the stream, a the velocity of sound corresponding to the general temperature, γ the ratio of the two specific heats, $\frac{1}{2}\rho u^2$ is replaced by

$$p_0\left\{\left(1+\frac{\gamma-1}{2}\frac{u^2}{a^2}\right)^{\frac{\gamma}{\gamma-1}}-1\right\},$$

which gives the resistance per unit of area. The compression is supposed (as in the theory of sound) to take place without loss of heat; and the numerical value of γ is 1·408.

When u is small in comparison with a, the resistance follows the same law as if the fluid were incompressible; but in the case of greater velocities the resistance increases more rapidly. The resistance to a meteor moving at speeds comparable with 20 miles per second must be enormous, as also the rise of temperature due to compression of the air. In fact it seems quite unnecessary to appeal to friction in order to explain the phenomena of light and heat attending the entrance of a meteor into the earth's atmosphere.

But although the old theory of resistance was not very wide of the mark in its application to the case of a lamina against which a stream impinges directly, the same cannot be said of the way in which the influence of obliquity was estimated. It was argued that inasmuch as a lamina moved edgeways through still fluid would create no disturbance (in the absence of friction), such an edgeways motion would produce no alteration in the resistance due to a stream perpendi-

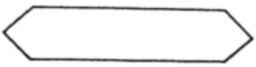


Fig. 2.

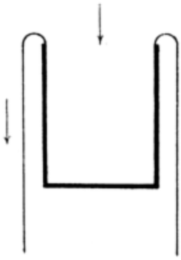


Fig. 1.

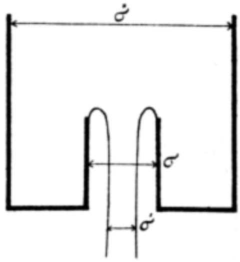


Fig. 3.

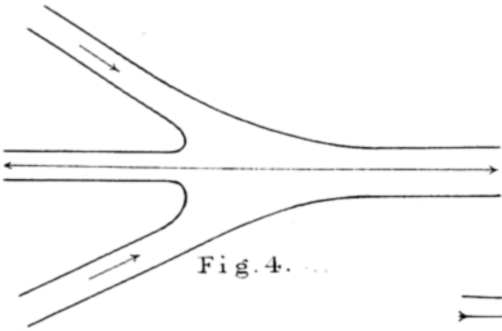


Fig. 4.

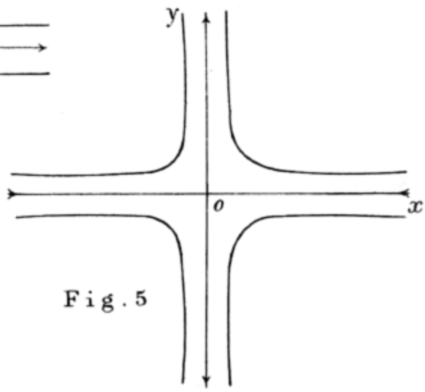
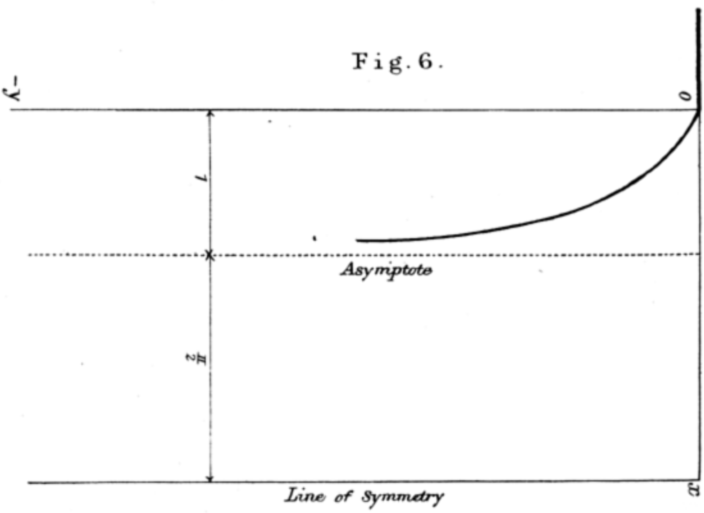


Fig. 5

Fig. 6.



