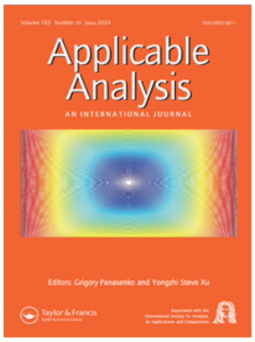

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A convergence criterion for elliptic variational inequalities

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ABSTRACT
We consider an elliptic variational inequality with unilateral constraints in a Hilbert space X which, under appropriate assumptions on the data, has a unique solution u. We formulate a convergence criterion to the solution u, i.e. we provide necessary and sufficient conditions on a sequence $\{u_n\} \subset X$ which guarantee the convergence $u_n \to u$ in the space X. Then we illustrate the use of this criterion to recover well-known convergence results and well-posedness results in the sense of Tykhonov and Levitin–Polyak. We also provide two applications of our results, in the study of a heat transfer problem and an elastic frictionless contact problem, respectively.

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1. Introduction

A large number of mathematical models in Physics, Mechanics and Engineering Science are expressed in terms of strongly nonlinear boundary value problems for partial differential equations which, in a weak formulation, lead to variational inequalities. The theory of variational inequalities was developed based on arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. Because of their importance in partial differential equations theory and engineering applications, a considerable effort has been put into the analysis, the control and the numerical simulations of variational inequalities. Basic references in the field are [1–5], for instance. Applications of variational inequalities in Mechanics can be found in the books [6–12].

In this paper, we study the convergence of an arbitrary sequence to the solution of an elliptic variational inequality. Our results below could be extended to more general inequalities in reflexive Banach spaces. Nevertheless, for simplicity, we restrict ourselves to the following functional framework: X is a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, $K \subset X$, $A : X \to X$, $j : X \to \mathbb{R}$ and $f \in X$. Then the inequality problem we consider in this paper is as follows.

Problem $\mathcal{P}$. *Find u such that*

$$u \in K, \quad (Au, v-u)_X + j(v) - j(u) \ge (f, v-u)_X \ \forall\, v \in K. \tag{1}$$

The unique solvability of Problem $\mathcal{P}$ follows from well-known results obtained in the literature, under various assumptions on the data. Here, we shall use the existence and uniqueness results that we recall

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in the following section, Theorem 2.6. We also present some convergence results to the solution u of inequality (1). Note that a large number of convergence results for inequality (1) have been obtained in the literature. The continuous dependence of the solution with respect to the data, the convergence of the solution to penalty problems when the penalty parameter converges to zero, the convergence of the solutions of discrete numerical schemes, the convergence of the solution of various perturbed problems when some parameters converge are several examples, among others. Note also that the concept of well-posedness (in the sense of Tyknonov or Levitin–Polyak) for inequality (1) is also based on the convergence to the solution u of the so-called approximating and generalized approximating sequences, respectively.

All these examples, together with various relevant applications in Optimal Control Theory, Physics and Mechanics, lead to the following question: is it possible to describe the convergence of a sequence of $\{u_n\} \subset X$ to the solution u of the variational inequality (1)? In other words, the question is to provide necessary and sufficient conditions for the convergence $u_n \to u$ in X, i.e. to provide a *convergence criterion*. The first aim of this paper is to provide an answer to this question. Here, we state and prove such a criterion of convergence expressed in terms of metric properties. The second aim is to illustrate how this criterion could be used in various examples and applications, to deduce some convergence results.

A short description of the rest of the manuscript is as follows. First, in Section 2 we present preliminary results concerning the unique solvability of Problem $\mathcal{P}$, together with some convergence results. In Section 3, we state and prove our main result, Theorem 3.1, which represents a criterion of convergence to the solution u of inequality (1). Section 4 is devoted to some theoretical applications of Theorem 3.1. Here we state and prove two convergence results, introduce a new well-posedness concept and show that it extends the classical Tykhonov and Levitin–Polyak well-posedness concepts for variational inequalities of the form (1). Finally, in Sections 5 and 6 we present applications of our theoretical results in the study of two specific boundary value problems, which model a static frictionless contact process for elastic materials and a stationary heat transfer problem, respectively. For these problems, we state and prove convergence results and provide their physical and mechanical interpretations.

2. Preliminaries

Everywhere in this paper, unless it is specified otherwise, we use the functional framework described in Introduction. Notations 0_X and I_X will represent the zero element and the identity operator of X, respectively. All the limits, upper and lower below, are considered as $n \to \infty$, even if we do not mention it explicitly. The symbols '$\rightharpoonup$' and '$\to$' denote the weak and strong convergence in various spaces which will be specified, except in the case when these convergence take place in $\mathbb{R}$. For a sequence $\{\boldsymbol{\varepsilon}_n\} \subset \mathbb{R}_+$ that converges to zero, we use the short hand notation $0 \le \boldsymbol{\varepsilon}_n \to 0$. Finally, we denote by $d(u, K)$ the distance between an element $u \in X$ to the set K, that is

$$d(u, K) = \inf_{v \in K} \|u - v\|_X. \tag{2}$$

For the convenience of the reader, we also recall the following definitions which can be found in many books and surveys, including [13, 14], for instance.

Definition 2.1: Let $\{K_n\}$ be a sequence of nonempty subsets of X and let K be a nonempty subset of X. We say that the sequence $\{K_n\}$ converges to K *in the sense of Mosco* ([15]) and we write $K_n \xrightarrow{M} K$, if the following conditions hold:

(a) for each $u \in K$, there exists a sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ in X;
(b) for each sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \rightharpoonup u$ in X, we have $u \in K$.

Definition 2.2: An operator $A\colon X \to X$ is called:

(a) *monotone* if $(Au - Av, u - v)_X \geq 0\ \forall\, u, v \in X$;
(b) *strongly monotone* if there exists $m > 0$ such that

$$(Au - Av, u - v)_X \geq m\|u - v\|_X^2 \quad \forall\, u, v \in X; \tag{3}$$

(c) *pseudomonotone*, if it is bounded and the convergence $u_n \rightharpoonup u$ in X together with inequality $\limsup\,(Au_n, u_n - u)_X \leq 0$ imply that

$$\liminf\,(Au_n, u_n - v)_X \geq (Au, u - v)_X \quad \forall\, v \in X;$$

(d) *hemicontinuous* if for all u, v, $w \in X$, the function $\lambda \mapsto (A(u + \lambda v), w)_X$ is continuous on $[0, 1]$.
(e) *demicontinuous* if $u_n \to u$ in X implies $Au_n \rightharpoonup Au$ in X;
(f) *Lipschitz continuous* if there exists $M > 0$ such that

$$\|Au - Av\|_X \leq M\,\|u - v\|_X \quad \forall\, u, v \in X. \tag{4}$$

Next, we follow [16, p.267] and introduce the following notion of a penalty operator.

Definition 2.3: An operator $G : X \to X$ is said to be a penalty operator of K if it is bounded, demicontinuous, monotone, and $Gu = 0_X$ if and only if $u \in K$.

Note that if K is a nonempty, closed, convex subset of X and P_K denotes the projection operator on K, then it is easy to see that the operator $G = I_X - P_K : X \to X$ is monotone and Lipschitz continuous. Therefore, using Definitions 2.2 and 2.3 it follows that G is a penalty operator of K. Moreover, the proposition below, proved in [14], shows that any penalty operator is pseudomonotone.

Proposition 2.4: *Let $G : X \to X$ be a bounded, hemicontinuous and monotone operator. Then G is pseudomonotone.*

In addition, the following result, stated and proved in [14], concerns the sum of two pseudomonotone operators.

