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A modified polynomial-based approach to obtaining the eigenvalues of a uniform Euler-Bernoulli beam carrying any number of attachments

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Abstract

Free vibration characteristics in uniform beams with several lumped attachments are an important problem in engineering applications that have to deal with mounting different equipment (e.g. motors, oscillators or engines) on a structural beam. In order to solve the lack of a generalized automatic procedure, this investigation presents a simple solving approach based on analytical means applied to a secular frequency equation for obtaining the natural frequencies of an arbitrarily supported single-span, or multi-span Euler-Bernoulli beam carrying any combination of miscellaneous attachments. The approach is obtained by solving a characteristic polynomial equation using a classical method for computing the roots of a polynomial. Interestingly, if the number of elements is greater than one a pole-zero cancellation is needed, but it does not require manual interventions such as initial values and iteration. The mathematical approach is validated with bibliographic references and evaluated for accuracy and computational effectiveness. A good agreement is observed with relative error values practically negligible mostly ranging between 10^{-3} and 10^{-9} in the first five natural frequencies, which confirms the validity of the presented approach in this paper. The MatLab code that has been developed with the solving approach is freely available as a supplementary material to this paper. Additionally, a MatLab graphical user interface (GUI) has also been developed in this work which allows to obtain the eigenvalues of a simply supported Euler-Bernoulli beam carrying an undetermined number of lumped elements. The GUI is also available for download, along with help facilities to be run in a Windows operating system and detailed instructions to reproduce the case studies presented here. The proposed scheme (and also the MatLab GUI) is very easy to code, and can be slightly modified to accommodate beams with arbitrary supports.

Keywords

Free vibrations analysis, Lumped parameter attachments, Assumed-modes, Eigenvalues, Continuous systems, MatLab GUI

Introduction

Combined dynamical systems consisting of beams with lumped attachments are often used as preliminary prototypes in very varied practical applications such as structural and mechanical design, active control systems or preventive maintenance. For example, structural designers need to assess natural frequencies to

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either be distinct from any expected excitation frequencies, avoiding system failure caused by resonance, or be as close to them in order to profit from resonance and get high-amplitude vibrations in the structure. The computation of eigenvalues is also mandatory to compute critical speeds in the preliminary design of flexible rotating shafts. Thus, the determination of natural frequencies and mode shapes in uniform beams with lumped elements has been widely studied in the literature with different mathematical formulations over the past decades (Cha 2001, 2002, 2005; Cha and Hu 2017; Mir Hosseini and Baddour 2017; Gürgöze 1997, 1998; Posiadała 1997; Wu and Chou 1999; Wu et al. 2013; Wu 2022). Among the pure analytical approaches previously applied to solve this problem we find the assumed modes (Meirovitch 1967; Cha 2002, 2005; Cha and Hu 2017; Mir Hosseini and Baddour 2017; Wu et al. 2013), Lagrange multipliers (Gürgöze 1998; Posiadała 1997) or the Laplace transform (Chang et al. 2001).

Despite the variety of the methods available and the conciseness of the results obtained, some of the mathematical schemes proposed to solve the frequency equation are rather laborious to implement. For example, the identification of Lagrange multipliers and constraint equations or the selection of admissible functions to represent the spatial domain. Moreover, the spatial configuration and material properties of the beam, the consideration of damping and both the number and nature of the attachments constitute additional sources of mathematical complexity to solve the problem. As a consequence, the available solving schemes need to be adapted to a certain configuration of the combined system and/or they rely on manual interventions in terms of initial seeds and iterations.

Significant efforts have already been done by previous researchers to develop generalized approaches (Gonçalves et al. 2019). For instance, Cha (2005) developed a simple and versatile approach based on the common assumed-modes method for an arbitrarily supported beam with different elements attached, including damping elements. With the proper algebraic manipulations he formulated an easy to code frequency equation that can be solved either graphically or numerically and generalized it to any combinations of attachments and boundary conditions. Limitations of this methodology in terms of the intensive computational cost for large number of component modes used in the approximation were later solved in Cha and Hu (2017) by formulating an alternative frequency equation and providing closed-form expressions for the infinite sum term for certain linear structures. Lin and Tsai (2007) developed a method to generate the exact natural frequencies and mode shapes of a uniform multi-span beam carrying multiple spring–mass systems using the numerical assembly method as previously done by Wu and Chou (1999) for single-span beams. Mir Hosseini and Baddour (2017) proposed the introduction of alternative admissible functions different from the common bare beams basis functions and penalty terms to be used together with the method of assumed modes to obtain the natural frequencies and mode shapes of a beam with lumped attachments. The main advantages of their methodology are its flexibility, as changes in boundary conditions only require a change in the penalty terms, and the numerical stability of the admissible functions. Wu et al. (2013) presented an analytical solution to compute the whirling speeds and mode shapes of a rotating shaft-disk system from the free vibration analysis of a stationary beam by replacing the effect of each rigid disk by a lumped mass and a frequency-dependent equivalent mass moment of inertia. However, even though all the above-mentioned studies have contributed to a great advance towards a generalized procedure, they are still either limited to a certain type of attachment, or certain manual interventions (e.g. initial guesses) are required under some specific configurations. For instance, Cha (2002, 2005), propose the use of graphical methods when damping is present due to the complex nature of eigenvalues.

In this paper, a purely polynomial-based solving approach to obtain the natural frequencies of an Euler-Bernoulli beam with lumped attachments of different nature is developed. The formulation is based on the frequency equation and attachments proposed in Cha (2005) which is modified with the proper algebraic manipulations leading to a characteristic polynomial equation. The latter is automatically solved using a classical method for computing the roots of a polynomial by means of a recursive algorithm for computing the determinant of a polynomial matrix. Besides, the formulation is adapted to allow standard PCs to solve the problem with large numbers of component modes. The effectiveness of the method is assessed and validated with previously published results in the literature for a simply supported beam. Moreover, an analysis of the influence of the main parameters of the attachment that control the first natural frequencies is carried out. Finally, a GUI interface that computes the natural frequencies of a simply-supported beam with any combination of lumped attachments is provided. The MatLab code and the GUI for a simply supported beam along with instructions to run it, as well as indications to accommodate any boundary conditions, are respectively available as a supplement to this paper and for free download at the web address: https://github.com/MarioRuz/UCOBeam.

Governing equations

Consider an arbitrarily supported uniform beam with the following lumped attachments shown in Fig. 1: a point or lumped mass m, a grounded viscous damper of coefficient c, a grounded translational spring of s tiffness k, an element with rotary inertia J, a grounded torsional viscous damper of coefficient c_t, a grounded torsional spring of stiffness k_t, and a damped oscillator that can have a rigid body degree of freedom $z(t)$ with parameters m_s, c_s, and k_s.

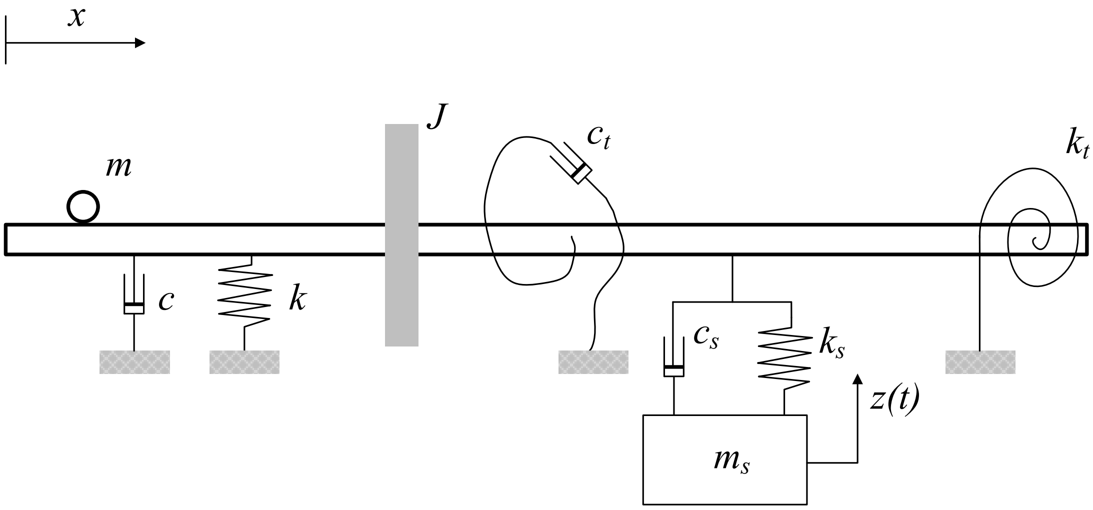

Figure 1. An arbitrarily supported beam carrying various lumped elements.