Mintern Bros lith.

cular to the plane of the lamina; and from this it would follow that when a lamina is exposed to an oblique stream, the resistance experienced would be that calculated from the same formula as before, on the understanding that u now represents the *perpendicular component* of the actual velocity of the stream. Or if the actual velocity of the stream be V, and α denote the angle between the direction of the stream and the lamina, the resistance would be per unit of area

$$\tfrac{1}{2}\rho V^2 \sin^2\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This force acts of course perpendicularly to the plane of the lamina; the component down the current is

$$\tfrac{1}{2}\rho V^2 \sin^3\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The argument by which this result is obtained, however, is quite worthless; and the law of the squares of the sines expressed in (1) is known to practical men to be very wide of the mark, especially for small values of α. The resistance at high obliquities is much greater than (1) would make it, being more nearly in proportion to the first power of $\sin\alpha$ than to the square.

As a proof that an edgeways motion of an elongated body through water is not without influence on the force necessary to move it with a given speed broadways, Mr. Froude says*, "Thus when a vessel was working to windward, immediately after she had tacked and before she had gathered headway, it was plainly visible, and it was known to every sailor, that her leeway was much more rapid than after she had begun to gather headway. The more rapid her headway became, the slower became the lee-drift, not merely relatively slower, but absolutely slower."

"Again, any one might obtain conclusive proof of the existence of this increase of pressure occasioned by the introduction of the edgeways component of motion, who would try the following simple experiment. Let him stand in a boat moving through the water, and, taking an oar in his hand, let him dip the blade vertically into the water alongside the boat, presenting its face normally to the line of the boat's motion, holding the plane steady in that position, and let him estimate the pressure of the water on the blade by the muscular effort required to overcome it. When he has consciously appreciated this, let him begin to sway the blade edgeways like a pendulum, and he will at once experience a very sensible increase of pressure. And if the edgeways sweep thus as-

* Proceedings of the Society of Civil Engineers, vol. xxxii., in a discussion on a paper by Sir F. Knowles on the Screw Propeller.

signed to the blade is considerable and is performed rapidly, the greatness of the increase in the pressure will be astonishing until its true meaning has been realized. Utilizing this proposition, many boatmen, when rowing a heavy boat with narrow-bladed oars, were in the habit of alternately raising and lowering the hand with a reciprocating motion, so as to give an oscillatory dip to the blade during each stroke, and thus obtained an equally vigorous reaction from the water with a greatly reduced slip or sternward motion of the blade."

It is not difficult to see that in the case of obliquity we have to do with the whole velocity of the current, and not merely with the resolved part. Behind the lamina there must be a region of dead water bounded by a surface of discontinuity, within which the pressure is the same as if there were no obstacle. On the front face of the lamina there must be an augmentation of pressure, vanishing at the edges and increasing inwards to a maximum at the point where the stream divides. At this point the pressure is $\frac{1}{2}\rho V^2$, corresponding to the loss of the *whole* velocity of the stream. It is true that the maximum pressure prevails over only an infinitely small fraction of the area; but the same may be said even when the incidence of the stream is perpendicular.

The exact solution of the problem in the case of two dimensions which covers almost all the points of practical interest, can be obtained by the analytical method of Helmholtz and Kirchhoff*. If an elongated blade be held vertically in a horizontal stream, so that the angle between the plane of the blade and the stream is α, the mean pressure is

$$\frac{\pi \sin \alpha}{4+\pi \sin \alpha} \rho V^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

varying, when α is small, as $\sin \alpha$, and not as $\sin^2 \alpha$. The proof will be found at the end of the present paper.

The fact that the resistance to the broadways motion of a lamina through still fluid can be increased enormously by the superposition of an edgeways motion is of great interest. For example, it will be found to be of vital importance in the problem of artificial flight.