Proposition 2.5: *Let A, $B : X \to X$ be pseudomonotone operators. Then the sum $A + B : X \to X$ is a pseudomonotone operator.*

In the study of Problem $\mathcal{P}$, we consider the following assumptions:

$$K \text{ is a nonempty, closed, convex subset of } X; \tag{5}$$

$$\begin{cases} A : X \to X \text{ is a pseudomonotone operator and} \\ \text{there exists } m > 0 \text{ such that (3) holds;} \end{cases} \tag{6}$$

$$j : X \to \mathbb{R} \text{ is convex and lower semicontinuous ;} \tag{7}$$

$$f \in X. \tag{8}$$

We now recall the following well-known existence and uniqueness result.

Theorem 2.6: *Assume* (5)–(8). *Then, the variational inequality* (1) *has a unique solution u.*

Theorem 2.6 represents a particular case of Theorem 84 in [17]. We now complete it with some convergence results of the form

$$u_n \to u \quad \text{in } X. \tag{9}$$

Here and everywhere in Sections 2 and 3, we keep notation u for the unique solution of Problem $\mathcal{P}$ obtained in Theorem 2.6, even if we do not mention explicitly. Moreover, $\{u_n\}$ represents a sequence of elements of X which will be specified.

Consider the sequences $\{K_n\}$, $\{\lambda_n\}$, $\{f_n\}$ such that the following conditions hold, for each $n \in \mathbb{N}$.

$$K_n \text{ is a nonempty, closed, convex subset of } X; \tag{10}$$

$$\lambda_n > 0; \tag{11}$$

$$f_n \in X. \tag{12}$$

Assume also that

$$G : X \to X \text{ is a penalty operator of } K. \tag{13}$$

Then, using Propositions 2.4 and 2.5 it follows that the operator $A + \frac{1}{\lambda_n} G : X \to X$ satisfies condition (6), for each $n \in \mathbb{N}$. Therefore, under the previous assumptions, Theorem 2.6 guarantees the unique solvability of the following three problems.

Problem $\mathcal{P}_n^1$. *Find u_n such that*

$$u_n \in K_n, \quad (Au_n, v - u_n)_X + j(v) - j(u_n) \geq (f, v - u_n)_X \;\forall\, v \in K_n. \tag{14}$$

Problem $\mathcal{P}_n^2$. *Find u_n such that*

$$\begin{aligned} u_n \in X, \quad & (Au_n, v - u_n)_X + \frac{1}{\lambda_n}(Gu_n, v - u_n)_X + j(v) - j(u_n) \\ & \geq (f, v - u_n)_X \;\forall\, v \in X. \end{aligned} \tag{15}$$

Problem $\mathcal{P}_n^3$. *Find u_n such that*

$$u_n \in K, \quad (Au_n, v - u_n)_X + j(v) - j(u_n) \geq (f_n, v - u_n)_X \;\forall\, v \in K. \tag{16}$$

Moreover, we have the following result.

Theorem 2.7: *Assume* (5)–(8). *Then the convergence* (9) *holds in each of the following three cases:*

(a) *Condition* (10) *holds, $K_n \xrightarrow{M} K$, and u_n denotes the solution to Problem $\mathcal{P}_n^1$.*
(b) *Conditions* (11) *and* (13) *hold, $\lambda_n \to 0$, and u_n denotes the solution to Problem $\mathcal{P}_n^2$.*
(c) *Condition* (12) *holds, $f_n \to f$ in X, and u_n denotes the solution to Problem $\mathcal{P}_n^3$.*

Theorem 2.7 represents a version of some convergence results obtained in [18, 19], for instance and, for this reason, we skip its proof. Nevertheless, we mention that a proof of the convergence results in Theorem 2.7 (b, c) will be provided in Section 4, under additional assumptions on the operator A and the function j. This proof is based on the convergence criterion we state and prove in Section 3. Moreover, for the convenience of the reader, we present below a sketch of the proof of the point (a) of this theorem, based on standard arguments of compactness, monotonicity, pseudomonotonicity and lower semicontinuity.

Proof of Theorem 2.7a.: The proof is structured in three steps, as follows.

Step (i) We use inequality (1) and the strong monotonicity of the operator A to prove that the sequence $\{u_n\}$ is bounded in X. Then, using the reflexivity of the space X, we deduce that this sequence contains a subsequence, again denoted by $\{u_n\}$, such that $u_n \rightharpoonup \widetilde{u}$ with some $\widetilde{u} \in X$.

Step (ii) We use the pseudomonotonicity of A and the properties of the function j to deduce that the element $\widetilde{u}$ satisfies inequality (1). Therefore, by the uniqueness of the solution, we deduce that $\widetilde{u} = u$. Moreover, a careful analysis reveals that any weakly convergent subsequence of the sequence $\{u_n\}$ converges weakly to u, in X. We then use a standard argument to deduce that the whole sequence $\{u_n\}$ converges weakly in X to u.

Step (iii) Finally, we use the strong monotonicity of the operator A, and the weak convergence $u_n \rightharpoonup u$ in X to deduce that the strong convergence holds (9), which concludes the proof.

■

We end this section with the remark that Theorem 2.7 provides relevant examples of sequences which converge to the solution u of the variational inequality (1). However, some elementary examples show that, besides the sequences introduced in parts (a), (b) and (c) of this theorem, there exists other sequences $\{u_n\}$ which converge to u. A criterion which could identify all such sequences is provided in the next section.

3. A convergence criterion

We now consider the following additional condition on the operator A and the function j:

$$\begin{cases} A : X \to X \text{ is a strongly monotone Lipschitz continuous operator,} \\ \text{i.e. there exist } m > 0 \text{ and } M > 0 \text{ such that (3) and (4) hold.} \end{cases} \tag{17}$$

$$\begin{cases} j : X \to \mathbb{R} \text{ is convex and for each } D > 0 \text{ there exists } L_D > 0 \text{ such that} \\ |j(u) - j(v)| \le L_D \|u - v\|_X \quad \forall\, u,\ v \in X \text{ with } \|u\|_X,\ \|v\|_X \le D. \end{cases} \tag{18}$$

Note that conditions (17) and (18) imply (6) and (7), respectively. Therefore, Theorems 2.6 and 2.7 still hold under assumptions (5), (8), (17) and (18). Moreover, condition (18) shows that the function j is Lipschitz continuous on each bounded sets of X. This implies that, in particular, j is locally Lipschitz. In addition, note that any continuous seminorm on the space X satisfies condition (18).

Our main result in this section is the following.

Theorem 3.1: *Assume* (5), (8), (17), (18) *and denote by u the solution of the variational inequality* (1) *provided by Theorem* 2.6*. Consider also an arbitrary sequence $\{u_n\} \subset X$, together with the statements* (19) *and* (20).

$$u_n \to u \quad \text{in } X. \tag{19}$$

$$\begin{cases} \text{(a) } d(u_n, K) \to 0\,; \\ \text{(b) } \textit{there exists } 0 \le \varepsilon_n \to 0 \textit{ such that} \\ \quad (Au_n, v - u_n)_X + j(v) - j(u_n) + \varepsilon_n(1 + \|v - u_n\|_X) \\ \quad \ge (f, v - u_n)_X \ \forall\, v \in K,\ n \in \mathbb{N}. \end{cases} \tag{20}$$

Then, these statements are equivalent, i.e. (19) *holds if and only if* (20) *holds.*

The proof of Theorem 3.1 is based on the following result.

