Product action

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Abstract

This paper studies the cycle indices of products of permutation groups. The main focus is on the product action of the direct product of permutation groups. The number of orbits of the product on n-tuples is trivial to compute from the numbers of orbits of the factors; on the other hand, computing the cycle index of the product is more intricate. Reconciling the two computations leads to some interesting questions about substitutions in formal power series. We also discuss what happens for infinite (oligomorphic) groups and give detailed examples. Finally, we briefly turn our attention to generalised wreath products, which are a common generalisation of both the direct product with the product action and the wreath product with the imprimitive action.

1 Introduction

Given two permutation groups (G_1, X_1) and (G_2, X_2), there are two 'natural' actions for the direct product and two for the wreath product, as follows. For the direct product $G_1 \times G_2$, we have the *intransitive action* $(G_1 \times G_2, X_1 \cup X_2)$, where the union is assumed disjoint; and the *product action* $(G_1 \times G_2, X_1 \times X_2)$. For the wreath product $G_1 \wr G_2$, we have the *imprimitive action* $(G_1 \wr G_2, X_1 \times X_2)$, and the *power action* $(G_1 \wr G_2, X_1^{X_2})$ (sometimes also called the product action).

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We are interested in calculating the cycle index of these products, and its specialisations including the number of orbits on n-tuples and on n-sets. For the intransitive and imprimitive actions, there are well-known techniques for this, which we outline in the next section. However, for the power and product action, things are less simple. For the product action of the direct product, the cycle index can be calculated by an operation which we describe. The number of orbits on n-tuples is obtained from the corresponding numbers for the factors simply by multiplying them. It is not obvious how these two operations are related; we discuss this in detail in the third section of the paper. In the fourth section we make some preliminary remarks on the more complicated problems for power action of wreath products.

Bailey *et al.* defined a *generalised wreath product* of a family of permutation groups indexed by a poset. This reduces to the product action for direct product and to the imprimitive action for wreath product. In the final section of the paper we discuss this construction and outline what is known about enumeration.

2 Preliminaries

This section contains definitions of the actions of products that we consider, and a summary of known material about cycle index.

2.1 Actions of direct and wreath products

Let (G_1,X_1) and (G_2,X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the disjoint union $X_1 \cup X_2$ by the rule

$$x(g_1,g_2) = \begin{cases} xg_1 & \text{if } x \in X_1, \\ xg_2 & \text{if } x \in X_2 \end{cases},$$

and on the Cartesian product $X_1 \times X_2$ by the rule

$$(x_1,x_2)(g_1,g_2) = (x_1g_1, x_2g_2).$$

Note that $X_1 \times X_2$ is naturally identified with the set of transversals of the two sets X_1 and X_2 in the disjoint union.

By $G_1 \wr G_2$ we mean the *permutational wreath product*, the split extension of the base group $B = G_1^{X_2}$ by G_2 (permuting the factors of the direct product in the way it acts on X_2). It acts on the Cartesian product $X_1 \times X_2$ by the rule

$$(x_1,x_2)f = (x_1 f(x_2), x_2), \qquad (x_1,x_2)g = (x_1, x_2g),$$

and on $X_1^{X_2}$ by the rule

$$(\phi f)(x_2) = (\phi(x_2))(f(x_2)), \qquad (\phi g)(x_2) = \phi(x_2 g^{-1}),$$

for $f \in B = G_1^{X_2}$, $g \in G_2$, and $\phi \in X_1^{X_2}$. Again, there is a natural identification of $X_1^{X_2}$ with the set of transversals for the copies $X_1 \times \{x_2\}$ of X_1 in $X_1 \times X_2$.

2.2 Cycle index of products

The *cycle index* of a finite permutation group (G,X) is

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{n} s_i^{c_i(g)},$$

where $n = |X|$, $s_1, \ldots, s_n$ are indeterminates, and $c_i(g)$ is the number of i-cycles in the cycle decomposition of g. We denote the result of substituting z_i for s_i in $Z(G)$ by $Z(G; s_i \leftarrow z_i)$.