The application of the assumed-modes method (Meirovitch 1967) is applied to determine the free vibration of a uniform Euler-Bernoulli beam carrying a number S of these lumped elements. After some algebraic manipulations to the frequency equation and according to Eq. (31) of Cha (2005), the characteristic determinant needed to be solved in order to obtain the eigenvalues can be written as:

$$\det\left(\lambda^2\,\mathbf{M}^d + \mathbf{K}^d + \sum_{j=1}^{S}\sigma_j\,\mathbf{u}_j\mathbf{u}_j^T\right) = 0, \tag{1}$$

where $\lambda = \delta + i\,\omega$ are the (complex) eigenvalues of the combined system, $\mathbf{M}^d$ and $\mathbf{K}^d$ are the mass and the stiffness matrices of the unconstrained beam (i.e., the beam without any attachment), and σ_j and u_j depend on the parameters of the j lumped element and the i^{th} eigenfunctions $\phi_i(x)$ of the unconstrained beam as detailed in Table 1. Normalization of the eigenfunctions $\hat{\phi}_i(x)$ with respect to the mass per unit length of the beam, ρ, is applied so that the mass matrix in Eq. (1) equals the identity matrix of order N, $\mathbf{I}_{N\times N}$, where N is the number of component modes considered in the analysis. Thus, Eq. (1) is rewritten as:

$$\det\left(\lambda^2\,\mathbf{I}_{N\times N} + \widehat{\mathbf{K}}^{\mathbf{d}} + \sum_{j=1}^{S}\sigma_j(\lambda)\,\mathbf{u}_j\mathbf{u}_j^T\right) = 0, \tag{2}$$

where $\widehat{\mathbf{K}}^{\mathbf{d}}$ is the normalized stiffness matrix of the unconstrained beam. In this study $\sigma_j(\lambda)$ has been reformulated in the following general fractional form while still being unique for each lumped element as:

$$\sigma_j(\lambda) = \frac{\sigma_j^N(\lambda)}{\sigma_j^D(\lambda)}. \tag{3}$$

where $\sigma_j^N(\lambda)$ and $\sigma_j^D(\lambda)$ are the numerator and denominator respectively of $\sigma(\lambda)$ for each attachment in Table 1 in vectorial polynomial notation in terms of λ.

ID	Lumped attachment	$\sigma(\lambda)$	$\mathbf{u}=\mathbf{u}(x_a)$	MatLab $\sigma^N(\lambda)$	MatLab $\sigma^D(\lambda)$
1	Point mass	$m\,\lambda^2$	$\boldsymbol{\phi}_i$	$[0,\ m,\ 0,\ 0]$	$[0,\ 0,\ 0,\ 1]$
2	Grounded translational viscous damper	$c\,\lambda$	$\boldsymbol{\phi}_i$	$[0,\ 0,\ c,\ 0]$	$[0,\ 0,\ 0,\ 1]$
3	Grounded translational spring	k	$\boldsymbol{\phi}_i$	$[0,\ 0,\ 0,\ k]$	$[0,\ 0,\ 0,\ 1]$
4	In-span simple support	$k\to\infty$	$\boldsymbol{\phi}_i$	$[0,\ 0,\ 0,\ k]$	$[0,\ 0,\ 0,\ 1]$
5	Rotary inertia	$J\,\lambda^2$	$\boldsymbol{\phi}'_i$	$[0,\ J,\ 0,\ 0]$	$[0,\ 0,\ 0,\ 1]$
6	Grounded torsional viscous damper	$c_t\,\lambda$	$\boldsymbol{\phi}'_i$	$[0,\ 0,\ c_t,\ 0]$	$[0,\ 0,\ 0,\ 1]$
7	Grounded torsional spring	k_t	$\boldsymbol{\phi}'_i$	$[0,\ 0,\ 0,\ k_t]$	$[0,\ 0,\ 0,\ 1]$
8	Undamped oscillator with no rigid dof	$m_s\,\lambda^2+k_s$	$\boldsymbol{\phi}_i$	$[0,\ m_s,\ 0,\ k_s]$	$[0,\ 0,\ 0,\ 1]$
9	Damped oscillator with no rigid dof	$m_s\,\lambda^2+c_s\,\lambda+k_s$	$\boldsymbol{\phi}_i$	$[0,\ m_s,\ c_s,\ k_s]$	$[0,\ 0,\ 0,\ 1]$
10	Undamped oscillator with rigid dof	$\frac{m_s\,k_s\,\lambda^2}{m_s\,\lambda^2+k_s}$	$\boldsymbol{\phi}_i$	$[0,\ m_s\cdot k_s,\ 0,\ 0]$	$[0,\ m_s,\ 0,\ k_s]$
11	Damped oscillator with rigid dof	$\frac{m_s\,\lambda^2(c_s\,\lambda+k_s)}{m_s\,\lambda^2+c_s\,\lambda+k_s}$	$\boldsymbol{\phi}_i$	$[m_s\cdot c_s,\ m_s\cdot k_s,\ 0,\ 0]$	$[0,\ m_s,\ c_s,\ k_s]$

Table 1. Expressions for σ and $\mathbf{u}$ for any lumped attachment located at x_a from Cha (2005) and the corresponding fractional terms of σ ($\sigma^N(\lambda)$ and $\sigma^D(\lambda)$) using vectorial polynomial notation to be coded in MatLab

Solving approach

Exploiting the Sherman-Morrison determinant formula (Sherman 1978), Eq. (2) can be shown to be identical to:

$$\det\left(\left(\lambda^2\,\mathbf{I}_{N\times N}+\widehat{\mathbf{K}}^{\mathbf{d}}\right)\left(\mathbf{I}_{N\times N}+\left(\lambda^2\,\mathbf{I}_{N\times N}+\widehat{\mathbf{K}}^{\mathbf{d}}\right)^{-1}\sum_{j=1}^{S}\sigma_j(\lambda)\,\mathbf{u}_j\mathbf{u}_j^T\right)\right)=$$
$$\det\left(\lambda^2\,\mathbf{I}_{N\times N}+\widehat{\mathbf{K}}^{\mathbf{d}}\right)\,\det(\mathbf{B})=\left(\prod_{i=1}^{N}\left(\lambda^2+\hat{K}_i^d\right)\right)\,\det(\mathbf{B})=0, \tag{4}$$

where the $(\xi,\kappa)^{th}$ entry of the squared matrix $\mathbf{B}$ of size S is given by:

$$B_{\xi\kappa}=\frac{1}{\sigma_\xi(\lambda)}\delta_\xi^\kappa+\sum_{i=1}^{N}\frac{u_i(x_\xi)\,u_i(x_\kappa)}{\lambda^2+\hat{K}_i^d}=\frac{1}{\sigma_\xi^N(\lambda)}\left[\sigma_\xi^D(\lambda)\,\delta_\xi^\kappa+\sigma_\xi^N(\lambda)\sum_{i=1}^{N}\frac{u_i(x_\xi)\,u_i(x_\kappa)}{\lambda^2+\hat{K}_i^d}\right]\qquad \xi,\kappa=1,2,\ldots,S, \tag{5}$$

and δ_ξ^κ represents the Kronecker delta. A straightforward algebra provides an alternative expression for the entries of matrix $\mathbf{B}$ as:

$$B_{\xi\kappa}=\frac{\sigma_\xi^D(\lambda)\,\left(\prod_{i=1}^{N}\gamma\left(\lambda^2+\hat{K}_i^d\right)\right)\,\delta_\xi^\kappa+\gamma\,\sigma_\xi^N(\lambda)\sum_{i=1}^{N}u_i(x_\xi)\,u_i(x_\kappa)\,\left(\prod_{l=1,\,l\neq i}^{N}\gamma\left(\lambda^2+\hat{K}_l^d\right)\right)}{\sigma_\xi^N(\lambda)\,\left(\prod_{i=1}^{N}\gamma\left(\lambda^2+\hat{K}_i^d\right)\right)}\qquad \xi,\kappa=1,2,\ldots,S. \tag{6}$$