Lemma 3.2: *Assume* (5), (8), (17), (18). *Then any sequence* $\{u_n\} \subset X$ *which satisfies condition* (20)(b) *is bounded.*

Proof: Let $n \in \mathbb{N}$. We test in (20) with a fixed element of K, say $v = u$. We have

$$(Au_n, u - u_n)_X + j(u) - j(u_n) + \varepsilon_n(1 + \|u - u_n\|_X) \geq (f, u - u_n)_X$$

which implies that

$$\begin{aligned}&(Au_n - Au, u_n - u)_X + j(u_n) \\ &\quad \leq (Au, u - u_n)_X + j(u) + \varepsilon_n(1 + \|u - u_n\|_X) + (f, u_n - u)_X\end{aligned}$$

and, moreover,

$$\begin{aligned}&m\,\|u - u_n\|_X^2 + j(u_n) \\ &\quad \leq \|Au\|_X\|u - u_n\|_X + j(u) + \varepsilon_n(1 + \|u - u_n\|_X) + \|f\|_X\|u_n - u\|_X.\end{aligned} \tag{21}$$

On the other hand, assumption (18) and a standard result on convex lower semicontinuous functions (see [20, p.208], for instance) implies that j is bounded from below by an affine continuous function. Hence, there exist $\alpha \in X$, $\beta \in \mathbb{R}$ such that

$$j(v) \geq (\alpha, v)_X + \beta \quad \forall\, v \in X. \tag{22}$$

This implies that

$$\begin{aligned}j(u_n) &\geq (\alpha, u_n)_X + \beta = (\alpha, u_n - u)_X + (\alpha, u)_X + \beta \\ &\geq -\|\alpha\|_X\|u_n - u\|_X - \|\alpha\|_X\|u\|_X - |\beta|\end{aligned}$$

and, substituting this inequality in (21) yields

$$\begin{aligned}m\,\|u - u_n\|_X^2 &\leq (\|\alpha\|_X + \|Au\|_X + \|f\|_X + \varepsilon_n)\|u - u_n\|_X \\ &\quad + j(u) + \|\alpha\|_X\|u\|_X + |\beta| + \varepsilon_n.\end{aligned}$$

Moreover, since $j(u) \leq |j(u)|$, we deduce that

$$\begin{aligned}m\,\|u - u_n\|_X^2 &\leq (\|\alpha\|_X + \|Au\|_X + \|f\|_X + \varepsilon_n)\|u - u_n\|_X \\ &\quad + |j(u)| + \|\alpha\|_X\|u\|_X + |\beta| + \varepsilon_n.\end{aligned}$$

Next, we use the implication

$$x^2 \leq ax + b \Longrightarrow x \leq a + \sqrt{b} \quad \forall\, x,\ a,\ b \geq 0 \tag{23}$$

and the convergence $\varepsilon_n \to 0$ to see that the sequence $\{\|u - u_n\|_X\}$ is bounded in $\mathbb{R}$. This implies that the sequence $\{u_n\}$ is bounded in X, which concludes the proof. ■

We now proceed with the proof of Theorem 3.1.

Proof: Assume that (19) holds. Then, since $u \in K$ it follows that $d(u_n, K) \leq \|u_n - u\|_X$ for each $n \in \mathbb{N}$ which implies that (20)(a) holds. To prove (20)(b) we fix $n \in \mathbb{N}$ and $v \in K$. We write

$$\begin{aligned}&(Au_n, v - u_n)_X + j(v) - j(u_n) - (f, v - u_n)_X \\ &\quad = (Au_n - Au, v - u_n)_X + (Au, v - u)_X + (Au, u - u_n)_X \\ &\qquad + j(v) - j(u) + j(u) - j(u_n) - (f, v - u)_X + (f, u_n - u)_X\end{aligned}$$

and, using (1) we deduce that

$$\begin{aligned}&(Au_n, v - u_n)_X + j(v) - j(u_n) - (f, v - u_n)_X \\ &\quad \geq (Au_n - Au, v - u_n)_X + (Au, u - u_n)_X + j(u) - j(u_n) + (f, u_n - u)_X.\end{aligned} \tag{24}$$

We now use (24) and inequalities

$$\begin{aligned}&(Au_n - Au, v - u_n)_X \geq -\|Au_n - Au\|_X \|v - u_n\|_X \geq -M\|u - u_n\|_X \|v - u_n\|_X, \\ &(Au, u - u_n)_X \geq -\|Au\|_X \|u - u_n\|_X, \\ &(f, u_n - u)_X \geq -\|f\|_X \|u - u_n\|_X\end{aligned}$$

to find that

$$\begin{aligned}&(Au_n, v - u_n)_X + j(v) - j(u_n) - (f, v - u_n)_X + j(u_n) - j(u) \\ &\quad + M\|u - u_n\|_X \|v - u_n\|_X + \|Au\|_X \|u - u_n\|_X + \|f\|_X \|u - u_n\|_X \geq 0.\end{aligned}$$

Therefore, with notation

$$\boldsymbol{\varepsilon}_n = \max\{M\|u - u_n\|_X, (\|Au\|_X + \|f\|_X)\|u - u_n\|_X + j(u_n) - j(u)\} \tag{25}$$

we see that

$$(Au_n, v - u_n)_X + j(v) - j(u_n) + \boldsymbol{\varepsilon}_n(1 + \|v - u_n\|_X) \geq (f, v - u_n)_X. \tag{26}$$

On the other hand, notation (25), assumption (19) and the continuity of the function j, guaranteed by hypothesis (18), show that

$$\boldsymbol{\varepsilon}_n \to 0. \tag{27}$$

We now combine (26) and (27) to see that condition (20)(b) is satisfied.

Conversely, assume now that (20) holds. Then, (20)(a) and definition (2) of the distance function show that for each $n \in \mathbb{N}$ there exist two elements v_n and w_n such that

$$u_n = v_n + w_n, \quad v_n \in K, \quad w_n \in X, \quad \|w_n\|_X \to 0. \tag{28}$$

We fix $n \in \mathbb{N}$ and use (20)(b) with $v = u \in K$ to see that

$$(Au_n, u - u_n)_X + j(u) - j(u_n) + \boldsymbol{\varepsilon}_n(1 + \|u - u_n\|_X) \geq (f, u - u_n)_X. \tag{29}$$

On the other hand, we use the regularity $v_n \in K$ in (28) and test with $v = v_n$ in (1) to find that

$$(Au, v_n - u)_X + j(v_n) - j(u) \geq (f, v_n - u)_X. \tag{30}$$

We now add inequalities (29), (30) to obtain that

$$\begin{aligned}&(Au_n, u - u_n)_X + (Au, v_n - u)_X + j(v_n) - j(u_n) \\ &\quad + \boldsymbol{\varepsilon}_n(1 + \|u - u_n\|_X) \geq (f, v_n - u_n)_X.\end{aligned} \tag{31}$$

Next, we use equality $u_n = v_n + w_n$ to see that

$$\begin{aligned}(Au_n, u - u_n)_X + (Au, v_n - u)_X &= (Au, v_n - u)_X - (Av_n, v_n - u)_X \\ &\quad + (Av_n, v_n - u)_X + (Au_n, u - v_n)_X - (Au_n, w_n)_X \\ &= (Au - Av_n, v_n - u)_X + (Au_n - Av_n, u - v_n)_X - (Au_n, w_n)_X\end{aligned}$$

and, therefore, (31) implies that

$$\begin{aligned}&(Au - Av_n, v_n - u)_X + (Au_n - Av_n, u - v_n)_X - (Au_n, w_n) + j(v_n) - j(u_n) \\ &\quad + \boldsymbol{\varepsilon}_n(1 + \|u - v_n - w_n\|_X) + (f, w_n)_X \geq 0.\end{aligned}$$

Using now assumption (17) and equality $u_n = v_n + w_n$ we deduce that

$$\begin{aligned}m\|u - v_n\|_X^2 &\leq M\|w_n\|_X\|u - v_n\|_X + \|Au_n\|_X\|w_n\|_X \\ &\quad + j(v_n) - j(u_n) + \boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_n\|u - v_n\|_X + \boldsymbol{\varepsilon}_n\|w_n\|_X + \|f\|_X\|w_n\|_X.\end{aligned} \tag{32}$$

On the other hand, Lemma 3.2 guarantees that the sequence $\{u_n\}$ is bounded in X. Therefore, (28) implies that there exists $D > 0$ such that

$$\|u_n\|_X \leq D, \quad \|v_n\|_X \leq D \;\forall\, n \in \mathbb{N}. \tag{33}$$

Moreover, using condition (17) we may assume that

$$\|Au_n\|_X \leq D \quad \forall\, n \in \mathbb{N}. \tag{34}$$

In addition, using (33), assumption (18) and equality $u_n = v_n + w_n$ in (28), we find that

$$j(v_n) - j(u_n) \leq L_D\|w_n\|_X. \tag{35}$$

We now combine the bounds (32), (34) and (35) to deduce that

$$\begin{aligned}m\|u - v_n\|_X^2 &\leq (M\|w_n\|_X + \boldsymbol{\varepsilon}_n)\|u - v_n\|_X \\ &\quad + (D + L_D + \boldsymbol{\varepsilon}_n + \|f\|_X)\|w_n\|_X + \boldsymbol{\varepsilon}_n.\end{aligned} \tag{36}$$

Next, we use (36), inequality (23) and the convergence $\|w_n\|_X \to 0, \boldsymbol{\varepsilon}_n \to 0$ to find that $\|u - v_n\|_X \to 0$. This implies that $v_n \to u$ in X and, using (28) we deduce that (19) holds, which concludes the proof. ■

Theorem 3.1 shows that, under assumptions (5)–(8), (17), (18), conditions (20)(a) and (20)(b) represent necessary and sufficient conditions for the convergence (19). The two examples below show that, in general, we cannot skip one of these conditions.