Knowledge of the cycle index enables various orbit-counting to be done. We let $f_n(G)$, $F_n(G)$ and $F_n^*(G)$ be the numbers of orbits of G on n-element subsets, n-tuples of distinct elements, and all n-tuples of elements of X respectively; and we let $f_G(t)$, $F_G(t)$, $F_G^*(t)$ be the ordinary generating function $\sum_{n \geq 0} f_n(G) t^n$ and the exponential generating functions $\sum_{n \geq 0} F_n(G) t^n / n!$ and $\sum_{n \geq 0} F_n^*(G) t^n / n!$ respectively. Then

$$\begin{aligned} f_G(t) &= Z(G; s_i \leftarrow t^i + 1), \\ F_G(t) &= Z(G; s_1 \leftarrow t + 1, s_i \leftarrow 1 \text{ for } i > 1), \\ F_G^*(t) &= Z(G; s_1 \leftarrow \mathrm{e}^t, s_i \leftarrow 1 \text{ for } i > 1). \end{aligned}$$

Note that

$$F_G^*(t) = F_G(\mathrm{e}^t - 1).$$

This equation can also be expressed as

$$F_n^*(G) = \sum_{k=1}^{n} S(n,k) F_k(G),$$

where $S(n,k)$ are the Stirling numbers of the second kind; in other words, the sequence $(F_n^*(G))$ is the *Stirling transform* of $(F_n(G))$ [3]. Hence we can recover the second sequence from the first by the *inverse Stirling transform*:

$$F_n(G) = \sum_{k=1}^{n} s(n,k) F_k^*(G),$$

where $s(n,k)$ are the Stirling numbers of the first kind.

The cycle indices of direct and wreath products, with the intransitive and imprimitive actions respectively, are given by

$$\begin{aligned} Z(G_1 \times G_2) &= Z(G_1)Z(G_2), \\ Z(G_1 \wr G_2) &= Z(G_2; s_i \leftarrow Z(G_1, s_j \leftarrow s_{i\cdot j})). \end{aligned}$$

This paper is mostly about the cycle indices of these groups in the product and power actions.

2.3 Oligomorphic groups

It is sometimes convenient to extend these definitions to infinite permutation groups. Such a group (G,X) is said to be *oligomorphic* if G has only a finite number of orbits on X^n for all natural numbers n.

For (G,X) a (finite or) oligomorphic permutation group, we define the *modified cycle index* $\tilde{Z}(G)$ by the rule

$$\tilde{Z}(G) = \sum_{\Delta} Z(G^{\Delta}_{\Delta}),$$

where G^{Δ}_{Δ} denotes the permutation group on Δ induced by its setwise stabiliser in G, and the sum is over a set of representatives of the G-orbits on finite subsets of X.

If it happens that G is a finite permutation group, then we have nothing new:

$$\tilde{Z}(G) = Z(G; s_i \leftarrow s_i + 1).$$

Some particular oligomorphic groups of interest to us are:

- S, the symmetric group on an infinite set;
- A, the group of order-preserving permutations of the rational numbers;
- C, the group of permutations preserving the cyclic order on the set of complex roots of unity.

See [2] for further details.

We note one example here. If $G = S$, then G is n-transitive for all $n \geq 0$, and so

$$F^*_n(S) = \sum_{k=1}^{n} S(n,k) = B(n)$$

the nth *Bell number* (the number of partitions of an n-set). Using the imprimitive action of the wreath product, we find also that

$$F_n(S \wr S) = B(n)$$

(it is not difficult to construct a bijection between $S \wr S$-orbits on n-tuples of distinct elements and S-orbits on arbitrary n-tuples); and so

$$F_n^*(S \wr S) = \sum_{k=1}^{n} S(n,k)B(k).$$

This is the number of (possibly improper) chains $\pi_1 \leq \pi_2$ in the poset of partitions of an n-set ordered by refinement, and is sequence A000258 in the *Encyclopedia of Integer Sequences* [5].

3 Product action of direct product

In this section we consider the product action of the direct product. Changing notation slightly, we have permutation groups (G,X) and (H,Y), and are interested in $G \times H$ in its action on $X \times Y$.

In what follows we shall discuss how the sequences associated with a direct product of permutation groups (in the product action) are related to the sequences of the factors. We shall see that the tamest sequence in this regard is (F_n^*), for which $F_n^*(G \times H) = F_n^*(G)F_n^*(H)$ holds. This is because an n-tuple of pairs is determined by the n-tuples of its first and second components, and this correspondence respects the action of $G \times H$.