The number of component modes used in the expansion (N) must be a finite number. As this number approaches infinity, the approximate eigenvalues converge to the exact solutions (Cha and Hu 2017), but the resulting terms within the products (i.e. Eq. (6)) may collapse. Thus, a very small parameter γ has been included in both numerator and denominator of every entry of the matrix $\mathbf{B}$ with the main objective to avoid the blow-up of any of the terms when N is considerably high. It is worthy to highlight that for $N<40$ then γ can be set to 1. The application of some properties of determinants ($\det(k\,\mathbf{A})=k^S\det(\mathbf{A})$ being $\mathbf{A}$ a matrix of order S) leads to rewrite $\det(\mathbf{B})$ as:

$$\det(\mathbf{B})=\frac{\det(\tilde{\mathbf{B}})}{\prod_{j=1}^{S}\sigma_j^N(\lambda)\cdot\left(\prod_{i=1}^{N}\gamma\left(\lambda^2+\hat{K}_i^d\right)\right)^S}, \tag{7}$$

where:

$$\tilde{B}_{\xi\kappa} = \sigma_\xi^D(\lambda) \left(\prod_{i=1}^{N} \gamma \left(\lambda^2 + \hat{K}_i^d \right) \right) \delta_\xi^\kappa + \gamma\, \sigma_\xi^N(\lambda) \sum_{i=1}^{N} u_i(x_\xi)\, u_i(x_\kappa) \left(\prod_{l=1,\, l\neq i}^{N} \gamma \left(\lambda^2 + \hat{K}_l^d \right) \right) \qquad \xi, \kappa = 1, 2, \ldots, S. \tag{8}$$

Combining Eq. (7) with Eq. (4), the characteristic equation to be solved in order to obtain the eigenvalues of the combined system can be, finally, written as:

$$\frac{\det(\tilde{\mathbf{B}})}{\prod_{j=1}^{S} \left(\gamma^N \sigma_j^N(\lambda) \right) \cdot \left(\prod_{i=1}^{N} \left(\lambda^2 + \hat{K}_i^d \right) \right)^{S-1}} = 0. \tag{9}$$

The main focus of the current scheme is to solve the polynomial characteristic equation of a mechanical system given by Eq. (9), whose functions $\sigma_j(\lambda)$ and u_j are set with the aid of Table 1, using only polynomial operations regardless both, the nature and location of the attachments. Thus, a recursive in-house code was developed for obtaining the determinant of a matrix ($\tilde{\mathbf{B}}$) whose entries are one variable (λ) polynomials (Polynomial Matrix). One may recall that this technique could also have been applied directly to Eq. (1) but its extreme computational cost for large N values is not affordable nowadays with standard PCs. There exist some other methods that could perform this task such as the Evaluation-Interpolation technique (DCEI) and the Evaluation-Interpolation technique in Complex basis (DCEIC) (Varsamis 2020). These methods constitute lines of current research of the authors and will be considered in the future. The main advantage of the current framework is that once the polynomial matrix $\tilde{\mathbf{B}}$ that describes the mechanical system is determined, the eigenvalues of the combined system are directly obtained by computing its roots. This can be easily solved using any existing prepackaged code such as the command `roots` available in MATLAB. This automatic solving proposal avoids the use of a graphical methodology or other complex iterative commands that require a very good seed (i.e. when damping is present) for each of the solutions in order to, finally, obtain an approximation of the eigenvalues of the combined system.

Single attachment

When the number of attachments added to the unconstrained beam is only one ($S = 1$) and located at x_s, then for a low number of component modes N the value of γ can be set to 1 and Eq. (9) reduces to:

$$\frac{\tilde{B}_{11}(\lambda)}{\sigma^N(\lambda)} = 0 \tag{10}$$

where:

$$\tilde{B}(\lambda) = \sigma^D(\lambda) \left(\prod_{i=1}^{N} \left(\lambda^2 + \hat{K}_i^d \right) \right) + \sigma^N(\lambda) \sum_{i=1}^{N} u_i(x_s)\, u_i(x_s) \left(\prod_{l=1,\, l\neq i}^{N} \left(\lambda^2 + \hat{K}_l^d \right) \right). \tag{11}$$

If the degree of the polynomial $\sigma^D(\lambda)$ is zero, which is common for the lumped elements with ID 1 to 9 collected in Table 1, the degree of the polynomial term given by Eq. (11) is $2\,N$. It must also be noted that the degree of the second term of the above equation is always lower than or equal to that of the first term. To illustrate this, consider the system of Fig. 2 which consists of an arbitrarily supported beam with a damped oscillator with no rigid dof (ID=9 in Table 1) with parameters m_s, k_s and c_s located at x_s.

Thus, $S = 1$ and from Table 1:

$$u(x_s) = \phi_i(x_s), \qquad \sigma^N(\lambda) = m_s\, \lambda^2 + c_s\, \lambda + k_s, \qquad \sigma^D(\lambda) = 1. \tag{12}$$

Substituting Eq. (12) into Eq. (11) and limiting the calculation to the two first modes ($N = 2$) yields the following 4^{th}-order polynomial:

$$\tilde{B}(\lambda) = \left[(\lambda^2 + \hat{K}_1^d)(\lambda^2 + \hat{K}_2^d) \right] + \left(m_s\, \lambda^2 + c_s\, \lambda + k_s \right) [\phi_1^2(x_s)(\lambda^2 + \hat{K}_2^d) + \phi_2^2(x_s)(\lambda^2 + \hat{K}_1^d)]. \tag{13}$$

Finally, as the term $\sigma^N(\lambda)$ from Eq. (12) is a second-order polynomial substituting Eq. (13) into Eq. (10) leads to a 2^{nd}-order polynomial characteristic equation whose two roots, λ_1 and λ_2, constitute the two first eigenvalues of the system represented in Fig. 2.

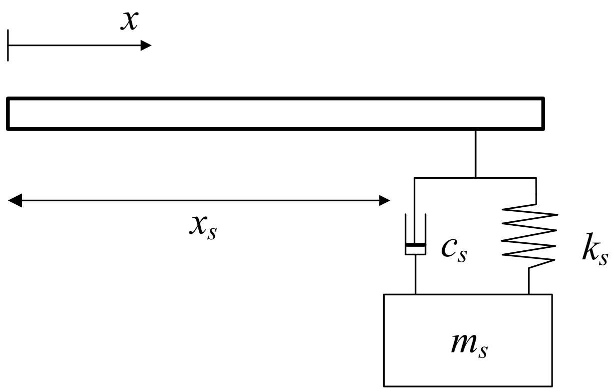

Figure 2. Arbitrarily supported beam with damped oscillator with no rigid dof.

Multiple attachments

In case that the number of elements attached to the unconstrained beam is more than 1, $S > 1$, a procedure of pole-zero cancellation is needed (see Eq. (9)). In order to perform this task, MATLAB provides a function called `minreal` which requires a transfer function object (SYS) which is constructed using `tf` from a polynomial numerator and denominator, $\sigma^N(\lambda)$ and $\sigma^D(\lambda)$, respectively. The `minreal` function produces, for a given model SYS, an equivalent model MSYS where all cancelling pole/zero pairs or non minimal state dynamics are eliminated. Additionally, a tolerance TOL is usually used for pole-zero cancellation. In our case, this value was set to 10^{-6}.

Validation

In order to evaluate the methodology the eigenvalues of a simply supported Euler-Bernoulli beam computed with the proposed scheme are compared against those obtained in previous studies with both, single and multiple attachments.