Example 3.3: Take $A = I_X$ and $j \equiv 0$. Then the solution of inequality (1) is $u = P_K f$ where, recall, P_K represents the projector operator on K. Assume now that $f \notin K$ and take $u_n = f$ for each $n \in \mathbb{N}$. It follows from here that (20)(b) is satisfied with $\boldsymbol{\varepsilon}_n = 0$. Nevertheless, (20)(a) does not hold since

$$d(u_n, K) = \|u_n - P_K u_n\|_X = \|f - P_K f\|_X > 0.$$

Moreover, the convergence (19) is not valid. We conclude from here that condition (20)(a) cannot be skipped, i.e. condition (20)(b) is not a sufficient condition to guarantee the convergence (19).

Example 3.4: Let $X = \mathbb{R}$, $K = [0, 1]$, $A = I_X$, $j \equiv 0$ and $f = \frac{1}{2}$. Then, it follows that $u = f = \frac{1}{2}$ and the sequence $\{u_n\}$ with $u_n = 0$ satisfies (20)(a) but does not satisfy (19). We conclude that condition (20)(a) is not a sufficient condition to guarantee the convergence (19).

We end this section with the remark that Theorem 3.1 was obtained under the assumptions (17), (18) which, however, are not necessary neither in the statement of Theorem 2.6 nor in the statement of Theorem 2.7. Removing or relaxing these conditions is an interesting problem which clearly deserves to be investigated into future.

4. Convergence and well-posedness results

In this section, we show how Theorem 3.1 can be used to deduce some theoretical convergence results. We also prove that the well-posedness of inequality (1), both in the Tykhonov and Levitin–Polyak sense, can be deduced as a consequence of this theorem. Finally we provide an interpretation of Theorem 3.1 in the context of the $\mathcal{T}$-well-posedness concept introduced in [21, 22] and used in various papers, including [23, 24].

Convergence results. A first consequence of Theorem 3.1 is the following continuous dependence result.

Corollary 4.1: *Assume* (5), (8), (12), (17) *and* (18). *Also, assume that* $f_n \to f$ *in* X *and denote by* u_n *the solution of Problem* $\mathcal{P}_n^3$. *Then, the convergence* (9) *holds.*

Proof: Let $n \in \mathbb{N}$ and $v \in K$. We use (16) and write

$$u_n \in K, \quad (Au_n, v - u_n)_X + j(v) - j(u_n) + (f - f_n, v - u_n)_X \geq (f, v - u_n)_X,$$

which implies that

$$u_n \in K, \quad (Au_n, v - u_n)_X + j(v) - j(u_n) + \|f - f_n\|_X \|v - u_n\|_X \geq (f, v - u_n)_X.$$

It follows from this inequality that conditions (20)(a) and (20)(b) are satisfied with $\boldsymbol{\varepsilon}_n = \|f - f_n\|_X \to 0$. We now use Theorem 3.1 to conclude the proof. ■

A second consequence of Theorem 3.1 concerns the penalty problem $\mathcal{P}_n^2$ and is as follows.

Corollary 4.2: *Assume* (5), (8), (11), (13), (17) *and* (18). *Also, assume that* $\lambda_n \to 0$ *and denote by* u_n *the solution of Problem* $\mathcal{P}_n^2$. *Then the convergence* (9) *holds.*

Proof: The proof is structured in several steps, as follows.

Step (i) We prove that the sequence $\{u_n\}$ satisfies condition (20)(b). Let $n \in \mathbb{N}$ and $v \in K$. Then, (13) and Definition 2.3 imply that $Gv = 0_X$, $(Gv - Gu_n, v - u_n)_X \geq 0$ and, therefore,

$$(Gu_n, v - u_n)_X \leq 0. \tag{37}$$

We now use inequalities (15) and (37) to see that

$$(Au_n, v - u_n)_X + j(v) - j(u_n) \geq (f, v - u_n)_X \tag{38}$$

which shows that (20)(b) holds with $\boldsymbol{\varepsilon}_n = 0$.

Step (ii) We prove that any weakly convergent subsequence of the sequence $\{u_n\}$ satisfies condition (20)(a). Indeed, consider a weakly convergent subsequence of the sequence $\{u_n\}$, again denoted by $\{u_n\}$. Then there exists an element $\widetilde{u} \in X$ such that

$$u_n \rightharpoonup \widetilde{u} \quad \text{in } X.$$

We shall prove that $\widetilde{u} \in K$ and $u_n \to \widetilde{u}$ in X. To this end, we fix $n \in \mathbb{N}$ and $v \in X$. We use (15) and (22) to write

$$\begin{aligned}\frac{1}{\lambda_n}(Gu_n, u_n - v)_X &\leq (Au_n, v - u_n)_X + j(v) - j(u_n) + (f, u_n - v)_X \\ &\leq \|Au_n\|_X\|v - u_n\|_X + j(v) + \|\alpha\|_X\|u_n\|_X + |\beta| + \|f\|_X\|v - u_n\|_X.\end{aligned}$$

On the other hand, Step (i) and Lemma 3.2 imply that the sequence $\{u_n\}$ is bounded. Therefore, from the previous inequality we deduce that there exists a constant $C(v) > 0$ which does not depend on n such that

$$(Gu_n, u_n - v)_X \leq \lambda_n C(v).$$

Passing to the upper limit in the above inequality and using assumption $\lambda_n \to 0$ we find that

$$\limsup\, (Gu_n, u_n - v)_X \leq 0. \tag{39}$$

Taking now $v = \widetilde{u}$ in (39), we deduce that

$$\limsup\, (Gu_n, u_n - \widetilde{u})_X \leq 0.$$

Then, using the convergence $u_n \rightharpoonup \widetilde{u}$ in X and the pseudomonotonicity of G, guaranteed by assumption (13) and Proposition 2.4, we deduce that

$$(G\widetilde{u}, \widetilde{u} - v)_X \leq \liminf\, (Gu_n, u_n - v)_X. \tag{40}$$

We now combine inequalities (39) and (40) to see that

$$(G\widetilde{u}, \widetilde{u} - v)_X \leq 0.$$

Recall that this inequality is valid for any $v \in X$. Therefore, we deduce that $G\widetilde{u} = 0_X$ and, using assumption (13) we find that

$$\widetilde{u} \in K. \tag{41}$$

Let $n \in \mathbb{N}$. Then, using (38) with $v = \widetilde{u}$ we find that

$$(Au_n, u_n - \widetilde{u})_X \leq j(\widetilde{u}) - j(u_n) + (f, u_n - \widetilde{u})_X$$

or, equivalently,

$$(Au_n - A\widetilde{u}, u_n - \widetilde{u})_X \leq (A\widetilde{u}, \widetilde{u} - u_n)_X + j(\widetilde{u}) - j(u_n) + (f, u_n - \widetilde{u})_X.$$

Finally, we use the strong monotonicity of the operator A, (3), to see that

$$m\|u_n - \widetilde{u}\|_X^2 \leq (A\widetilde{u}, \widetilde{u} - u_n)_X + j(\widetilde{u}) - j(u_n) + (f, u_n - \widetilde{u})_X.$$

We now pass to the upper limit in this inequality, use the convergence $u_n \rightharpoonup \widetilde{u}$ in X and the continuity of j to infer that

$$u_n \to \widetilde{u} \quad \text{in } X. \tag{42}$$

We now combine (41) and (42) to see that $d(u_n, K) \to 0$ which concludes the proof of this step.