The sequence (F_n) and the cycle index are also in principle easy to compute, although less immediately, while (f_n) tends to be, more often than not, quite wild.

In the former part we deal mostly with finite groups. In the latter part we shall study the sequences for groups obtained as products of the groups S, A and C; in particular, for $S \times S$, $A \times A$, and $C \times C$.

3.1 Cycle index

Take an i-cycle in a permutation $g \in G$ and a j-cycle in a permutation $h \in H$. The pair (g,h) acts on the product of the supports of these two cycles as $\gcd(i,j)$ cycles each of length $\mathrm{lcm}(i,j)$. Hence the cycle index of $G \times H$ can be computed as follows: define $s_i \circ s_j = (s_{\mathrm{lcm}(i,j)})^{\gcd(i,j)}$, and extend multiplicatively to arbitrary monomials and then additively to arbitrary polynomials. Then

$$Z(G\times H) = Z(G)\circ Z(H).$$

The equality $F_n^*(G\times H) = F_n^*(G)F_n^*(H)$ will be deduced from this fact.

In what follows, we often have to substitute $s_1 \leftarrow \mathrm{e}^t$ and $s_i \leftarrow 1$ for $i > 1$ into a cycle index; we denote this particular substitution by (C). We also use the notation $[x^n]A(x)$, where $A(x)$ is a power series, to denote the coefficient of x^n in $A(x)$. Now we have:

$$\begin{aligned}
F_n^*(G\times H) &= \left[\frac{t^n}{n!}\right] F_{G\times H}^*(t) && (1)\\
&= \left[\frac{t^n}{n!}\right] F_{G\times H}(\mathrm{e}^t - 1) && (2)\\
&= \left[\frac{t^n}{n!}\right] Z(G\times H;(C)) && (3)\\
&= \left[\frac{t^n}{n!}\right] \left(Z(G)\circ Z(H);(C)\right). && (4)
\end{aligned}$$

The equality in (1) is just the definition of the exponential generating function $F_G^*(t)$. That in (2) relates the sequences (F_n) and (F_n^*), and that in (3) relates them to the cycle index of G, as described earlier.

On the other hand, we have:

$$\begin{aligned}
F_n^*(G)F_n^*(H) &= \left[\frac{t^n}{n!}\right] F_G^*(t) \left[\frac{t^n}{n!}\right] F_H^*(t) && (5)\\
&= \left[\frac{t^n}{n!}\right] \left(F_G^*(t) \bullet F_H^*(t)\right) && (6)\\
&= \left[\frac{t^n}{n!}\right] \left(Z(G;(C)) \bullet Z(H;(C))\right). && (7)
\end{aligned}$$

We have denoted by $\bullet$ the operation between exponential generating functions given by

$$\sum \frac{a_n t^n}{n!} \bullet \sum \frac{b_n t^n}{n!} := \sum \frac{a_n b_n t^n}{n!},$$

that is, the operation induced on the e.g.f. by the termwise product of the corresponding sequences.

So we have to prove the equality between (4) and (7). Here it is, slightly rephrased.

Proposition 3.1 *If A and B are polynomials in s_1, s_2,...,*

$$(A \circ B)((C)) = A((C)) \bullet B((C)).$$

Proof Firstly, the thesis holds for the s_is:

$$(s_1 \circ s_1)((C)) = s_1((C)) = \mathrm{e}^t, \text{ and } s_1((C)) \bullet s_1((C)) = \mathrm{e}^t \bullet \mathrm{e}^t = \mathrm{e}^t;$$

for $i > 1$,

$$(s_1 \circ s_i)((C)) = s_i((C)) = 1, \text{ and } s_1((C)) \bullet s_i((C)) = \mathrm{e}^t \bullet 1 = 1;$$

and finally, for i and j both greater than 1,

$$(s_i \circ s_j)((C)) = (s_{\mathrm{lcm}(i,j)})^{\mathrm{gcd}(i,j)}((C)) = 1, \text{ and } (s_i((C)) \bullet s_j((C)) = 1 \bullet 1 = 1.$$