The eigenfunctions $\phi_i(x)$ of an unconstrained simply supported beam are (Meirovitch 1967):

$$\phi_i(\tilde{x}) = \mathcal{C}\,\sin(i\,\pi\,\tilde{x}) \qquad i = 1, 2, \ldots, N, \tag{14}$$

where $\tilde{x} = x/L \in [0, 1]$. The constant $\mathcal{C}$ is selected by normalization of the mass matrix of the bare beam as stated before such that $\hat{M}_{ii} \equiv \hat{M}_i = 1$. Thus,

$$\hat{M}_{ij} = \mathcal{C}^2 \int_0^1 \sin(i\,\pi\,\tilde{x})\,\sin(j\,\pi\,\tilde{x})\,dx = \left\{ \begin{array}{cc} \frac{\mathcal{C}^2}{2}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{array} \right. , \tag{15}$$

The normalization condition enforces that $\mathcal{C} = \sqrt{2}$. Therefore, the normalized eigenfunctions $\hat{\phi}_i(\tilde{x})$ are:

$$\hat{\phi}_i(\tilde{x}) = \sqrt{2}\,\sin(i\,\pi\,\tilde{x}) \qquad i = 1, 2, \ldots, N, \tag{16}$$

And the components of the normalized stiffness matrix $\hat{K}_d$ are:

$$\hat{K}_{ij} = \left\{ \begin{array}{cc} (i\pi)^4, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{array} \right. , \tag{17}$$

The results of the validation process of different combinations of lumped attachments to a simply supported beam are tabulated and presented in the following section. Previously, all the variables and parameters were properly non-dimensionalized with respect to the physical parameters of the beam so that the results of Eq. (9) correspond to the non-dimensional eigenvalues, $\tilde{\lambda}_i$, of the combined system. Thus,

$$\tilde{m} = \frac{m}{\rho L}, \quad \frac{m_s}{\rho L}, \qquad \tilde{J} = \frac{J}{\rho L^3}, \qquad \tilde{c} = \frac{c}{\sqrt{\frac{EI\rho}{L^2}}}, \quad \frac{c_s}{\sqrt{\frac{EI\rho}{L^2}}}, \quad \frac{c_t}{\sqrt{EI\rho\,L^2}} \qquad \tilde{k} = \frac{k}{\frac{EI}{L^3}}, \quad \frac{k_s}{\frac{E\,I}{L^3}}, \quad \frac{k_t}{\frac{E\,I}{L}} \tag{18}$$

and

$$\tilde{\lambda}_i = \frac{\lambda_i}{\sqrt{\frac{EI}{\rho L^4}}} \tag{19}$$

where L is the length of the beam, ρ is the mass per unit length, E is the Young's modulus and I is the moment of inertia of the beam.

It must be noted that if the lumped element is undamped, $\lambda_i = \pm j\,\omega_i$ where ω_i is the i^{th} natural frequency of the combined system and j is the complex unity equal to $\sqrt{-1}$; whereas for systems involving damping the results are complex eigenvalues, $\lambda_i = \delta_i \pm j\,\omega_i$, where δ_i and ω_i are the real (damping coefficient) and imaginary (frequency) components of the eigenvalue, respectively. Therefore, to validate the solving algorithm, we computed the relative error in each natural frequency (ϵ_i) or in each real (ϵ_{iR}) and imaginary (ϵ_{iI}) components of the eigenvalues as:

$$\epsilon_i = \frac{|\omega_{i,Eq.(9)} - \omega_{i,Literature}|}{\omega_{i,Literature}} \qquad \epsilon_{iR} = \frac{|\delta_{i,Eq.(9)} - \delta_{i,Literature}|}{\delta_{i,Literature}} \qquad \epsilon_{iI} = \frac{|\omega_{iI,Eq.(9)} - \omega_{iI,Literature}|}{\omega_{iI,Literature}} \tag{20}$$

where the subscript Eq. (9) refers to the values obtained with the proposed algorithm versus those obtained in the above-mentioned literature.

Results

Firstly, the analysis of the solving scheme for large numbers of component modes was carried out in terms of the relative error and the computational time effort. Tests were executed on an Intel i5-10400 CPU running at 2.90 GHz. Then, the first natural frequencies (for undamped systems) or eigenvalues (for damped systems) of several case studies previously validated in the literature (Cha 2002, 2005; Lin and Tsai 2007; Cha and Hu 2017; Dawson and Cha 2019) were computed using the proposed scheme. All of them correspond to simply supported Euler-Bernoulli uniform beams with different lumped attachment configurations as denoted by ID in Table 1. Secondly, a parametric analysis of two case studies with different lumped attachments is carried out. Finally, the GUI and the application of a previously validated case study is presented.

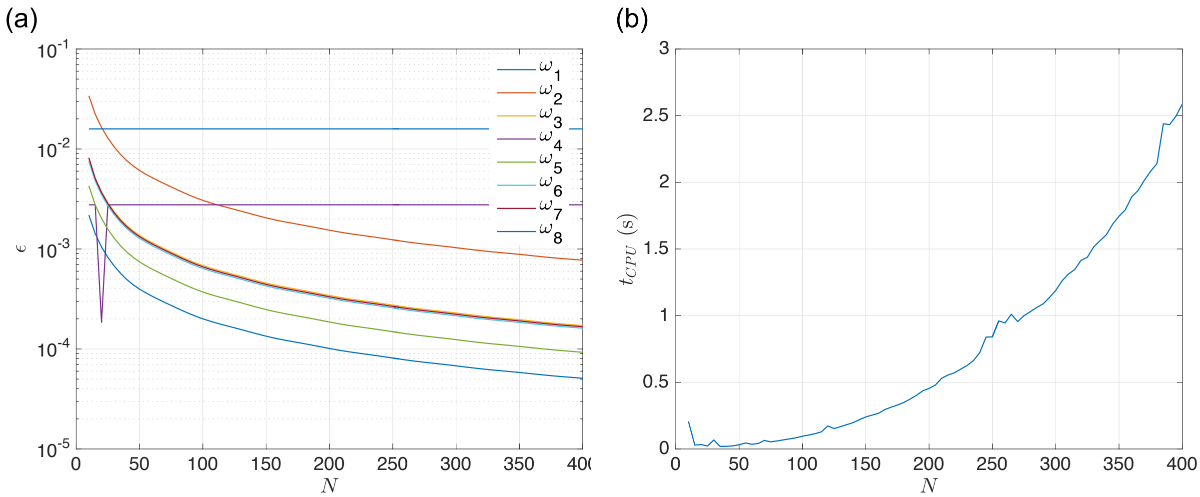

Figure 3. Relative error ϵ (a) and computational time $t_{CPU}(s)$ (b) for the first eight natural frequencies of Example 4 of Cha and Hu (2017) in terms of the number of component modes, N.

Assessment of the number of component modes

Fig. 3 shows the relative error of the first eight natural frequencies computed for N values of up to 400 of a simply supported beam with two lumped elements: i) a translational element ($ID = 10$) located at $\tilde{x} = 1/3$ with parameters $\tilde{m} = 0.25$ and $\tilde{k} = 9$; and ii) a torsional spring ($ID = 7$) at $\tilde{x} = 9/11$ with $\tilde{k}_t = 20$. The error was computed relative to the exact values obtained by solving the boundary value problem

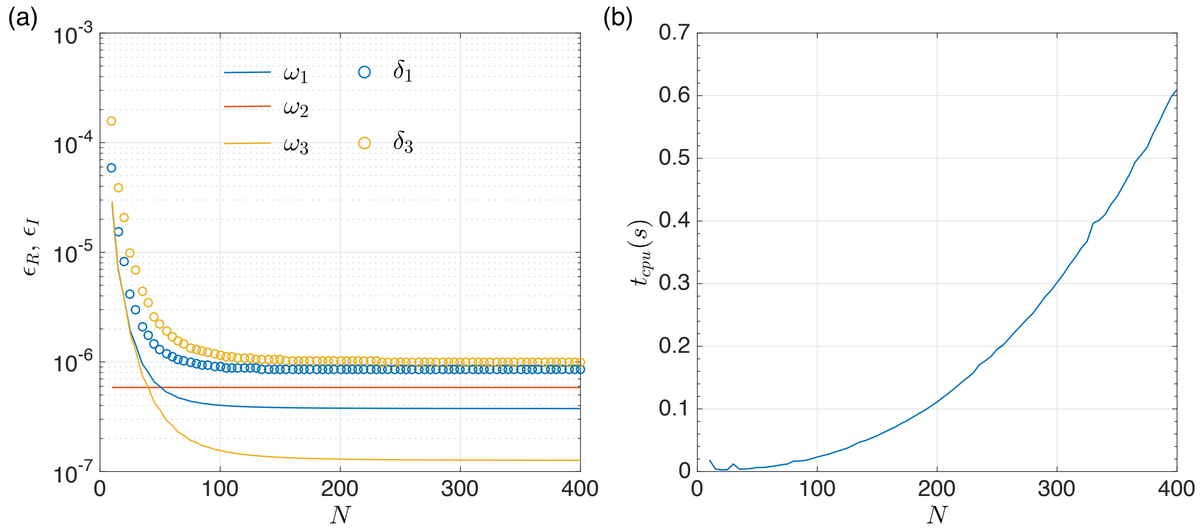

Figure 4. Relative error of the real (round symbols) ϵ_R and imaginary (solid lines) ϵ_I components (a), and computational time $t_{CPU}(s)$ (b) for the first three eigenvalues of case study in Table 3 of Cha (2002) in terms of the number of component modes, N. It must be noted that $\delta_2 = 0$.

developed in Example 4 in (Cha and Hu 2017). The convergence of the eigenvalues towards the exact natural frequencies obviously increases with large N values (Fig. 3a) together with the computational processing time (Fig. 3b). However, the rate of improvement in accuracy significantly slows down for N values over 200 components (Fig. 3a).