Step (iii) We prove that any weakly convergent subsequence of the sequence $\{u_n\}$ converges to the solution u of inequality (1). This step is a direct consequence of Steps (i), (ii) and Theorem 3.1.

Step (iv) We prove that the whole sequence $\{u_n\}$ converges to the solution u of inequality (1). To this end, we argue by contradiction. Assume that the convergence (9) does not hold. Then there exists $\delta_0 > 0$ such that for all $k \in \mathbb{N}$ there exists $u_{n_k} \in X$ such that

$$\|u_{n_k} - u\|_X \geq \delta_0. \tag{43}$$

Note that the sequence $\{u_{n_k}\}$ is a subsequence of the sequence $\{u_n\}$ and, therefore, Step (i) and Lemma 3.2 imply that it is bounded in X. We now use a compactness argument to deduce that there exists a subsequence of the sequence $\{u_{n_k}\}$, again denoted by $\{u_{n_k}\}$, which is weakly convergent in X. Then Step (iii) guarantees that $u_{n_k} \to u$ as $k \to \infty$. We now pass to the limit when $k \to \infty$ in (43) and find that $\delta_0 \leq 0$. This contradicts inequality $\delta_0 > 0$ and concludes the proof.

■

Note that Corollaries 4.1 and 4.2 represent a version of Theorem 2.7 (b and c), obtained under assumptions (17) and (18) instead to (6) and (7), respectively. Similar arguments can be used to prove the result in Theorem 2.7(a).

Well-posedness results. Convergence results for the solution of optimization problems and variational inequalities are strongly related to the well-posedness of these problems. References in the field are [25–29] and, more recently [21]. Here we restrict ourselves to mention only two classical well-posedness concepts in the study of the variational inequality (1) and, to this end, following [21, p. 40, 42], we recall the following definitions.

Definition 4.3: (a) A sequence $\{u_n\} \subset X$ is called an *approximating sequence* for inequality (1) if there exists a sequence $0 \leq \boldsymbol{\varepsilon}_n \to 0$ such that

$$\begin{aligned} &u_n \in K, \quad (Au_n, v - u_n)_X + j(v) - j(u_n) + \boldsymbol{\varepsilon}_n \|v - u_n\|_X \\ &\quad \geq (f, v - u_n)_X \ \forall\, v \in K,\ n \in \mathbb{N}. \end{aligned} \tag{44}$$

(b) Inequality (1) is *well-posed in the sense of Tykhonov* (or, equivalently, is *Tykhonov well-posed*) if it has a unique solution and any approximating sequence converges in X to u.

Definition 4.4: (a) A sequence $\{u_n\} \subset X$ is called a *generalized (or LP) approximating sequence* for inequality (1) if there exist two sequences $\{w_n\} \subset X$ and $\{\boldsymbol{\varepsilon}_n\} \subset \mathbb{R}_+$ such that $w_n \to 0_X$ in X, $\boldsymbol{\varepsilon}_n \to 0$ and, moreover,

$$\begin{aligned} &u_n + w_n \in K, \quad (Au_n, v - u_n)_X + j(v) - j(u_n) + \boldsymbol{\varepsilon}_n \|v - u_n\|_X \\ &\quad \geq (f, v - u_n)_X \ \forall\, v \in K,\ n \in \mathbb{N}. \end{aligned} \tag{45}$$

(b) Inequality (1) is *well-posed in the sense of Levitin–Polyak* (or, equivalently, is *Levitin–Polyak well-posed*) if it has a unique solution and any *LP*-approximating sequence converges in X to u.

Remark 4.5: It is easy to see that a sequence $\{u_n\} \subset X$ is an *LP* approximating sequence for inequality (1) if and only if

$$\begin{cases} \text{(a) } d(u_n, K) \to 0\,; \\ \text{(b) there exists } 0 \le \varepsilon_n \to 0 \text{ such that} \\ \quad (Au_n, v - u_n)_X + j(v) - j(u_n) + \varepsilon_n \|v - u_n\|_X \\ \quad \ge (f, v - u_n)_X \;\forall\, v \in K,\; n \in \mathbb{N}. \end{cases} \tag{46}$$

Indeed, if (46) holds, then taking $w_n = P_K u_n - u_n$ where, recall, P_K denotes the projection operator on K, we have $u_n + w_n \in K$ for all $n \in \mathbb{N}$ and $\|w_n\|_X = d(u_n, K) \to 0$. Therefore, it follows that $\{u_n\} \subset X$ is an *LP* approximating sequence in the sense of Definition 4.4(a). Conversely, if $\{u_n\} \subset X$ is an *LP* approximating sequence in the sense of Definition 4.4(a), then $d(u_n, K) \le \|w_n\|_X$ for each $n \in \mathbb{N}$ and, since $w_n \to 0_X$, we deduce that condition (46)(a) is satisfied. Moreover, (45) shows that condition (46)(b) is satisfied, too.

Remark 4.6: The concept of LP approximating sequence for inequality problems governed by the set of constraints K varies from paper to paper and from problem to problem. Thus a concept of LP approximating sequence similar to that in Definition 4.4(a) was used in [30, 31], based on the decomposition of the form $u_n + w_n \in K$ with $w_n \to 0_X$. A different concept used in the literature is based on the assumption $d(u_n, K) \to 0$. References in the field include [32, 33]. Nevertheless, in the case of inequality (1) the two concepts are equivalent, as proved in Remark 4.5.

It is easy to see that if (44) holds then both conditions (20)(a) and (20)(b) are satisfied. Therefore, using Definition 4.3 and Theorem 3.1 it is easy to deduce the following result.

Corollary 4.7: *Assume* (5), (8), (17) *and* (18). *Then inequality* (1) *is well-posed in the sense of Tykhonov.*

The Levitin–Polyak well-posedness of inequality (1) is a consequence of Remark 4.5 and Theorem 3.1, too, as it follows from the following result.

Corollary 4.8: *Assume* (5), (8), (17) *and* (18). *Then inequality* (1) *is Levitin–Polyak well-posed.*

Definitions 4.3(a) and 4.4(a) show that any approximating sequence is a generalized approximating sequence, too. Therefore, using Corollary 4.8 we obtain the following implications:

$$\begin{aligned} &\{u_n\} \text{ is an approximating sequence} \\ &\quad \Longrightarrow \{u_n\} \text{ is a generalized approximating sequence} \\ &\quad \Longrightarrow u_n \to u \text{ in } X. \end{aligned}$$

The following elementary examples show that the converse of these implications is not valid.

Example 4.9: Consider Problem $\mathcal{P}$ in the particular case $X = \mathbb{R}, K = [0,1], A = I_X, j \equiv 0$ and $f = 2$. Then (1) becomes: find u such that

$$u \in [0,1], \quad u(v - u) \ge f(v - u) \;\forall\, v \in [0,1]. \tag{47}$$

The solution of this inequality is $u = P_K f = 1$. Let the sequence $\{u_n\} \subset \mathbb{R}$ be given by $u_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $u_n \to u$ but $\{u_n\}$ is not a generalized approximating sequence for (47). Indeed, assume that there exists $0 \le \varepsilon_n \to 0$ such that, for all $n \in \mathbb{N}$

$$u_n(v - u_n) + \varepsilon_n |v - u_n| \ge f(v - u_n) \quad \forall\, v \in [0,1]. \tag{48}$$

We fix $n \in \mathbb{N}$ and take $v = 1 - \frac{1}{2n}$ in (48). Then, using equalities $u_n = 1 - \frac{1}{n}, f = 2$ we deduce that $\varepsilon_n \ge 1 + \frac{1}{n}$. This inequality is valid for each $n \in \mathbb{N}$, which contradicts the convergence $\varepsilon_n \to 0$.