This holds for monomials as well. In fact, assuming $a < b < \ldots < z$,

$$(s_a^{m_a} s_b^{m_b} \ldots s_z^{m_z})((C)) = s_a^{m_a}((C))$$

(that is, is equal to 1 if $a > 1$, or to e^{m_a} if $a = 1$). So, we can limit ourselves to considering polynomials consisting only of monomials in which a single indeterminate appears.

$$\begin{aligned}
&\Big((s_1^l + s_i^m) \circ (s_1^p + s_j^q)\Big)((C)) \\
&\qquad = \Big(s_1^l \circ s_1^p + s_1^l \circ s_j^q + s_i^m \circ s_1^p + s_i^m \circ s_j^q\Big)((C)) \\
&\qquad = \Big(s_1^{lp} + s_j^{lq} + s_i^{mp} + (s_{\mathrm{lcm}(i,j)})^{mq \cdot \mathrm{gcd}(i,j)}\Big)((C)) \\
&\qquad = \mathrm{e}^{lpt} + 3;
\end{aligned}$$

and

$$\begin{aligned}
& (s_1^l + s_i^m)((C)) \bullet (s_1^p + s_j^q)((C)) \\
= \;& (\mathrm{e}^{lt} + 1) \bullet (\mathrm{e}^{pt} + 1) \\
= \;& \left\{2, \sum_{r_1+\ldots+r_l=n} \binom{n}{r_1, \ldots, r_l}\right\}_{n=1}^{\infty} \bullet \left\{2, \sum_{s_1+\ldots+s_p=n} \binom{n}{s_1, \ldots, s_p}\right\}_{n=1}^{\infty} \\
= \;& \left\{4, \sum_{a_1+\ldots+a_{lp}=n} \binom{n}{a_1, \ldots, a_{lp}}\right\}_{n=1}^{\infty} \\
= \;& \mathrm{e}^{lpt} + 3.
\end{aligned}$$

$\diamondsuit$

Here we have identified a sequence and its exponential generating function, and used the notation (from Wilf [9]) that denotes by $\{b_n\}_{n=0}^{\infty}$ the sequence corresponding to the e.g.f. $\sum_n b_n t^n/n!$. Expressions for the terms of products and powers of e.g.f.s can also be found in Wilf's book.

The fact that the equality $\sum\binom{n}{r_1,\ldots,r_l}\sum\binom{n}{s_1,\ldots,s_p} = \sum\binom{n}{a_1,\ldots,a_{lp}}$ holds is for instance a consequence of it being just $l^n \cdot p^n = (lp)^n$ in disguise.

3.2 Which substitutions work?

One could ask what happens to the equality in Prop. 3.1 if one substitutes for the indeterminates s_i generic functions $f_i(t) = \sum_{n\geq 0} f_{i,n}t^n/n!$. Here is a partial answer.

Let us see for which f_is one gets

$$(s_j \circ s_k; s_i \leftarrow f_i(t)) = (s_j; s_i \leftarrow f_i(t)) \bullet (s_k; s_i \leftarrow f_i(t)).$$

We have

$$\begin{aligned}
& (s_j \circ s_k; s_i \leftarrow f_i(t)) \\
= \; & ((s_{\mathrm{lcm}(j,k)})^{\gcd(j,k)}; s_i \leftarrow f_i(t)) \\
= \; & \left(f_{\mathrm{lcm}(j,k)}(t)\right)^{\gcd(j,k)} \\
= \; & \left(\sum_{n\geq 0} \frac{f_{\mathrm{lcm}(j,k),n}\, t^n}{n!}\right)^{\gcd(j,k)} \\
= \; & \sum_{n\geq 0} \left(\sum_{r_1+\cdots+r_{\gcd(j,k)}=n} \binom{n}{r_1 \cdots r_{\gcd(j,k)}} f_{\mathrm{lcm}(j,k),r_1} \cdots f_{\mathrm{lcm}(j,k),r_{\gcd(j,k)}}\right) \frac{t^n}{n!}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (s_j; s_i \leftarrow f_i(t)) \bullet (s_k; s_i \leftarrow f_i(t)) \\
= \; & f_j(t) \bullet f_k(t) \\
= \; & \sum_{n\geq 0} \left[\frac{t^n}{n!}\right] f_j(t) \left[\frac{t^n}{n!}\right] f_k(t) \frac{t^n}{n!} \\
= \; & \sum_{n\geq 0} \frac{f_{j,n} f_{k,n}}{n!} t^n.
\end{aligned}$$