Regarding damped systems, Fig. 4a represents the relative error of the real and imaginary components of the three first eigenvalues of a simply supported beam with a grounded translational viscous damper ($ID = 2$) located at $\tilde{x} = 0.5$ with parameter $\tilde{c} = 0.3\pi^2$. Error values were computed relative to the values obtained for the same case study in Table 3 in (Cha 2002). Here, the rate of improvement in accuracy in both the real and imaginary components was even faster than in the undamped system shown in Fig. 3a. and they also converged earlier for N values around 100 in both components of the eigenvalues. Also, the errors of both real and imaginary components were one order of magnitude lower than in the undamped system falling below 10^{-6} for N values around 40. In this case study, as the location of the attachment coincides with the node of the second mode ($\tilde{x}_2$ in Table 8), the eigenvalue λ_2 of the combined system only presents the imaginary component, ω_2, whose magnitude is identical to the second natural frequency of the linear structure ($\tilde{\omega}_2$ in Table 8). Therefore, the relative error of the imaginary component is a constant value regardless of N. As for the processing time, a similar trend was found (Fig. 4b) with a significant decrease in one third over the undamped multiple attachment case study shown in Fig. 3b.

Thus, in view of the order of magnitude of the gain in accuracy of the eigenvalues obtained with the present solving scheme and considering the number of modes of the case studies published in the literature, a fixed $N = 40$ value was used for the validation of the case studies presented in the following section.

Validation of single and multiple attachments systems

To begin investigating the performance of the proposed algorithm, we will follow the procedure in the work of Mir Hosseini and Baddour (2017) for generating test cases with solutions validated in the literature that are used as benchmark against which to compare the results obtained using the proposed algorithm. Different case studies representing both, symmetrical and non-symmetrical locations of the lumped elements also allow to analyze the method regardless the symmetry in the configuration of the simply supported boundary condition, and thus the location of nodes. Note that in the following Tables of results, the parameters $\tilde{m}$, $\tilde{k}$ and $\tilde{c}$ correspond to the non-dimensionalized parameters according to Eq.(18) depending on the type of attachment (ID in Table 1). For example, for the lumped element with ID 7 (grounded torsional spring) in Table 2, the $\tilde{k}$ parameter is computed from the k_t value of the attachment.

Table 2 presents the results for beams with single attachment undamped configurations that enable direct comparison among the studies. The excellent agreement between the natural frequencies calculated with the proposed solving scheme and those given in the references can be noted, with relative error values practically negligible and mostly ranging between 10^{-9} and 10^{-3} for all the case studies and frequencies computed. Also, for a single lumped mass element (both ID 1 case studies in Table 2) a heavier mass parameter $\tilde{m}$ leads to higher discrepancies as already pointed out by Mir Hosseini and Baddour (2017), with an increase of two orders of magnitude from 10^{-7} to 10^{-5} in the relative error values of the two first natural frequencies when $\tilde{m}$ rises from 0.1 to 1. For single attachment damped configurations (Tables 3 and 4) similar conclusions can be drawn and relative error values were again very small (lower than $8e^{-2}$) for both the natural frequencies and the damping coefficients.

An increase in the discrepancy with the number of mode was always found for test cases with heavy lumped masses parameters $\tilde{m} = 1$ when the mass of the attachment is equal to that of the beam (i.e ID 1 and 8 case studies in Table 2 and ID 9 in Tables 3 and 4). In those situations there is a clear increasing trend in the error values with the number of modes.

Finally, the preservation of certain natural frequencies of the bare beam ($\tilde{\omega}_i$ in Table 8) when a single lumped attachment is located at a node location can also be seen with the present solving approach. Thus, the results obtained for the second and fourth natural frequencies remain unaltered when $\tilde{x}_a = 0.5$ (i.e both ID 1 and 3 case studies in Table 2, and ID 2 and 9 in Table 3) as both the second and fourth modes present a node at such location (see $\tilde{x}_n$ in Table 8).

Table 5 presents the results for beams with multiple attachment undamped configurations. The solving approach proposed in this paper leads again to very good agreement to all case studies with relative error values again very small and stable between 10^{-6} and 10^{-3}. Finally, Tables 6 and 7 show the eigenvalues of two case studies of beams with multiple attachment damped configurations which are in excellent agreement with the solutions validated in the literature in both their real and imaginary components. Thus, this general solving approach yields accurate estimations of the eigenvalues and significant performance benefits as it does not require any manual intervention and it can be easily applied in any combined damped system.

All of the numerical examples presented in this paper are computed in MatLab in less than 1 second. When there are multiple attachments and when damping is present, the proposed method may require a longer computational time. However, this extra computational effort is negligible considering that the exact natural frequencies or eigenvalues can be obtained with a completely automatic procedure regardless the number and nature of the attachments.

Parametric analysis

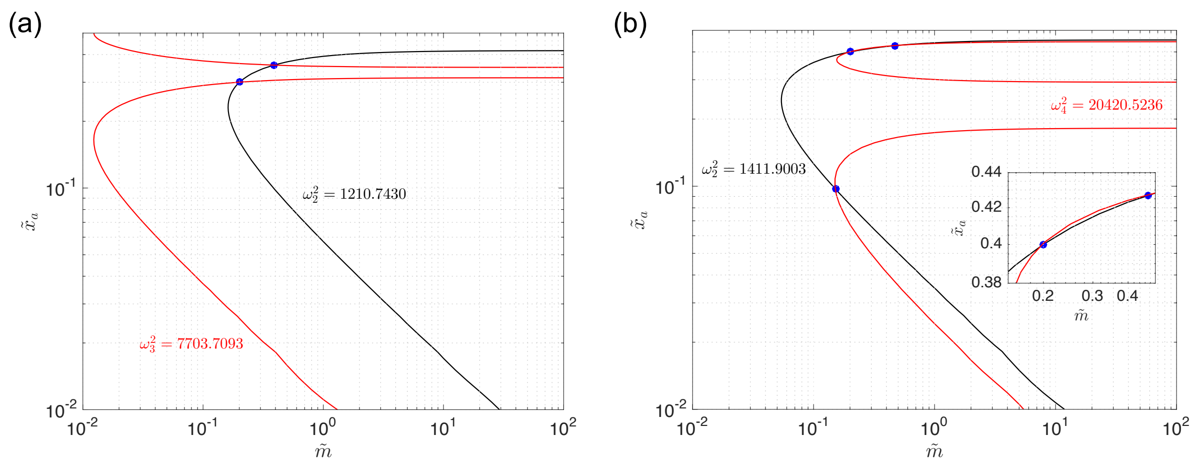

Figure 5. Graphical solution of case studies collected in Table 1 (a) and Table 2 (b) of Mir Hosseini and Baddour (2014).