Example 4.10: Consider Problem $\mathcal{P}$ in the particular case $X = \mathbb{R}$, $K = [0, 1]$, $A = I_X$, $j \equiv 0$ and $f = 1$ and note that the solution of the corresponding inequality (1) is $u = P_K f = 1$. Let $\{u_n\} \subset \mathbb{R}$ be the sequence given by $u_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ is not an approximating sequence, since condition $u_n \in K$ for each $n \in \mathbb{N}$ is not satisfied. Nevertheless, $\{u_n\}$ is a generalized approximating sequence for (47). Indeed, it is easy to see that conditions in Definition 4.4(a) hold with $w_n = -\frac{1}{n}$ and $\boldsymbol{\varepsilon}_n = \frac{1}{n}$, for all $n \in \mathbb{N}$.

The examples above show that Tykhonov and Levitin–Polyak well-posedness concepts are not optimal, in the sense that the approximating sequences and generalized approximate sequences they generate, respectively, do not recover all the sequences of X which converge to the solution u of the variational inequality (1). This remark leads in a natural way to the following question: how to identify a class of sequences, say the class of $\mathcal{T}$-approximating sequences, such that the following equivalence holds:

$$\{u_n\} \text{ is a } \mathcal{T} - \text{approximating sequence} \Longleftrightarrow u_n \to u \text{ in } X.$$

A possible answer to this question is provided by the following definition.

Definition 4.11: A sequence $\{u_n\} \subset X$ is called a $\mathcal{T}$*-approximating sequence* for inequality (1) if there exists a sequence $0 \leq \boldsymbol{\varepsilon}_n \to 0$ such that

$$\begin{aligned} d(u_n, K) \leq \boldsymbol{\varepsilon}_n, \quad & (Au_n, v - u_n)_X + j(v) - j(u_n) + \boldsymbol{\varepsilon}_n(1 + \|v - u_n\|_X) \\ & \geq (f, v - u_n)_X \ \forall\, v \in K, \ n \in \mathbb{N}. \end{aligned} \tag{49}$$

Inspired by Definitions 4.3(b) and 4.4(b), we complete Definition 4.11 as follows.

Definition 4.12: Inequality (1) is $\mathcal{T}$*-well-posed* if it has a unique solution and any $\mathcal{T}$-approximating sequence converges in X to u.

Adopting these definitions, we are in a position to state the following two theorems, which represent equivalent formulations of Theorem 3.1.

Theorem 4.13: *Assume* (5), (8), (17) *and* (18). *Then, a sequence* $\{u_n\} \subset X$ *converges to the solution of inequality* (1) *if and only if it is a* $\mathcal{T}$*-approximating sequence.*

Theorem 4.14: *Assume* (5), (8), (17) *and* (18). *Then, inequality* (1) *has a unique solution if and only if it is* $\mathcal{T}$*-well-posed.*

Note that Definition 4.12 introduces a new concept of well-posedness for the variational inequality (1). It can be extended to the study of various nonlinear problems like hemivariational inequalities, inclusions, minimization problems, various classes of time-dependent and evolutionary inequalities. Details can be found in the recent book [21]. There the concept of Tykhonov triple, denoted by $\mathcal{T}$, was introduced. Moreover, the main properties of Tykhonov triples have been stated and proved, together with various examples and counter examples. Then, given a metric space (X, d), Problem $\mathcal{P}$ and a Tykhonov triple $\mathcal{T}$, both defined on X, the abstract concept of $\mathcal{T}$-well-posedness for Problem $\mathcal{P}$ has been introduced, based on two main ingredients: the existence of a unique solution to Problem $\mathcal{P}$ and the convergence to it to a special kind of sequences, the so-called $\mathcal{T}$-approximating sequences. Moreover, various applications in Functional Analysis and Contact Mechanics have been provided.

We end this section with the remark that well-posedness concepts can be extended to abstract problems for which the set of solutions (assumed to be not empty) is not reduced to a singleton. For such problems, various concepts of generalized well-posedness have been introduced in the literature. They are based on the definition of a family of so-called generalized approximating sequences and the condition that every sequence of this family has a subsequence which converges to some point of the solution set. A recent reference on this topic is [34], where the Levitin–Polyak well-posedness of the so-called split equilibrium problems is studied. Additional details can be found in this book [21].

5. A frictionless contact problem

The abstract results in Sections 3 and 4 are useful in the study of various mathematical models which describe the equilibrium of elastic bodies in contact with an obstacle, the so-called foundation. In this section, we introduce and study an example of such model and, to this end, we need some notations and preliminaries.

Let $d \in \{2,3\}$. We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ and use the notation "$\cdot$", $\|\cdot\|$, $\mathbf{0}$ for the inner product, the norm and the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a domain with smooth boundary Γ divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. A generic point in $\Omega \cup \Gamma$ will be denoted by $\boldsymbol{x} = (x_i)$. We use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, for an element $\boldsymbol{v} \in H^1(\Omega)^d$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ to Γ. In addition, we consider the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\},$$

which is a real Hilbert space endowed with the canonical inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x$$

and the associated norm $\|\cdot\|_V$. Here and below $\boldsymbol{\varepsilon}$ represents the deformation operator, i.e.

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}),$$

where an index that follows a comma denotes the partial derivative with respect to the corresponding component of $\boldsymbol{x}$, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. The completeness of the space V follows from the assumption $meas\,(\Gamma_1) > 0$ which allows us to use Korn's inequality. We denote by $\mathbf{0}_V$ the zero element of V and we recall that, for an element $\boldsymbol{v} \in V$, its normal and tangential components on Γ are given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu} \quad \textit{and} \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu},$$

respectively. Here and below $\boldsymbol{\nu}$ denote the unitary outward normal to Γ. We also recall the trace inequality

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \le d_0 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V \tag{50}$$

in which d_0 represents a positive constant.

For the inequality problem we consider in this section we use the data $\mathcal{F}$, F, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and k which satisfy the following conditions:

$$\left\{\begin{array}{l} \text{(a) } \mathcal{F}\colon \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } M_{\mathcal{F}} > 0 \text{ such that} \\ \qquad \|\mathcal{F}\boldsymbol{\varepsilon}_1 - \mathcal{F}\boldsymbol{\varepsilon}_2\| \le M_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d. \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \qquad (\mathcal{F}\boldsymbol{\varepsilon}_1 - \mathcal{F}\boldsymbol{\varepsilon}_2) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}}\, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d. \end{array}\right. \tag{51}$$

$$F \in L^\infty(\Gamma_3),\ F(\boldsymbol{x}) \ge 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{52}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d,\ \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{53}$$

$$k > 0. \tag{54}$$

Moreover, we use K for the set defined by

$$K = \{\boldsymbol{v} \in V : v_\nu \le k \text{ a.e. on } \Gamma_3\} \tag{55}$$

and r^+ for the positive part of $r \in \mathbb{R}$, that is $r^+ = \max\{r, 0\}$.

Then the inequality problem we consider is the following.

Problem $\mathcal{P}^c$. *Find* $\boldsymbol{u}$ *such that*

$$\begin{aligned} \boldsymbol{u} \in K, \quad & \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\,\mathrm{d}x + \int_{\Gamma_3} F u_\nu^+ (v_\nu - u_\nu)\,\mathrm{d}a \\ & \ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}a \quad \forall\, \boldsymbol{v} \in K. \end{aligned}$$

Following the arguments in [17, 18], it can be shown that Problem $\mathcal{P}^c$ represents the variational formulation of a mathematical model that describes the equilibrium of an elastic body Ω which is acted upon by external forces, is fixed on Γ_1, and is in frictionless contact on Γ_3. The contact takes place with a rigid foundation covered by a layer of rigid-plastic material of thickness k. Here $\mathcal{F}$ is the elasticity operator, $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ denote the density of applied body forces and tractions which act on the body and the surface Γ_2, respectively and F is a given function which describes the yield limit of the rigid-plastic material.

