KZ and Worldsheet Supersymmetry

Lorenzo R. Florés and Ryan J. Buchanan

October 24, 2024

Abstract

This note touches upon the Knizhnik-Zamolodchikov connection, and its relationship to supersymmetry. A special type of spin structure for a string called $spin(\mathfrak{s})$ is used to explain the quantized bosonic and fermionic actions as, respectively, holomorphic and anti-holomorphic functions of the connection.

1 The Setup

Fix a complex projective manifold $\mathcal{L}$ of real dimension four, furnished with a distinguished vertical line with measure homeomorphic to the unit interval. We shall denote this dimension by z. Define a sequence $s_0, s_2, ..., s_{2n} \in z$, and another $s_1, s_3, ..., s_n \in \bar{z}$. We can abuse notation by writing z for the first sequence and $\bar{z}$ for the second. Define Lagrangian submanifold ℓ and $\bar{\ell}$ containing, respectively, z and $\bar{z}$, alongside two inclusion functors $\iota : \ell \hookrightarrow \mathcal{L}$ and $\bar{\iota} : \bar{\ell} \hookrightarrow \mathcal{L}$, satisfying

$$Triv_{KZ} = Id_{\mathcal{L}} \circ \iota = Id_{\mathcal{L}} \circ \bar{\iota}$$

where KZ is the Knizhnik-Zamolodchikov connection.

Fix a category $\mathfrak{Tors}$ whose objects are the torsion elements of ZK and denote the objects of this category by $t_0, ..., t_n$. Then, we will cheekily denote the morphisms by $P_0, P_1, ..., P_n$ so that they resemble the CFT momentum operator. The p-velocity of an inelastic body becomes:

$$p_v = \frac{t_\mu P_\mu}{\varepsilon} \tag{1.1}$$

for some $\varepsilon \in z \cup \bar{z}$. Because the holomorphic and antiholomorphic components of the KZ connection have empty intersection, this means that we essentially have to add in a centralizer "by hand" in order to serve as the zero-object for teleparallel torsion; this element can be denoted by ν_k, where k is our ground field. Usually this is $\mathbb{C}$, but we can also explore other avenues, such as setting k to be the Euclidean space $\mathbb{E}$, the sphere spectrum $\mathbb{S}$, or even an extension of $\mathbb{C}$ such as the Levi-Civita field which includes infinitesimals.

Of course, we want to include the Lorentz factor

$$\gamma = \frac{1+i}{\sqrt{1-(z-\bar{z})^{-1}}}$$

in order to guaruntee the appropriate on-shell behavior. In the most convenient case, when $t_\mu = (1-(z-\bar{z})^{-1})^{1/4} = P_\mu$, our terms cancel, and we are left with

$$\gamma p_v = \frac{1+i}{\varepsilon}$$

and so

$$\lim_{\varepsilon \to 1+i} \gamma p_v = c^{\mathcal{N}-\Delta}$$

where c is the speed of light in a vacuum and $\mathcal{N}$ is the number of supersymmetry modes and Δ is given by summing over the conformal weights of a given BPS state. When this value is zero, we obtain a pure charge term:

$$Q_\psi^\rho = c^{\mathcal{N}-|\Omega\rangle} \tag{1.2}$$

The term ρ here formally belongs to the Koszul complex of $\mathcal{L}$:

$$\rho \subset \bigwedge^{\mathcal{N}} z_i^{\pm} \tag{1.3}$$

and is known in the literature as the "flavor index." This term essentially characterizes the electroweak symmetry breaking, in the sense that it is determined by the resonance window generated by the bulk-boundary interactions.

1.1 Model Structure

Formally, given an orientable manifold $\mathcal{M}$ with orientations $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$, if there is an isomorphism $\alpha \times z \xrightarrow{\sim} \beta$, we should be able to detect the level of divergence as $\alpha(\rho) \to \beta$. The left-moving component is given by inverting the model structure. So, in a rather hand-wavy way:

$$\mathcal{M}_{\alpha\to\beta} \coloneqq D_b^{op}(\ell,\bar{\ell}) \leftarrow D_b(\bar{\ell},\ell) \tag{1.4}$$

where D_b is the derived category of bounded sheaves. In the below category:

$$\begin{tikzcd}
\mathcal{M} \arrow[loop above, "\alpha_{Pre}"] \arrow[r] \arrow[dr] & D_b(\ell,\bar{\ell}) \arrow[r] \arrow[dr, phantom, "\Leftarrow\varphi\Rightarrow"] & D_b^{op}(\ell,\bar{\ell}) \arrow[d, "\beta"] & \mathcal{M} \arrow[loop above, "\beta_{Pre}"] \arrow[l] \arrow[dl] \\
 & Fuk_{\mathcal{L}}^{op} \arrow[u, "\alpha"] & Fuk_{\mathcal{L}} \arrow[l] &
\end{tikzcd}$$

$Fuk_{\mathcal{L}}$ is the category of codimension one Lagrangian submanifolds of $\mathcal{L} \cong Rep(\mathcal{M})$.

