The Jordan-Hölder Theorem

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Let G be a finite group, and $S_1, \dots, S_q$ a sequence of simple finite groups such that any simple quotient of the form H/N, where H is a subgroup of G, and N is a normal subgroup of H, is isomorphic to S_j for a unique $j \in Q := \{1, \dots, q\}$. For any $j \in Q$ and any increasing sequence $G_1 \subset \cdots \subset G_k$ (with $k \geq 2$) of subgroups of G, with G_i normal in G_{i+1} for $i = 1, \dots, k-1$, let $f_j(G_1, \dots, G_k)$ be the number of quotients G_{i+1}/G_i isomorphic to S_j:

$$f_j(G_1, \dots, G_k) = |\{i \mid 1 \leq i \leq k-1 \text{ and } G_{i+1}/G_i \simeq S_j\}|.$$

Recall that $(G_1, \dots, G_k)$ as above is a *composition series* for G_k if $G_1 = 1$, and if all the quotients G_{i+1}/G_i are simple. In this case we clearly have

$$\sum_{j \in Q} f_j(G_1, \dots, G_k) = k - 1. \tag{1}$$

Theorem 1 (Jordan-Hölder Theorem)**.** *If $(G_1, \dots, G_k)$ and $(H_1, \dots, H_m)$, with $k, m \geq 2$, are composition series for $G_k = G = H_m$, then we have $f_j(G_1, \dots, G_k) = f_j(H_1, \dots, H_m)$ for all $j \in Q$. In particular* (1) *implies $k = m$.*

It suffices to prove:

Lemma 2. *If G is a finite group admitting a composition series $(G_1, \dots, G_k)$ with $k \geq 2$, then Theorem 1 holds for G.*

Proof of Lemma 2. We prove Lemma 2 by induction on $k \geq 2$. The case $k = 2$ is obvious. Assume that $k \geq 3$ and that the Lemma holds for n with $2 \leq n \leq k-1$. Let $(G_1, \dots, G_k)$ and $(H_1, \dots, H_m)$ be two composition series for G. If $G_{k-1} = H_{m-1}$ we can apply the induction hypothesis to this group, and the result follows. Hence we can assume $G_{k-1} \neq H_{m-1}$. Setting $K := G_{k-1} \cap H_{m-1}$, and letting $(K_1, \dots, K_p)$ be a composition series for $K = K_p$, we get

$$f_j(G_1, \dots, G_k) = f_j(G_1, \dots, G_{k-1}) + f_j(G_{k-1}, G_k)$$

for all $j \in Q$. Applying the induction hypothesis to G_{k-1} yields $p = k-2$ and

$$\begin{aligned} f_j(G_1, \dots, G_k) &= f_j(K_1, \dots, K_{k-2}, G_{k-1}) + f_j(G_{k-1}, G_k) \\ &= f_j(K_1, \dots, K_{k-2}) + f_j(K_{k-2}, G_{k-1}) + f_j(G_{k-1}, G_k) \\ &= f_j(K_1, \dots, K_{k-2}) + f_j(K_{k-2}, H_{m-1}) + f_j(H_{m-1}, H_m) \\ &= f_j(K_1, \dots, K_{k-2}, H_{m-1}) + f_j(H_{m-1}, H_m), \end{aligned}$$

the third equality following from the Diamond Isomorphism Theorem. Arguing as above, we conclude that $m = k$ and that

$$\begin{aligned} f_j(G_1, \dots, G_k) &= f_j(H_1, \dots, H_{k-1}) + f_j(H_{k-1}, H_k) \\ &= f_j(H_1, \dots, H_k). \end{aligned}$$

□

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