Next, consider the sequences $\{F_n\}$, $\{\mu_n\}$, $\{\boldsymbol{f}_{0n}\}$, $\{\boldsymbol{f}_{2n}\}$, $\{k_n\}$ such that, for each $n \in \mathbb{N}$, the following hold:

$$F_n \in L^\infty(\Gamma_3), \quad F_n(\boldsymbol{x}) \ge 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{56}$$

$$\mu_n \in L^\infty(\Gamma_3), \quad \mu_n(\boldsymbol{x}) \ge 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{57}$$

$$\boldsymbol{f}_{0n} \in L^2(\Omega)^d, \quad \boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d. \tag{58}$$

$$k_n \ge k. \tag{59}$$

$$d_0^2 \|\mu\|_{L^\infty(\Gamma_3)} \|F_n\|_{L^\infty(\Gamma_3)} < m_{\mathcal{F}}. \tag{60}$$

Recall that in (60) and below d_0 and $m_{\mathcal{F}}$ represent the constants introduced in (50) and (51), respectively. Finally, for each $n \in \mathbb{N}$ define the set

$$K_n = \{\boldsymbol{v} \in V : v_\nu \le k_n \text{ a.e. on } \Gamma_3\}, \tag{61}$$

and consider the following perturbation of Problem $\mathcal{P}^c$

Problem $\mathcal{P}_n^c$. *Find* $\boldsymbol{u}_n$ *such that*

$$\begin{aligned} \boldsymbol{u}_n \in K_n, \quad & \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n))\,\mathrm{d}x + \int_{\Gamma_3} F u_{n\nu}^+ (v_\nu - u_{n\nu})\,\mathrm{d}a \\ & + \int_{\Gamma_3} \mu_n F_n u_{n\nu}^+ (\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{n\tau}\|)\,\mathrm{d}a \\ & \ge \int_\Omega \boldsymbol{f}_{0n} \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_{2n} \cdot (\boldsymbol{v} - \boldsymbol{u}_n)\,\mathrm{d}a \quad \forall\, \boldsymbol{v} \in K_n. \end{aligned}$$

Problem $\mathcal{P}_n^c$ represents the variational formulation of a mathematical model of contact, similar to that associated to Problem $\mathcal{P}^c$. Nevertheless, two differences exist between the corresponding models. The first one arises in the fact that for the model in Problem $\mathcal{P}_n^c$ the contact is assumed to be frictional and is described with the classical Coulomb law of dry friction, governed by the friction coefficient μ_n. The second difference arises in the fact that in the statement of Problem $\mathcal{P}_n^c$ we use the data $\boldsymbol{f}_{0n}$,

$\boldsymbol{f}_{2n}$ and k_n which represent a perturbation of the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and k, respectively, used in the statement of Problem $\mathcal{P}^c$.

Our main result in this section, is the following.

Theorem 5.1: *Assume* (51)–(54), (56)–(60). *Then Problem $\mathcal{P}^c$ has a unique solution and, for each $n \in \mathbb{N}$, Problem $\mathcal{P}_n^c$ has a unique solution. Moreover, if*

$$\begin{cases} k_n \to k, \quad \mu_n \to 0 \text{ in } L^\infty(\Gamma_3),\ F_n \to F \text{ in } L^\infty(\Gamma_3), \\ \boldsymbol{f}_{0n} \to \boldsymbol{f}_0 \text{ in } L^2(\Omega)^d,\ \boldsymbol{f}_{2n} \to \boldsymbol{f}_2 \text{ in } L^2(\Gamma_2)^d \quad \text{as } n \to \infty, \end{cases} \tag{62}$$

then the solution of Problem $\mathcal{P}_n^c$ converges to the solution of Problem $\mathcal{P}^c$, i.e.

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } V \text{ as } n \to \infty. \tag{63}$$

Proof: We start with some additional notation. First, we consider the operator $A : V \to V$, the functions j, $j_n : V \to \mathbb{R}$, $\varphi_n : V \times V \to \mathbb{R}$ and the elements $\boldsymbol{f}, \boldsymbol{f}_n \in V$ defined as follows:

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x, \tag{64}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} F v_\nu^+ \,\mathrm{d}a, \quad j_n(\boldsymbol{v}) = \int_{\Gamma_3} F_n v_\nu^+ \,\mathrm{d}a, \tag{65}$$

$$\varphi_n(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu_n F_n u_\nu^+ \|\boldsymbol{v}_\tau\|\,\mathrm{d}a, \tag{66}$$

$$\begin{cases} (\boldsymbol{f}, \boldsymbol{v})_V = \displaystyle\int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\,\mathrm{d}a, \\ (\boldsymbol{f}_n, \boldsymbol{v})_V = \displaystyle\int_\Omega \boldsymbol{f}_{0n} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_{2n} \cdot \boldsymbol{v}\,\mathrm{d}a \end{cases} \tag{67}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $n \in \mathbb{N}$. Then it is easy to see that

$$\begin{cases} \boldsymbol{u} \text{ is a solution of Problem } \mathcal{P}^c \text{ if and only if} \\ \boldsymbol{u} \in K, \quad (A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V\ \forall\, \boldsymbol{v} \in K. \end{cases} \tag{68}$$

Moreover, for each $n \in \mathbb{N}$, the following equivalence holds:

$$\begin{cases} \boldsymbol{u}_n \text{ is a solution of Problem } \mathcal{P}_n^c \text{ if and only if} \\ \boldsymbol{u}_n \in K_n, \quad (A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) \\ \quad +\varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \geq (\boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V\ \forall\, \boldsymbol{v} \in K_n. \end{cases} \tag{69}$$

Equivalence (68) suggests us to use the abstract results in Sections 2 and 3 with $X = V$, K defined by (55), A defined by (64), j defined by (65) and $\boldsymbol{f}$ given by (67). It is easy to see that in this case conditions (5), (8), (17) and (18) are satisfied. For instance, using assumption (51) we see that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geq m_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2, \quad \|A\boldsymbol{u} - A\boldsymbol{v}\|_V \leq M_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$. Therefore, conditions (3) and (4) hold with $m = m_{\mathcal{F}}$ and $M = M_{\mathcal{F}}$, respectively which shows that A satisfies condition (17). Condition (18) is also satisfied since j is a continuous seminorm on the space V.

Therefore, we are in a position to apply Theorem 2.6 in order to deduce the existence of a unique solution of the variational inequality in (68). The unique solvability of the variational inequality in (69) follows from a standard argument of quasivariational inequalities. The proof can be found in [18], for instance and, therefore, we skip it. Note that here, besides the regularities (56)–(58) and condition $k_n > 0$, we need the smallness assumption (60).