One of the capabilities of the mathematical approach developed in this research work is that a sensitivity analysis of the parameters of the combined structure can be automatically carried out by a sweep in the parameters space of the problem. This procedure allows to easily solve the inverse problem (e.g. Mir

ID	$\tilde{x}_a$	$\tilde{m}$	$\tilde{k}$	Method	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_4$	$\tilde{\omega}_5$
1	0.5	0.1		Proposed	9.00782	39.47842	82.07560	157.91367	230.16712
				Cha (2002)	9.00782	39.47844	82.07565		
				ϵ_i	5.8×10^{-7}	5.9×10^{-7}	5.8×10^{-7}		
1	0.5	1		Proposed	4.39315	39.47842	65.17537	157.91367	203.59029
				Mir Hosseini and Baddour (2017)	4.3933	39.4821	65.1983	157.9754	203.9526
				ϵ_i	3.5×10^{-5}	9.3×10^{-5}	3.5×10^{-4}	3.9×10^{-4}	1.8×10^{-3}
3	0.5		$0.1\pi^2$	Proposed	9.96907	39.47842	88.83755	157.91367	346.74411
				Cha (2002)	10.8089	39.47844	88.93627		
				ϵ_i	7.8×10^{-2}	5.9×10^{-7}	1.1×10^{-3}		
4	0.4		4.5×10^{15}	Proposed	33.43868	76.87658	127.98675	246.74011	300.35487
				Lin and Tsai (2007)	33.43842	76.87508	127.98167	246.73936	300.31488
				ϵ_i	7.7×10^{-6}	2×10^{-5}	4×10^{-5}	3×10^{-6}	1.3×10^{-4}
7	0.75		10	Proposed	12.94225	39.47842	92.40276	166.22391	251.42718
				Cha (2005)	12.91225	39.47842	92.40276	166.22391	251.42718
				ϵ_i	1.2×10^{-8}	1×10^{-8}	3.6×10^{-9}	2.4×10^{-9}	8.3×10^{-10}
8	0.3	1	3	Proposed	6.53221	29.75921	86.72981	143.23034	209.38551
				Cha (2005)	6.53224	29.76006	86.73123	143.25706	209.46378
				ϵ_i	4×10^{-6}	2.8×10^{-5}	1.6×10^{-5}	1.9×10^{-4}	3.7×10^{-4}
10	0.75	0.2	3	Proposed	3.79678	10.04559	39.55518	88.84338	157.91367
				Lin and Tsai (2007)	3.79678	10.04558	39.55517	88.84337	157.91365
				ϵ_i	4.1×10^{-6}	3.5×10^{-6}	3.6×10^{-6}	3.4×10^{-6}	3.4×10^{-6}
10	0.3	0.1	1	Proposed	3.13694	9.94300	39.50149	88.82752	157.91586
				Dawson and Cha (2019)	3.1369	9.943	39.5015		
				ϵ_i	1.4×10^{-5}	1.5×10^{-8}	2.9×10^{-7}		

Table 2. Dimensionless first five natural frequencies ($\tilde{\omega}_i$) of a simply supported Euler-Bernoulli beam carrying a single undamped attachment located at $\tilde{x}_a$. ID represents the nature of the attachment whose parameters are detailed in Table 1

ID	$\tilde{x}_a$	$\tilde{m}$	$\tilde{c}$	$\tilde{k}$	Method	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_4$	$\tilde{\omega}_5$
2	0.5		$0.3\pi^2$		Proposed	9.43779	39.47842	88.61785	157.91367	246.60924
					Cha (2002)	9.43780	39.47844	88.61791		
					ϵ_{iI}	8×10^{-7}	5.9×10^{-7}	5.9×10^{-7}		
9	0.23	1	0.5	5	Proposed	7.24914	27.43030	77.36542	155.98269	233.49742
					Cha (2005)	7.24916	27.43110	77.36909	155.98546	233.54820
					ϵ_{iI}	3.2×10^{-6}	2.9×10^{-5}	4.7×10^{-5}	1.8×10^{-5}	2.1×10^{-4}
9	0.5	0.1	$0.1\pi^2$	$0.1\pi^2$	Proposed	9.06304	39.47842	82.06938	157.91367	230.16312
					Cha (2002)	9.83696	39.47844	82.13612		
					ϵ_{iI}	8×10^{-2}	5.9×10^{-7}	8×10^{-4}		
11	0.3	0.1	0.1	1	Proposed	3.10228	9.93416	39.49876	88.82738	157.91560
					Dawson and Cha (2019)	3.1023				
					ϵ_{iI}	5.5×10^{-6}				

Table 3. Dimensionless first five natural frequencies ($\tilde{\omega}_i$) of a simply supported Euler-Bernoulli beam carrying a single damped attachment located at $\tilde{x}_a$. ID represents the nature of the attachment whose parameters are detailed in Table 1

ID	$\tilde{x}_a$	$\tilde{m}$	$\tilde{c}$	$\tilde{k}$	Method	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$	$\tilde{\delta}_5$
2	0.5		$0.3\pi^2$		Proposed	-2.97674		-2.95554		-2.95804
					Cha (2002)	-2.97674		-2.95555		
					ϵ_{iR}	1.8×10^{-6}		5.4×10^{-6}		
9	0.23	1	0.5	5	Proposed	-0.12589	-0.06498	-0.01315	-0.00121	-0.00663
					Cha (2005)	-0.12589	-0.06498	-0.01315	-0.00121	-0.00658
					ϵ_{iR}	1.4×10^{-5}	1×10^{-4}	5×10^{-4}	3.1×10^{-3}	6.7×10^{-3}
9	0.5	0.1	$0.1\pi^2$	$0.1\pi^2$	Proposed	-0.82580		-0.61483		-0.47064
					Cha (2002)	-0.82275		-0.61726		
					ϵ_{iR}	3.7×10^{-3}		3.9×10^{-3}		
11	0.3	0.1	0.1	1	Proposed	-0.48434	-0.07974	-0.09169	-0.00958	-0.00346
					Dawson and Cha (2019)	-0.4843				
					ϵ_{iR}	8.8×10^{-5}				

Table 4. Dimensionless first five damping coefficients of a simply supported Euler-Bernoulli beam carrying a single damped attachment located at $\tilde{x}_a$. ID represents the nature of the attachment whose parameters are detailed in Table 1

ID	$\tilde{x}_a$	$\tilde{m}$	$\tilde{c}$	$\tilde{k}$	Method	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_4$	$\tilde{\omega}_5$
10	1/3	0.25		9	Proposed	5.76342	16.46170	41.92097	88.82644	163.21179
7	9/11			20	Cha and Hu (2017)	5.76057	13.3370	41.8493	89.0727	163.060
					ϵ_i	5×10^{-4}	7.6×10^{-3}	1.7×10^{-3}	2.8×10^{-3}	9×10^{-4}
4	1/4			4.5×10^{15}	Proposed	157.90185	184.23681	246.69027	318.87496	627.26416
4	2/4			4.5×10^{15}	Cha and Hu (2017)	157.914	184.224	246.691	318.740	631.655
4	3/4			4.5×10^{15}	ϵ_i	7.7×10^{-5}	7×10^{-5}	3×10^{-6}	4×10^{-4}	7×10^{-3}
10	0.3	0.1		1	Proposed	3.13642	9.08554	32.24952	41.96732	88.82644
10	0.4	0.1		100	Dawson and Cha (2019)	3.1364	9.0855	32.2499	41.9675	89.2816
					ϵ_i	6.6×10^{-6}	4×10^{-6}	1.2×10^{-5}	4.3×10^{-6}	5×10^{-3}
10	0.1	0.2		3	Proposed	2.37802	2.88187	3.85865	10.54994	39.47838
10	0.4	0.5		4.5	Lin and Tsai (2007)	2.37801	2.88186	3.85865	10.54993	39.68171
10	0.8	1		6	ϵ_i	4.6×10^{-6}	3.8×10^{-6}	3.7×10^{-6}	3.5×10^{-6}	5×10^{-3}
10	0.1	0.2		0.3	Proposed	2.35035	2.63863	2.92577	3.38061	3.86107
10	0.2	0.3		3.5	Lin and Tsai (2007)	2.35035	2.63863	2.92577	3.38060	3.86107
10	0.4	0.5		4.5	ϵ_i	3.4×10^{-6}	3.9×10^{-6}	3.8×10^{-6}	4×10^{-6}	3.6×10^{-6}
10	0.6	0.65		5						
10	0.8	1		6						
4	0.4			4.5×10^{15}	Proposed	3.85561	33.58198	76.88647	127.98976	246.74011
10	0.75	0.2		3	Lin and Tsai (2007)	3.85560	33.58184	76.88521	127.98508	246.74618
					ϵ_i	4.2×10^{-6}	7.5×10^{-6}	1.9×10^{-5}	4×10^{-5}	2×10^{-5}
10	0.1	0.2		3	Proposed	2.99797	3.41352	3.87047	81.77047	133.79572
4	0.3			4.5×10^{15}	Lin and Tsai (2007)	2.99796	3.41351	3.87093	81.77960	133.82198
10	0.4	0.3		3.5	ϵ_i	3.8×10^{-6}	3.8×10^{-6}	4×10^{-6}	1.6×10^{-5}	4×10^{-5}
4	0.7			4.5×10^{15}						
10	0.8	0.5		4.5						