1.1.1 Line Elements

The line elements are given by $f : \gamma p_v \circ \iota$ and $\bar{\iota} \circ \gamma p_v : g$ and satisfy

$$f \circ g = Id_{\gamma p_v}$$

and

$$g \circ f = Id_{\iota\bar{\iota}} = Id_{\bar{\iota}\iota}$$

so that

$$(f \circ g)[z] = (g \circ f)^{op}[\bar{z}]$$

The line elements satisfy distributivity:

$$(f \circ g)[z] = f(z) \circ g(z)$$

associativity:

$$f \circ (g[z]) = (f \circ g)[z]$$

and obey the Leibniz-type rule

$$f \circ (g \circ f) = g^{op} \circ (f^{op} \circ f)$$

2 Spinors

Fix a groupoid $Fuk(\mathcal{M})$, and suppose $\mathcal{M}$ has the same isotopy class as $\mathcal{L}$. The objects of this category are Lagrangian submanifolds, and the morphisms are connectivity-preserving deformations. So, if $\mathcal{M}$ is convex, then the convexity should survive after applying a smooth transformation $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$. Of course, the notation here is a bit abusive, since these are actually distinct objects, rather than, say, the same object with multiplicity two.

Suppose we are given some operator $\mathscr{O}$ which "counts" the number of BPS states. In order to do this, we need to pass from $Fuk(\mathcal{M})$ to the Scholze-Česnavičius category $Ani_k((Fuk(\mathcal{M})))$. This is where our topological invariants, such as the famous Gromov-Witten invariant, actually get encoded. The minimal data for this category is given by the original manifold $\mathcal{M}$, *along with* a *Laurent series* which determines the "animation" of our kinematic objects with respect to a dynamical variable.

Example 2.1. *Let ψ be a spinorial matrix, $\mathfrak{s}$ a string, α and β be the poles of our manifold, and define a function*

$$Ind(\psi) = \begin{cases} 1, & d(\alpha, \mathfrak{s}) < \varepsilon \\ 0, & otherwise \end{cases}$$

This is called the "indicator function." The data $\psi, Ind(\psi)$ forms an animated spinorial matrix, and the spin of a particle is given by

$$Spin(\mathfrak{s}) = \begin{cases} \frac{1}{2} Ind(\psi), & \textit{for fermions} \\ Even, & \textit{for bosons} \\ \pi \curvearrowright W, & \textit{otherwise} \end{cases} \tag{2.1}$$

In the third case, $\pi \in B_n$ is an action of the braid group on n strands, and this describes the behavior of anyons in a topological insulator..

On a brane π is called the *canonical involution*, and it maps antipodal points to one another on the nose. It is an involution, because $\pi = -\pi$, such that $[\pi \circ \pi](x) = x$, and it is canonical in this case because it is the only such non-trivial transformation. More generally, we can define a map π^+ which maps critical points to critical points; this might be useful for describing phase transitions, or tipping points of a network.

Because the animated structure of the spinor field is non-abelian by nature, the structure of $Spin(\mathfrak{s})$ is quantized by default. This meas that the commutators and anticommutators will always be well-defined, and typically will be non-trivial. In the *classical approximation*, the spinor structure of the string is lifted

$$\mathscr{C} = \begin{tikzcd} Spin(\mathfrak{s}) \arrow[d] \arrow[dr] & \\ SO(n, \mathfrak{s}) \arrow[r] & \mathfrak{su}(2) \rtimes I \end{tikzcd}$$

to the above product, where I is the moment of inertia, which is given by the Pochhammer

$$I = \frac{\sum_{n=\mu}^{N} \gamma^{\mu}(x)_n}{(\mu + N)^{\frac{3}{2}}} \qquad n \in z \cup \bar{z} \tag{2.2}$$

2.1 Categorical Data

Note that in the category $\mathscr{C}^{op}$, there may be an obstruction which prevents us from passing from the Lie group structure to the spinorial string structure. The technical details are rather involved, but essentially the come from “ghost terms” of I, which are trivializations of the connections that are degenerate in some well-specified sense; namely, their pure charge is an animated Noetherian ring which admits a zero-fold covering

$$\mathbb{0}_{\mathscr{R}} \coloneqq \bigcup_{\nu} Ani(\mathscr{R})$$

where ν is the underlying Weitzenböck torsion, and there is an inclusion

$$\iota_! : \nu \hookrightarrow spt_* : 0 \in T_\infty$$

For more information, see [6] and [7]. Recall that spt_* is the following diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[rrrr, bend left, "f"] & \perp \arrow[r] & * \arrow[r] & \top \arrow[r] & 0 \arrow[llll, bend left, "g"] \end{tikzcd}$$

where $\perp$ is the bottom element of a Heyting lattice and $\top$ is the top, and T_∞ is an ∞-topos. We have $LocSys(T_\infty) = \mathfrak{Lcl}$, the category of locales, and $\mathfrak{Lcl}^{op} = \mathfrak{Frm}$, the category of frames.

A References

[1] K. Česnavičius, P. Scholze, *Purity for Flat Cohomology*, (2023)
[2] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, *Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes*, (1993)
[3] A. Brini, Y. Schuler, *Refined Gromov-Witten Invariants*, (2024)
[4] C.H. Fu, Y. Wang, *BCJ, worldsheet quantum algebra and KZ equations*, (2020)
[5] L. Bel, *Connecting Connections*, (2008)
[6] R.J. Buchanan, *The Quantum Koszul-Plücker Picture*, (2024)
[7] P. Emmerson, *Integration of Buchanan’s Method with Koszul-Plücker Theory*, (2024)