According to the old theory the component of resistance transverse to the stream varied as $\sin^2 \alpha \cos \alpha$, and attained its maximum for $\alpha = 55^\circ$ nearly. The substitution of expression

* Formulæ (3) and (4) were given at the Glasgow Meeting of the British Association. I was then only acquainted with Kirchhoff's "Vorlesungen über mathematische Physik," and was not aware that the case of an oblique stream had been considered by him (*Crelle*, Bd. 70, 1869). However, Kirchhoff has not calculated the forces; so that the formulæ are new.

(3) for $\sin^2 \alpha$ will materially modify the angle at which the transverse force is greatest. The quantity to be made a maximum is

$$\frac{\sin\alpha\cos\alpha}{4+\pi\sin\alpha};$$

and the value of α for which the maximum is attained is $\alpha=39°$ nearly, being considerably less than according to the old theory, on account of the increased value of the normal pressure at high obliquities.

The pressure, whose mean amount is given in (3), is far from symmetricalty distributed over the breadth of the blade, as might be anticipated from the fact that the region of maximum pressure, where the stream divides, is evidently nearer to the anterior or up-stream edge. If the breadth of the blade be called l, the distance (x) of the centre of pressure, reckoned from the middle, is

$$x=\frac{3}{4}\cdot\frac{\cos\alpha\, l}{4+\pi\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the blade be pivoted so as to be free to turn about an axis parallel to its edges, (4) gives the position of the axis corresponding to any angle of inclination α. If $\alpha=90°$, $x=0$, as is evident from symmetry. As α diminishes, the corresponding value of x increases and reaches a maximum, viz. $\frac{3}{16}l$, when $\alpha=0$. The axis then divides the breadth of the blade in the ratio 11 : 5.

These results may be stated in another form as follows. If the axis of suspension divide the width in a more extreme ratio than 11 : 5, there is but one position of stable equilibrium, that namely in which the blade is parallel to the stream with the narrower portion directed upwards. If the axis be situated exactly at the point which divides the width in the ratio 11 : 5, this position becomes neutral, in the sense that for small displacements the force of restitution is of the second order, but the equilibrium is really stable. When the axis is still nearer the centre of figure, the position parallel to the stream becomes unstable, and is replaced by two inclined positions given by (4), making with the stream equal angles, which increase from zero to a right angle as the axis moves in towards the centre. With the centre line itself for axis, the lamina can only remain at rest when transverse to the stream, though of course with either face turned upwards.

The fact, rather paradoxical to the uninitiated, that a blade free to turn about its centre line sets itself transversely, may be easily proved by experiment. For this purpose it is sufficient to take a piece of thin brass plate shaped as in the figure (Plate V. fig. 2), and mount it with its points bearing in two small indentations in a U-shaped strip of thicker plate, easily made by striking the strip with a conically pointed piece of steel driven by the hammer. When this little apparatus is moved through the water, the movable piece at once sets itself across the direction of motion. The same result may be observed when the apparatus is exposed to the wind; but in this case an unexpected phenomenon often masks the stability of the transverse position. It is found that when the plate is set rotating, the force of the wind will maintain or accelerate the motion. This effect might be supposed to be due to a want of symmetry, were it not that the rotation occurs in either direction. It is evidently connected with the disturbance of the fluid due to the motion of rotation, and is not covered by the calculation leading to formula (4), which refers to the forces experienced when the blade is *at rest* in any position.

I am not aware of any experimental measurements with which (4) could be compared; but the result that the equilibrium parallel to the stream is indifferent when the axis is situated in the position defined by the ratio 11 : 5, is in agreement with the construction of balanced rudders, of which the front part is usually made of about one half the width of the hinder part.

The accompanying Table contains some numerical examples of the general formulæ. The first column gives the angle between the lamina and stream, the second the value of $\sin^2\alpha$, to which, on the old theory, the resistance should be proportional; the third column is derived from some experiments by Vince on water, published in the 'Philosophical Transactions' for 1798.