Table 5. Dimensionless first five natural frequencies of a simply supported Euler-Bernoulli beam carrying multiple undamped attachments located at $\tilde{x}_a$. ID represents the nature of the attachment whose parameters are detailed in Table 1

ID	$\tilde{x}_a$	$\tilde{m}$	$\tilde{c}$	$\tilde{k}$	Method	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_4$	$\tilde{\omega}_5$
1	2/5	0.3			Proposed	7.79574	35.95237	83.32763	139.61037	246.49874
2	7/10		5		Cha and Hu (2017)	7.81297	35.9529	83.3276	139.610	246.499
					ϵ_{iI}	2×10^{-3}	1.5×10^{-5}	3.9×10^{-7}	2.6×10^{-6}	1.1×10^{-6}
3	0.2			5	Proposed	0.85515	6.60373	31.75571	90.76723	131.77866
1	0.35	1.75			Cha (2005)	0.85837	6.61739	31.75586	90.76638	131.77892
11	0.5	5	2	4	ϵ_{iI}	3.75×10^{-3}	2.06×10^{-3}	4.72×10^{-6}	9.36×10^{-6}	1.97×10^{-6}
7	0.75			10						

Table 6. Dimensionless first five natural frequencies of a simply supported Euler-Bernoulli beam carrying multiple damped attachments located at $\tilde{x}_a$. ID represents the nature of the attachment whose parameters are detailed in Table 1

ID	$\tilde{x}_a$	$\tilde{m}$	$\tilde{c}$	$\tilde{k}$	Method	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$	$\tilde{\delta}_5$
1	2/5	0.3			Proposed	-2.08586	-5.85048	-0.54448	-0.97166	-5.01267
2	7/10		5		Cha and Hu (2017)	-2.09966	-5.85079	-0.544465	-0.97166	-5.01267
					ϵ_{iR}	6.6×10^{-3}	5.3×10^{-5}	1.7×10^{-5}	7.4×10^{-7}	3.8×10^{-7}
3	0.2			5	Proposed	-0.18308	-0.46289	-0.59933	-1.39999	-1.55597
1	0.35	1.75			Cha (2005)	-0.18379	-0.46564	-0.59939	-1.40048	-1.55610
11	0.5	5	2	4	ϵ_{iR}	3.86×10^{-3}	5.90×10^{-3}	1.00×10^{-4}	3.50×10^{-4}	8.35×10^{-5}
7	0.75			10						

Table 7. Dimensionless first five damping coefficients of a simply supported Euler-Bernoulli beam carrying multiple damped attachments located at $\tilde{x}_a$. ID represents the nature of the attachment whose parameters are detailed in Table 1

Hosseini and Baddour (2014); Dawson and Cha (2019)) with numerous engineering applications in order to either impose or avoid certain frequencies of the vibrational system by modifying the characteristics of the combined system.

In this way, Fig. 5 shows the graphical solution for the second and third (Fig. 5a) and for the second and four (Fig. 5b) squared natural frequencies of a simply supported beam with a lumped mass. The blue dots represent the unique solutions for parameters $\tilde{m} = 0.2$ and $\tilde{x}_a = 0.3$ (Fig. 5a) and $\tilde{m} = 0.1516$ and $\tilde{x}_a = 0.0972$ (Fig. 5b) obtained in Tables 1 and 2 in Mir Hosseini and Baddour (2014). The isocontours of the square of the natural frequencies considered are plotted and allow to identify the changes in the location along the beam $\tilde{x}_a$ and the magnitude of the mass $\tilde{m}$ of the element that allow to keep unaltered those natural frequency values.

Fig. 6 shows the isocontours for the three first eigenvalues of a simply supported beam with two attachments: i) a lumped mass ($ID = 1$) located at $\tilde{x} = 2/5$; and ii) a translational damper ($ID = 2$) at $\tilde{x} = 7/10$. The red dots represent the unique solutions for parameters $\tilde{m} = 0.3$ and $\tilde{c} = 5$ obtained in Example 3 in Cha and Hu (2017). Here for a fixed position $\tilde{x}$ different combinations of the magnitude of the parameters of the lumped attachments help to impose a fixed natural frequency or damping coefficient. Also, Figure 6 (a), (c) and (e) clearly show that the natural frequency is mostly conditioned by the magnitude of the mass element, $\tilde{m}$, as denoted by the vertical lines. Similarly, the horizontal isocontours of the damping coefficient indicate that they are mostly influenced by the magnitude of the translational damping parameter, $\tilde{c}$, specially for higher frequencies (Figure 6 (d) and (f))

The interpretation of the results in terms of the above graphical solutions provide a very valuable tool for mechanical designers as they can easily identify the required changes in the location of the lumped attachments and/or in the magnitude of their parameters for a certain desired eigenvalue.