Thus, we are asking for conditions on the functions $f_i(t)$ under which the following happens:

$$f_{j,n}\, f_{k,n} = \sum_{r_1+\cdots+r_{\gcd(j,k)}=n} \binom{n}{r_1 \cdots r_{\gcd(j,k)}} f_{\mathrm{lcm}(j,k),r_1} \cdots f_{\mathrm{lcm}(j,k),r_{\gcd(j,k)}}. \tag{8}$$

If we examine what happens for the first few coefficients, i.e., for $n = 0, 1, 2, \ldots$ we find, not too surprisingly:

$$f_{j,0}\, f_{k,0} = (f_{\mathrm{lcm}(j,k),0})^{\gcd(j,k)}, \tag{9}$$

that is an analogue of the defining relations for the product between s_is (but one must remark that here we are considering numbers, not indeterminates). The next steps are less enlightening:

$$\begin{aligned} f_{j,1}\, f_{k,1} &= D \cdot f_{L,1}\, (f_{L,0})^{D-1}, \\ f_{j,2}\, f_{k,2} &= D \cdot f_{L,2}\, (f_{L,0})^{D-1} + D(D-1) \cdot (f_{L,1})^2\, (f_{L,0})^{D-2}, \end{aligned}$$

having denoted $\gcd(j,k)$ by D and $\mathrm{lcm}(j,k)$ by L.

We can describe quite explicitly the terms of the sequence $(f_{i,0})$ by means of the following proposition, which describes the consequences of the relation (9).

Proposition 3.2 *Let (a_i) be a sequence of natural numbers such that*

$$a_i a_j = (a_{\mathrm{lcm}(i,j)})^{\gcd(i,j)}.$$

Then:

1. *all terms in the sequence are 0 or 1 (except possibly a_2);*
2. *the sequence is multiplicative (i.e., for i, j coprime, $a_{ij} = a_i a_j$); so it is determined by its terms of prime power index;*
3. *if p is a prime and $a_{p^k} = 0$ then $a_{p^l} = 0$ for each $l > k$ (thus, for p odd, one has $\ldots = a_{p^{N-1}} = a_{p^N} = 1$ and $a_{p^{N+1}} = a_{p^{N+2}} = \ldots = 0$ for some N).*

Proof 1. For each i, $a_i a_i = (a_i)^i$: so, if $a_i \neq 0$, $i = 2$ or else $(a_i)^{i-2} = 1$. In the latter case a_i is a $(i-2)$th root of unity; if we restrict ourselves to natural numbers, it has to be 1. In the former case we have no restraints on the values of a_2; but, as $a_2\, a_{2k} = (a_{2k})^2$ (for any natural k), if there is a k such that $a_{2k} \neq 0$, we have $a_2 = a_{2k} = 1$.

2. Obvious.

3. If $k < l$, $a_{p^k} a_{p^l} = (a_{p^l})^{p^k} = a_{p^l}$ etc.

$\diamondsuit$

Analogous, but less neat, descriptions can be given for the sequences $(f_{i,1})$ (whose terms turn out to be 0 or i), $(f_{i,2})$ (with terms 0, i, $i(1-i)$ or i^2) etc.

We can also fix our attention on a sequence $(f_{i,n})$ for a fixed i (which is more meaningful, as this is the sequence of the coefficient of $\sum_{n\geq 0} f_{i,n}t^n/n! = f_i(t)$). The equation (8), setting $j = k = i$, gives a recursion for the terms of the sequence $(f_{i,n})$ (fixed i):

$$f_{i,n}^2 = \sum_{r_1+\cdots+r_i=n} \binom{n}{r_1 \cdots r_i} f_{i,r_1} \cdots f_{i,r_i}.$$

Unfortunately, this recursion is quite unwieldy due to the appearance in it of products of i terms. However, at least for $i = 1$ and $i = 2$ it yields useful descriptions of $(f_{i,n})$.

For $i = 1$ it becomes just $f_{1,n}^2 = f_{1,n}$; thus, each term of the sequence has to be 0 or 1 (when they are all equal to 1, we get back $f_1(t) = \mathrm{e}^t$, where we started from).