α.	$\sin^2\alpha$.	Vince.	$\dfrac{\sin\alpha(4+\pi)}{4+\pi\sin\alpha}$.	$\dfrac{3}{4}\dfrac{\cos\alpha}{4+\pi\sin\alpha}$.	
90	1·0000	1·000	1·0000	·0000	·5000
70	·8830	·974	·9652	·0369	·2676
50	·5868	·873	·8537	·0752	·0981
30	·2500	·663	·6411	·1166	·0173
20	·1170	·458	·4814	·1389	·0040
10	·0302	·278	·2728	·1625	·0004

The quantity directly measured by Vince was the resolved part of the resistance in the direction of the stream, from which the tabulated number is derived by division by $\sin\alpha$. The fourth column represents the law of resistance according to the formula now proposed, a factor being introduced so as to make the maximum value unity. The fifth column gives the distance between the centre of pressure and the middle line of the blade, expressed as a fraction of the total width. The sixth column is the value of

$$\frac{2(1-2\cos\alpha+\cos^3\alpha)+\alpha\sin\alpha}{4+\pi\sin\alpha},$$

which is the distance from the anterior edge of the point where the stream divides, and where accordingly the pressure attains its greatest value. It will be seen that, as might be expected, this distance becomes small at moderate obliquities.

The result of Vince's experiments agrees with theory remarkably well; and the contrast with $\sin^2\alpha$ is especially worthy of note. The experiments were made with a whirling machine, and appear to have been carefully conducted; but they were on too small a scale to be quite satisfactory. The subject might now be resumed with advantage.

From theory it would appear that any part of the region of dead water behind the lamina might be filled up with solid matter without in any way disturbing the motion or altering the resistance; but in practice with actual fluids this statement must not be taken without qualification. If the boundary of the solid approach too nearly the natural position of the surface of separation, the intervening fluid appears to be sucked out until the lines of flow follow the surface of the obstacle. This is the state of things aimed at, and approximately attained, in well-designed ships, round which the water flows nearly according to the electrical law. The resistance is then of an entirely altered character, and depends only upon the friction against the skin.

It was observed by Sir William Thomson at Glasgow, that motions involving a surface of separation are unstable. This is no doubt the case, and is true even of a parallel jet moving with uniform velocity. If from any cause a slight swelling occurs at any point of the surface, an *increase* of pressure ensues tending not to correct but to augment the irregularity. I had occasion myself to refer to a case of this kind in a paper on Waves, published in the 'Philosophical Magazine' for April 1876. But it may be doubted whether the calculations of resistance are materially affected by this circumstance, as the pressures experienced must be nearly independent of what happens at some distance in the rear of the obstacle, where the instability would first begin to manifest itself.

The formulæ proposed in the present paper are also liable to a certain amount of modification from friction which it would be difficult to estimate beforehand, but which cannot be very considerable, if the experiments of Vince are to be at all relied on.

In the following analysis ϕ and ψ are the potential and stream functions, $z=x+iy$, $\omega=\phi+i\psi$; and it is known that the general conditions of fluid motion in two dimensions are satisfied by taking z as an arbitrary function of ω. If

$$\frac{dz}{d\omega}=\zeta=\rho\,(\cos\theta+i\sin\theta), \quad . \quad . \quad . \quad (A)$$

Kirchhoff shows that ζ represents the velocity of the steam at any point, with the exception that its modulus ρ is proportional to the reciprocal of the velocity instead of to the velocity itself. If the general velocity of the stream be unity, the condition to be satisfied along a surface of separation bounding a region of dead water is $\rho=1$. The value of ψ must of course also preserve a constant value along the same surface.