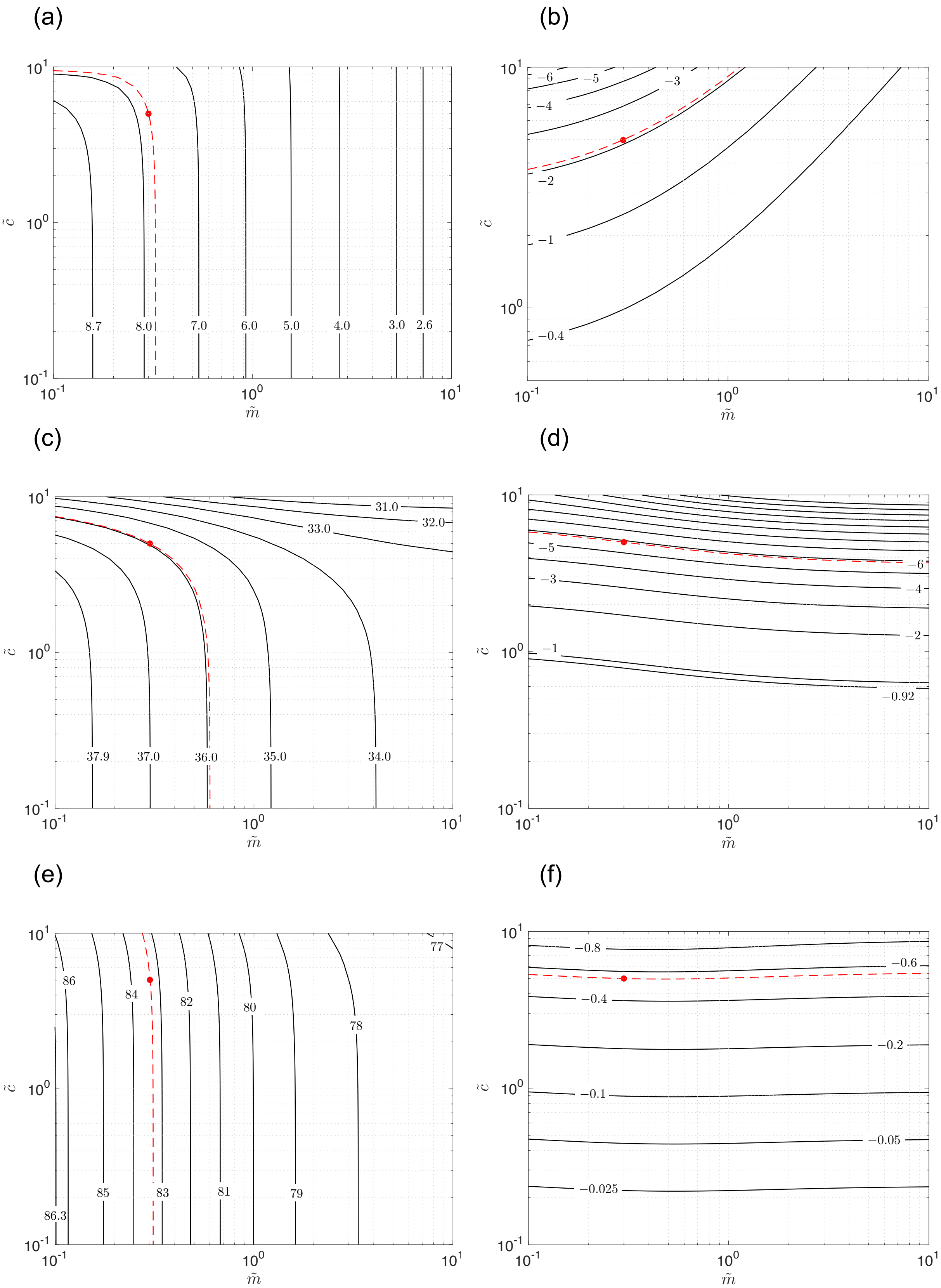

Figure 6. Graphical solution of the case studies collected in Cha and Hu (2017) in terms of the first three dimensionless natural frequencies (panels (a), (c) and (e)) and damping coefficients (panels (b), (d) and (f)). The isocontour (dashed red) lines correspond to the solution given in Cha and Hu (2017) (the red filled circle symbol corresponding to $\tilde{m} = 0.3$ and $\tilde{c} = 5$) in terms of both the dimensionless lumped mass, $\tilde{m}$, and translational damper, $\tilde{c}$, selected.

MatLab GUI: UCOBeam

An interactive tool that integrates the proposed solving scheme named UCOBeam has been developed. The tool allows to define and attach any of the lumped elements shown in Table 1. In this way, the corresponding matrices are constructed and the proposed algorithm is executed, providing the eigenvalues of the specified configuration. The GUI is freely available for download at the web address https://github.com/MarioRuz/UCOBeam along with instructions to compile it under the Windows operating system.

Fig. 7 shows the main window of the tool. The differentiated areas (a-d) as well as the main features are described as follows:

- Shaft/beam properties (a). This menu allows the user to define the main properties of the beam, specifically the Young Modulus, length, unit mass density and moment of inertia.
- Element type (b). This table provides the lumped elements defined in Table 1 that can be attached to the beam. Depending on the selected element, the corresponding properties in section (c) can be configured.
- Element properties (c). In this section, the user can define the parameters related with the selected element type in section (b). The "Add" button includes the defined element at the defined location in 'Element location x(m)' field and allows to introduce as many elements as desired by the end user .
- Results (d). This section provides the first five eigenvalues of the defined setup. Both, values in radians per second as well as the dimensionless values are shown.

Each time the user press the "Simulate" button the results section is updated. Finally, the "Reset" button deletes all the included lumped elements and their parameters and thus allows the user to implement a new setup.

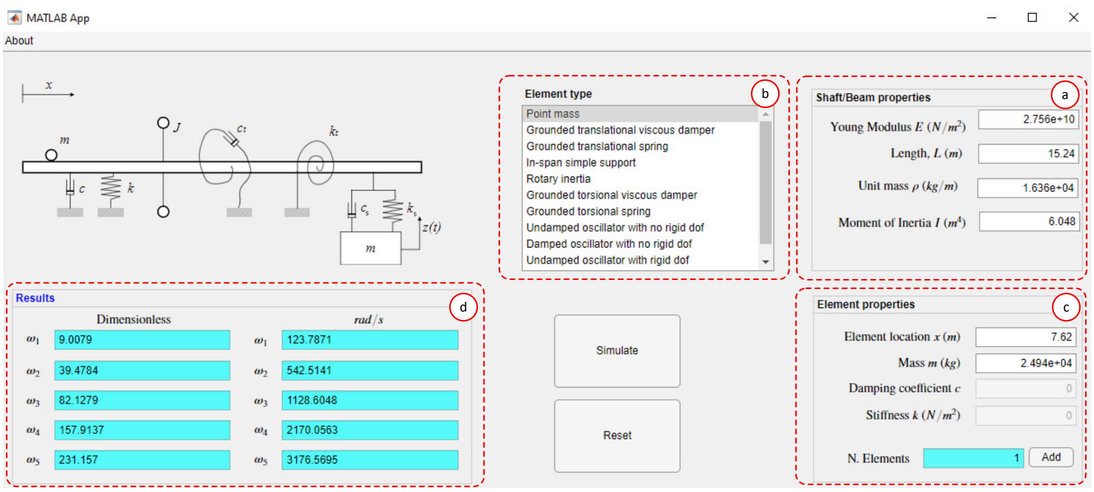

Figure 7. UCOBeam main window. The user can compute the 5 first natural frequencies of a mechanical system in terms of the dimensional characteristics of the Euler-Bernoulli beam.

The results shown in Fig. 7 correspond to the example provided by Cha (2002), whose results were summarized in ID 1 in Table 2. In that case study, a point mass element is located in the middle of an uniform simply supported Euler-Bernoulli beam. The system parameters are as follows: Young's modulus $E = 2.756 \times 10^{10}$ N/m^2; the beam length $L = 15.24$ m; $I = 6.048$ m^4; the mass per unit length $\rho = 1.636 \times 10^4$ kg/m. The mass point element is located in the middle of the beam, at $x = 7.62$ m, and has a mass of $m = 0.1\,\rho\,L = 24937.212$ kg. Once the beam properties are defined, the user only has to select the element type, specify its properties and press the "Add" button located in section (c). The same procedure can be applied to attach more components to the beam to simulate multiple attachments configurations.

Conclusions

In this paper, a novel solving approach applied to a secular equation is proposed for obtaining the eigenvalues of an arbitrarily supported Euler-Bernoulli beam carrying any combination of lumped elements using the assumed modes method. Rather than obtaining the eigenvalues of the mechanical system either numerically or graphically, an algebraic manipulation was developed providing a characteristic polynomial equation which can be directly solved using a classical method for computing the roots of a polynomial. Thus, the proposed solving method is automatic and avoids the use of a graphical methodology or more complex commands as `fsolve` that require a very good seed (when damping is present) for each of the solutions in order to, finally, obtain an approximation of the eigenvalues of the system. The proposed solving scheme yielded accurate and stable eigenvalues and were in excellent agreement with the results previously obtained for identical case studies with relative errors mostly ranging between 10^{-3} and 10^{-9} in the first five natural frequencies. Therefore, results obtained using the proposed methodology can serve as accuracy standards for other numerical methods.

Event though the solving approach developed in this study was originally conceived as a method to determine the natural frequencies of a combined system whose parameters are known (forward problem), the automatic character of the solving procedure allows to easily carry out inverse problem studies by a sweep in the parameters space of the problem. Thus, the generation of graphs with isocontours allow to easily carry out sensitivity analysis of the parameters of the combined system and they also constitute very helpful tools to either impose certain natural frequencies or to avoid others by modifying the characteristics of the lumped elements.

The MatLab code for a simply supported Euler-Bernoulli beam is freely available as supplementary material to this paper. Also, a GUI was developed to generate the eigenvalues of a simply supported beam carrying an undetermined number of lumped attachments with the proposed solving methodology. Both, the MatLab code and the GUI can be easily modified to accommodate beams with any kind of boundary conditions with just simple modifications to set the appropriate analytical eigenfunctions.

Declaration of conflicting interests

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Supplementary material

The MatLab script that has been coded with the solving approach developed in this work could be provided in a `.zip` file. This compressed file contains the main script, whose name is `ucobeam.m` as well as other functions required to solve the problem. The header of the file `ucobeam.m` contains all the parameters belonging to cases studies of Tables 2 to 7.

Properties of the beam

The dimensionless 5 natural frequencies $\tilde{\omega}_i = (i\,\pi)^2$ from $i = 1$ to $i = 5$ and the location of nodes $\tilde{x}_n$ for a simply supported Euler-Bernoulli beam is reminded in Table 8.

Mode number	$\tilde{\omega}_i$	$\tilde{x}_n$
1	9.86960	
2	39.47842	0.5
3	88.82644	0.33333, 0.66667
4	157.91367	0.25, 0.5, 0.75
5	246.74011	0.2, 0.4, 0.6, 0.8

Table 8. Dimensionless natural frequencies $\tilde{\omega}_i$ and location of nodes $\tilde{x}_n$ for a simply supported bare beam.

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