Taking $i = 2$ gives

$$f_{2,n}^2 = \sum_{r=0}^{n} \binom{n}{r} f_{2,r} f_{2,n-r}.$$

If we take $f_{2,0} = 0$ or 1, we get respectively $f_{2,n} = \pm\sqrt{\sum_{r=1}^{n-1} \binom{n}{r} f_{2,r} f_{2,n-r}}$ and $f_{2,n} = 1 \pm \sqrt{1 + \sum_{r=1}^{n-1} \binom{n}{r} f_{2,r} f_{2,n-r}}$. The solution obtained by taking $f_{2,0} = 1$ and then always the sign "+" is $f_{2,n} = 2^n$.

3.3 The general case

The equality $F_n^*(G \times H) = F_n^*(G)F_n^*(H)$ holds in general (for finite or oligomorphic permutation groups). This makes computing the number of orbits on n-tuples of a direct product a somewhat easy task.

Given any two oligomorphic groups G and H acting on X and Y respectively, if we know their F_n-sequences, there is a straightforward way to work out the number of orbits on n-tuples of distinct elements of $X \times Y$:

- take $(F_n(G))$ and $(F_n(H))$;

- Stirling-transform them to obtain $(F_n^*(G))$ and $(F_n^*(H))$;
- multiply them to obtain $(F_n^*(G \times H))$;
- Stirling-invert it to obtain $(F_n(G \times H))$.

Example: $S \times S$ Let us turn our attention to the action of $S \times S$ on $\Omega \times \Omega$.

We start with the action on n-sets. The group is clearly transitive, so that $f_1 = 1$. There are three orbits on 2-sets: denoting by $\{(a,\alpha),(b,\beta)\}$ a generic 2-set, the orbits correspond to $a = b$, $\alpha = \beta$ or neither. A set of representatives for the six orbits on 3-sets is obtained, denoting by $\{(a,\alpha),(b,\beta),(c,\gamma)\}$ a generic set, from the following possibilities:

1. $a \neq b \neq c \neq a$ and $\alpha \neq \beta \neq \gamma \neq \alpha$;
2. $a = b = c$ and $\alpha \neq \beta \neq \gamma \neq \alpha$;
3. $a \neq b \neq c \neq a$ and $\alpha = \beta = \gamma$;
4. $a = b \neq c$ and $\alpha \neq \beta \neq \gamma \neq \alpha$;
5. $a \neq b \neq c \neq a$ and $\alpha = \beta \neq \gamma$;
6. $a \neq b = c$ and $\alpha = \beta \neq \gamma$.

In general, an orbit on n-sets in this action is determined by how many of the first components are equal to each other, plus the same for second components, plus how elements of equality classes of first components appear paired with those for second components. In other words, an orbit identifies (not univocally) two partitions of n.

The set of orbits on n-sets is in bijection with at least two other easily described sets: the set of binary (0-1) matrices with exactly n entries equal to 1 and no zero row or column, up to row and column permutations; and the set of bipartite graphs with a distinguished block, with n edges and no isolated vertex, up to isomorphism.

For the orbits on n-tuples we have pretty analogous correspondences, this time with labelled versions of those matrices or graphs. The analogue of considering binary matrices is taking matrices as above, with exactly one entry equal to 1, one equal to 2, ..., one equal to n, and the rest zero. The analogue of the graph interpretation is considering bipartite graphs as above with the edges labelled 1 to n.

While calculating the numbers $f_n(S \times S)$ appears to be difficult, we can use the procedure given above to work out $F_n(S \times S)$. We know that $F_n^*(S) = B_n$, the nth Bell number, and it is easy to see that with each partition of $\{1,2,\dots,n\}$ we can associate an orbit on n-tuples of not necessarily distinct elements, and vice versa. For instance, with the partition $\{\{1,3,4\},\{2,5\}\}$ we associate the orbit containing (a,b,a,a,b) ($a \neq b$).

So $F_n^*(S \times S)$ is equal to B_n^2 and an orbit on n-tuples of pairs corresponds to a pair of partitions of $\{1,2,\dots,n\}$: for instance with the pair of partitions

$$(\{\{1,3,4\},\{2,5\}\},\{\{1,4\},\{2,5\},\{3\}\})$$

we may associate the orbit containing

$$((a,x),(b,y),(a,z),(a,x),(b,y))$$

($a \neq b$, $x \neq y \neq z \neq x$).