The form of ζ applicable to the present problem is

$$\zeta=\cos\alpha+\frac{1}{\sqrt{\omega}}+\sqrt{\left(\cos\alpha+\frac{1}{\sqrt{\omega}}\right)^2-1}. \quad . \quad (B)$$

When $\omega=\infty$,

$$\zeta=\cos\alpha-i\sin\alpha.$$

The surface of separation corresponds to $\psi=0$, for which value of ψ ω becomes real; and the point at which the stream divides corresponds to $\omega=0$, for which $\zeta=\infty$. For $\psi=0$ and real values of $\cos\alpha+\frac{1}{\sqrt{\omega}}$ less than unity, $\rho=1$. This portion therefore corresponds to the surface of separation, for which the pressure is constant. When $\cos\alpha+\frac{1}{\sqrt{\omega}}$ is real and greater than unity, ζ is real, indicating that the direction of motion is parallel to the axis of x. This part corresponds to the anterior face of the lamina.

The augmentation of pressure at any point is represented by $\frac{1}{2}\left(1-\frac{1}{\rho^2}\right)$, if the density of the fluid be taken as unity; and thus the whole resistance is measured by the integral

$$\int\frac{1}{2}\left(1-\frac{1}{\rho^2}\right)dl,$$

if dl represents an element of the width of the lamina. Kirchhoff shows how to change the variable of integration from l to ω. The velocity of the fluid is $\frac{d\phi}{dl}$, or, since ψ is here zero,

$\frac{d\omega}{dl}$. Thus, since ζ is real, $\pm\zeta=\rho=\frac{dl}{d\omega}$; and therefore the integral may be replaced by

$$\int \pm\frac{1}{2}\left(\zeta-\frac{1}{\zeta}\right)d\omega, \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}$$

in which all the elements are to be taken positive.

From the form of ζ in (B), it appears that

$$\frac{1}{2}\left(\zeta-\frac{1}{\zeta}\right)=\sqrt{\left(\cos\alpha+\frac{1}{\sqrt{\omega}}\right)^2-1}. \quad . \quad . \quad \text{(D)}$$

The width of the lamina l is $\int\zeta d\omega$, where the limits of integration are such as make

$$\cos\alpha+\frac{1}{\sqrt{\omega}}=\pm 1.$$

The integration may be effected by the introduction of a new variable β, where

$$\beta=\sin^2\alpha\sqrt{\omega}-\cos\alpha,$$

and the limits for β are ± 1. Thus

$$\int\left(\cos\alpha+\frac{1}{\sqrt{\omega}}\right)d\omega=\frac{\beta^2\cos\alpha}{\sin^4\alpha}+\frac{2\beta}{\sin^4\alpha}+\text{const.};$$

and therefore between the limits ± 1 we have

$$4\div\sin^4\alpha.$$

The second part of ζ may be written $\sqrt{1-\beta^2}\div\sin\alpha\sqrt{\omega}$, giving the integral

$$\int\frac{d\omega}{\sin\alpha\sqrt{\omega}}\sqrt{1-\beta^2}=\frac{\beta\sqrt{1-\beta^2}}{\sin^3\alpha}+\frac{\sin^{-1}\beta}{\sin^3\alpha}+\text{const.}$$

Thus the complete value of z between the limits, or l, is

$$l=\frac{4}{\sin^4\alpha}+\frac{\pi}{\sin^3\alpha}=\frac{4+\pi\sin\alpha}{\sin^4\alpha}. \quad . \quad . \quad . \quad \text{(E)}$$

By (C) and (D) the whole pressure on the lamina is represented by the second part of l in (E), or $\pi\div\sin^3\alpha$; so that the mean pressure is

$$\frac{\pi}{\sin^3\alpha}\div\frac{4+\pi\sin\alpha}{\sin^4\alpha}=\frac{\pi\sin\alpha}{4+\pi\sin\alpha},$$

as was to be proved.

Again, the elementary moment of pressure about $z=0$ is

$$\pm\frac{1}{2}\left(\zeta-\frac{1}{\zeta}\right)d\omega\,.\,z=\frac{2\sqrt{1-\beta^2}}{\sin^3\alpha}\,z\,d\beta.$$

Now if the arbitrary constant be taken suitably, the complete value of z is

$$z = \frac{\beta^2 \cos\alpha + 2\beta}{\sin^4\alpha} + \frac{\beta\sqrt{1-\beta^2} + \sin^{-1}\beta}{\sin^3\alpha}.$$

The odd terms in z will contribute nothing to the integral; and therefore we may take for the moment of pressure about $z=0$,

$$\int_{-1}^{+1} \frac{2\sqrt{1-\beta^2}}{\sin^3\alpha} \cdot \frac{\beta^2 \cos\alpha}{\sin^4\alpha} d\beta = \frac{\pi}{\sin^3\alpha} \cdot \frac{\cos\alpha}{4\sin^4\alpha}.$$

In this result the first factor represents the total pressure, and therefore $\frac{\cos\alpha}{4\sin^4\alpha}$ expresses the distance of the centre of pressure from the point $z=0$. With the same origin the value of z for the middle of the lamina is $\frac{\cos\alpha}{\sin^4\alpha}$; and thus the displacement of the centre of pressure from the middle of the lamina is

$$-\frac{3}{4}\frac{\cos\alpha}{\sin^4\alpha}.$$

This distance must now be expressed in terms of l or

$$4+\pi\sin\alpha \div \sin^3\alpha,$$

which gives as the final result,

$$-\frac{3}{4}\ \frac{\cos\alpha \,.\, l}{4+\pi\sin\alpha}.$$

The negative sign indicates that the centre of pressure is on the upstream side of the middle point.

As to the form of the surface of separation, its intrinsic equation is given at once by the value of ζ in terms of ω. The real part of ζ is $\cos\theta$ (since $\rho=1$), where θ is the angle between the tangent at any point and the plane of the lamina. Along the surface of separation ω is identical with ϕ, and $\frac{d\phi}{ds}=1$. Thus if s be the length of the arc of either branch measured from the point where it joins the lamina, the intrinsic equation is

$$\cos\theta = \cos\alpha \pm \frac{1}{\sqrt{s+c}},$$

and the constant is to be determined by the condition that $s=0$ when $\cos\theta = \pm 1$. Since $\cos\theta = \frac{dx}{ds}$, the relation between x and s is readily obtained on integration; but the relation between y and s is more complicated.

In the case of perpendicular incidence $\cos\alpha=0$, $c=1$, so that

$$\cos\theta=\pm\frac{1}{\sqrt{s+1}},$$

giving on integration

$$x=2\sqrt{s+1}+\text{const.}$$

It appears that the value of x does not approach a finite limit as s increases indefinitely.

LIV. *Notes on Hydrodynamics.* *By* Lord Rayleigh, *F.R.S.**

[Plate V.]

The Contracted Vein.

THE contraction of a jet of fluid in escaping from a higher to a lower pressure through a hole in a thin plate has been the subject of much controversy. Of late years it has been placed in a much clearer light by a direct application of the principle of momentum to the circumstances of the problem by Messrs. Hanlon and Maxwell † among others.

For the sake of simplicity the liquid will be supposed to be unacted upon by gravity, and to be expelled from the vessel by the force of compressed air through a hole of area σ in a thin plane plate forming part of the sides of the vessel. After passing the hole the jet contracts, and at a little distance assumes the form of a cylindrical bar of reduced area σ'. The ratio $\sigma':\sigma$ is called the coefficient of contraction.

The velocity acquired by the fluid in escaping from the pressure p is determined, in the absence of friction, by the principle of energy alone. If the density of the fluid be unity, and the acquired velocity v,

$$v^2=2p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The product of v, as given by (1), and σ is sometimes, though very improperly, called the theoretical discharge; and it differs from the true discharge for two reasons. In the first place, the velocity of the fluid is not equal to v over the whole of the area of the orifice. At the edge, where the jet is free, the velocity is indeed v; but in the interior of the jet the pressure is above atmosphere, and therefore the velocity less than v. And, secondly, it is evident that the quantity of fluid passing the orifice depends, not upon the whole velocity with

* Communicated by the Author.

† Proceedings of the Mathematical Society, November 11, 1869.