Stirling-inverting $F_n^*(S \times S)$, we find that $F_n(S \times S) = \sum_{i=1}^n s(n,i) B_i^2$.

A generic pair of partitions corresponds to an n-tuple with repeated elements; to obtain n-tuples of distinct elements, we have to add the condition that the two partitions have meet $\{\{1\},\{2\},\dots,\{n\}\}$ (where meet means the coarsest common refinement). (See the papers by Pittel ([7]) and Canfield ([4]).) The sequence $F_n(S \times S)$ is sequence A059849 in Sloane [5].

The above generalises in a natural way to the product of k copies of S in the product action: one has $F_n^*(S^k) = B_n^k$, and $F_n(S^k) = \sum_{i=1}^n s(n,i) B_i^k$.

Example: $A \times A$ The links between the sequences counting orbits on n-sets and n-tuples can be well described for the groups $G = A$, $A \times A$, $A \times A \times A$, ...

The key observation is that the group induced by such a G on n points (elements of Ω, $\Omega \times \Omega$, ...) is trivial. Therefore each orbit on n-sets gives rise to exactly $n!$ orbits on n-tuples of distinct elements, so that the ratio between the $F_n(G)/f_n(G)$ is equal to $n!$ for each n.

Let us now apply the procedure described above to the group $G = A \times A$ acting on $\mathbf{Q} \times \mathbf{Q}$.

Recall that for A one has $f_n = 1$ and $F_n = n!$ for each n. Applying the Stirling transform to $F_n(A)$, we get $F_n^*(A)$, which also gives the number of labelled total preorders, also called weak orders or preferential arrangements (this is sequence A000670 in [5]). The remaining steps of the procedure give $F_n(A \times A)$; dividing by $n!$ we obtain $f_n(A \times A)$. Using GAP [6], we find the first terms to be 1, 4, 24, 196, 2016, 24976, 361792, ...

Also in this situation one can give bijections between orbits and other structures: matrices, bipartite graphs, pairs of partitions.

Here we have one orbit on n-sets for each binary matrix with exactly n entries 1 (without allowing permutations on rows or columns); and one orbit on n-tuples of distinct elements for each matrix with entries 1, 2,..., n (one each) and zero elsewhere.

As for graphs, we consider here bipartite graphs with a total ordering on each of the blocks; label the edges to get the correspondence with orbits on n-tuples.

Lastly, the correspondence with pairs of partitions with meet $\{\{1\},\{2\},\ldots,\{n\}\}$ requires the additional condition for each of the partitions to be ordered (that is to be an ordered list of subsets of $\{1,2,\ldots,n\}$).

Example: $C\times C$ We may finally sketch what happens for the group $C\times C$; recalling that $f_n(C) = 1$ and $F_n(C) = (n-1)!$, one can apply the procedure to work out $F_n(C\times C)$. It is also straigthforward to describe the analogue of the bijections: for instance, orbits on n-sets correspond to binary matrix as above up to cyclic permutations of rows and columns.

4 Power action of wreath product

We do not have a convenient expression for the cycle index of a wreath product in the power action. For the orbits on n-tuples, we have the following result.

Proposition 4.1 *Let $G = G_1 \wr G_2$, in the power action. Then*

$$F_n^*(G) = Z(G_2; s_i \leftarrow F_n(G_1)^i).$$

Proof If $B = G_1^m$ is the base group, then each orbit of B on n-tuples is indexed by an m-tuple of orbits of G_1 on n-tuples. Taking the G_1-orbits on n-tuples as figures, each B-orbit is a function from $\{1,\ldots,m\}$ to the set of figures, and G-orbits on n-tuples correspond to G_2-orbits on such functions. The result follows from the Cycle Index Theorem. $\diamondsuit$

5 Generalised wreath products

Let I be a set with partial order ρ. Suppose that a permutation group (G_i, X_i) is associated with each element $i\in I$. Bailey *et al.* [1] defined the *generalised wreath product* $(G,X) = \prod_{i\in I}(G_i, X_i)$, in such a way that

- X is the Cartesian product $\prod_{i\in I} X_i$;
- if I is an antichain of size 2, then the generalised wreath product is the direct product with the product action;
- if I is a chain of size 2, then the generalised wreath product is the wreath product with the imprimitive action.

The generalised wreath product is defined as follows. For each $i \in I$, we define the group F_i to be the direct product of copies of G_i indexed by $\prod_{j>i} X_j$. The factor corresponding to an element $(x_j : j > i)$ in the product acts as follows. Take any element $(x'_k : k \in I)$ of X. If $x'_j = x_j$ for all $j > i$, then G_i acts on the ith coordinate; otherwise, G_i acts trivially.

Now the generalised wreath product $\prod_{i\in I}(G_i, X_i)$ is the group generated by the subgroups F_i for $i \in I$. For further information on the structure of this group we refer to [1]. We leave it as an exercise to check that it coincides with the product action of the direct product if I is a 2-element antichain, and with the imprimitive action of the wreath product if I is a 2-element chain.

The obvious question now is to calculate, if possible, the cycle index, or at least the orbit-counting series, for a generalised wreath product.

Some results are already known. Bailey *et al.* showed that, if all (G_i, X_i) are transitive, then (G, X) is transitive, and gave a description of the orbits of G on X^2 in terms of the orbits of G_i on X_i^2 and the antichains of the poset (I, ρ). Their result was as follows:

Theorem 5.1 *Let $(G, X) = \prod_{i\in I}(G_i, X_i)$ be a generalised wreath product. For each antichain S of I, and each choice of an orbit O_i of G_i on pairs of distinct elements of X_i for $i \in S$, there is an orbit of G on pairs $((x_i), (y_i))$ satisfying*

- *$x_i = y_i$ if i is not below any element of S;*
- *$(x_i, y_i) \in O_i$ if $i \in S$;*
- *no condition if $i < j$ for some $j \in S$.*

These are all the orbits of G on X^2.

This list includes the case where $x_i = y_i$ for all i (with $S = \emptyset$). Since $F_2^*(G) = 1 + F_2(G)$ for a transitive group G, we have the following result:

Theorem 5.2 *Let $(G,X) = \prod_{i\in I}(G_i,X_i)$, where each (G_i,X_i) is transitive. Then*

$$1+F_2(G) = \sum_S \prod_{i\in S} F_2(G_i),$$

where the sum is over all antichains of I.

Example If each G_i is 2-transitive on X_i, then $F_2(G)$ is equal to the number of antichains in I. This number is also equal to the number of poset homomorphisms from I to the 2-element chain.

Example If I is the 2-element chain, then $1+F_2(G) = 1+F_2(G_1)+F_2(G_2)$. If I is the 2-element antichain then $1+F_2(G) = (1+F_2(G_1))(1+F_2(G_2))$. These agree with our earlier results for imprimitive and product actions.

Subsequently, Praeger *et al.* [8] showed the following:

Theorem 5.3 *Let $(G,X) = \prod_{i\in I}(G_i,X_i)$. If (G_i,X_i) is n-transitive for all $i \in I$, then the number of orbits of G on X^n is equal to the number of poset homomorphisms from (I,ρ) to the poset $\mathcal{P}(n)$ of partitions of an n-set (ordered by refinement).*

In particular, $F_n^*(S^2, X^2) = B(n)^2$ (where $B(n) = |\mathcal{P}(n)|$ is the Bell number), and $F_n^*(S \wr S, X \times X)$ is the number of chains of length 2 in $\mathcal{P}(n)$ (including trivial chains (π,π)). These of course agree with our earlier results.

The main problem we wish to pose is to find a common generalisation of these two results to count orbits on n-tuples of an arbitrary generalised wreath product, or (better) to calculate its cycle index.

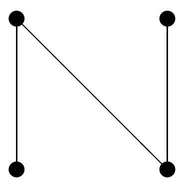

Figure 1: The poset N

Note that, if the poset (I,ρ) is *N-free* (that is, if it does not contain the poset shown in Figure 1 as induced subposet), then it can be constructed from singleton posets by the operations of disjoint union and ordered sum, and so the generalised wreath product can be built from its factors by the operations of direct product (with the product action) and wreath product (with the imprimitive action). In these cases, the cycle index can be calculated in principle. However, the proportion of n-element posets which are N-free tends to 0 as $n \to \infty$.

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