We now move to the proof of the convergence (63). Let $n \in \mathbb{N}$ and $\boldsymbol{v} \in K$. We write

$$\begin{aligned}&(A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j(\boldsymbol{v}) - j(\boldsymbol{u}_n) - (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n)_V \\ &\quad = (A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) + \varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) - (\boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V \\ &\qquad + \big[j(\boldsymbol{v}) - j(\boldsymbol{u}_n) - j_n(\boldsymbol{v}) + j_n(\boldsymbol{u}_n)\big] + [\varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{v})] + (\boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n)_V. \end{aligned} \tag{70}$$

Then we use assumption $k_n \geq k$ to see that $K \subset K_n$ and, therefore, we are allowed to test in (69) with $\boldsymbol{v} \in K$. We obtain

$$(A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) + \varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) - (\boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V \geq 0 \tag{71}$$

and combining (70) and (71) we deduce that

$$\begin{aligned}&(A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j(\boldsymbol{v}) - j(\boldsymbol{u}_n) + \big[j(\boldsymbol{u}_n) - j(\boldsymbol{v}) + j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n)\big] \\ &\quad + [\varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n)] + (\boldsymbol{f} - \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n)_V. \end{aligned} \tag{72}$$

Next, using definitions (65)–(67) and standard embedding and trace arguments we find that

$$j(\boldsymbol{u}_n) - j(\boldsymbol{v}) + j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) \leq c_0 \|F_n - F\|_{L^\infty(\Gamma_3)} \|\boldsymbol{v} - \boldsymbol{u}_n\|_V, \tag{73}$$

$$\begin{aligned}&(\boldsymbol{f} - \boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V \\ &\leq c_0 \left(\|\boldsymbol{f}_{0n} - \boldsymbol{f}_0\|_{L^2(\Omega)^d} + \|\boldsymbol{f}_{2n} - \boldsymbol{f}_2\|_{L^2(\Gamma_2)^d}\right) \|\boldsymbol{v} - \boldsymbol{u}_n\|_V, \end{aligned} \tag{74}$$

$$\varphi_n(\boldsymbol{u}_n, \boldsymbol{v}) - \varphi_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \leq c_0 \|\mu_n\|_{L^\infty(\Gamma_3)} \|F_n\|_{L^\infty(\Gamma_3)} \|\boldsymbol{v} - \boldsymbol{u}_n\|_V, \tag{75}$$

where c_0 represents a positive constant which does not depend on n. We now substitute (73)–(75) in (72) and use notation

$$\begin{aligned}\boldsymbol{\varepsilon}_n = \max\, &\Big\{c_0 \|F_n - F\|_{L^\infty(\Gamma_3)}, c_0 \left(\|\boldsymbol{f}_{0n} - \boldsymbol{f}_0\|_{L^2(\Omega)^d} + \|\boldsymbol{f}_{2n} - \boldsymbol{f}_2\|_{L^2(\Gamma_2)^d}\right), \\ &c_0 \|\mu_n\|_{L^\infty(\Gamma_3)} \|F_n\|_{L^\infty(\Gamma_3)}\Big\} \end{aligned} \tag{76}$$

to deduce that

$$(A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j(\boldsymbol{v}) - j(\boldsymbol{u}_n) + \boldsymbol{\varepsilon}_n \|\boldsymbol{v} - \boldsymbol{u}_n\|_V \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n)_V. \tag{77}$$

Note that definition (76) and assumptions (62) imply that $\boldsymbol{\varepsilon}_n \to 0$. Therefore, inequality (77) shows that condition (20)(b) is satisfied. We are now in a position to use Lemma 3.2 to see that

$$\text{there exists } D > 0 \text{ such that } \|\boldsymbol{u}_n\|_V \leq D \quad \forall\, n \in \mathbb{N}. \tag{78}$$

On the other hand, using definitions (55) and (61) it is easy to see that $\frac{k}{k_n}\boldsymbol{u}_n \in K$ and, therefore

$$d(\boldsymbol{u}_n, K) \leq \left\| \boldsymbol{u}_n - \frac{k}{k_n}\boldsymbol{u}_n \right\|_V = \left| 1 - \frac{k}{k_n} \right| \|\boldsymbol{u}_n\|_V \quad \forall\, n \in \mathbb{N}. \tag{79}$$

We now use (79), (78) and assumption $k_n \to k$ to see that $d(\boldsymbol{u}_n, K) \to 0$ which shows that condition (20)(a) is satisfied, too.

It follows from above that we are in a position to use Theorem 3.1 to deduce the convergence (63). These results combined with equivalences (68) and (69) allows us to conclude the proof of the theorem. ■

We end this section with the following physical interpretation of Theorem 5.1. First, the existence and uniqueness part in the theorem prove the unique weak solvability of the contact problems considered: the frictionless contact with a rigid foundation covered by a layer of rigid-plastic material of thickness k and the frictional contact with a rigid foundation covered by a layer of rigid-plastic material of thickness k_n. Second, the weak solution of the frictionless contact problem with a rigid foundation covered by a layer of rigid-plastic material depends continuously on the density of body forces and surface tractions as well as on the yield limit and the thickness of the layer. In addition, it can be approached by the solution of the corresponding frictional problem with a small coefficient of friction.

6. A heat transfer problem

In this section, we apply the abstract results in Sections 2–4 in the study of a mathematical model which describes a heat transfer phenomenon. The problem we consider represents a version of the problems already considered in [24, 35] and, for this reason, we skip the details. Its classical formulation is the following.

Problem $\mathcal{C}^t$. *Find a temperature field $u : \Omega \to \mathbb{R}$ such that*

$$-\Delta u = g \quad \text{a.e.in } \Omega, \tag{80}$$

$$u = 0 \text{ a.e. on } \Gamma_1, \tag{81}$$

$$-\frac{\partial u}{\partial \nu} = q \text{ a.e. on } \Gamma_2. \tag{82}$$

$$u = b \text{ a.e. on } \Gamma_3. \tag{83}$$

Here, as in Section 5, Ω is a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$ in applications) with smooth boundary Γ, divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. We denote by $\boldsymbol{\nu}$ and outer normal unit to Γ and recall that in (80)–(83) we do not mention the dependence of the different functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. The functions g, q and b are given and will be described below. Here we mention that g represents the internal energy, q is a prescribed heat flux and b denotes a prescribed temperature. Moreover, $\frac{\partial u}{\partial \nu}$ denotes the normal derivative of u on the boundary Γ.

Now, let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lambda_n > 0$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we consider the following boundary problem.

Problem $\mathcal{C}_n^t$. *Find a temperature field $u_n : \Omega \to \mathbb{R}$ such that*

$$-\Delta u_n = g \text{ a.e. in } \Omega, \tag{84}$$

$$u_n = 0 \text{ a.e. on } \Gamma_1, \tag{85}$$

$$-\frac{\partial u_n}{\partial \nu} = q \text{ a.e. on } \Gamma_2, \tag{86}$$

$$-\frac{\partial u_n}{\partial \nu} = \frac{1}{\lambda_n}(u_n - b) \text{ a.e. on } \Gamma_3. \tag{87}$$

Note that Problem $\mathcal{C}_n^t$ is obtained from Problem $\mathcal{C}^t$ by replacing the Dirichlet boundary condition (83) with the Neumann boundary condition (87). Here λ_n is a positive parameter, and its inverse $h_n = \frac{1}{\lambda_n}$ represents the heat transfer coefficient on the boundary Γ_3. In contrast to Problem $\mathcal{C}^t$ (in which the temperature is prescribed on Γ_3), in Problem $\mathcal{C}_n^t$ this condition is replaced by a condition on the flux of the temperature, governed by a positive heat transfer coefficient.

For the variational analysis of Problem $\mathcal{C}^t$, we consider the space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1\}.$$

Note that, here and below in this section, we still write v for trace of the element v to Γ. Denote in what follows by $(\cdot,\cdot)_V$ the inner product of the space $H^1(\Omega)$ restricted to V and by $\|\cdot\|_V$ the associated norm. Since *meas* $\Gamma_1 > 0$, it is well known that $(V,(\cdot,\cdot)_V)$ is a real Hilbert space. Next, we assume that

$$g \in L^2(\Omega), \quad q \in L^2(\Gamma_2), \quad b \in H^{\frac{1}{2}}(\Gamma_3), \tag{88}$$

$$\text{there exists } v_0 \in V \text{ such that } v_0 = b \textit{ a.e. on } \Gamma_3, \tag{89}$$

and, finally, we introduce the set

$$K = \{v \in V : v = b \text{ a.e. on } \Gamma_3\}. \tag{90}$$

Note that assumption (89) represents a compatibility assumption on the data b which guarantees that the set K is not empty. Then it is easy to see that the variational formulation of problems $\mathcal{C}^t$ and $\mathcal{C}^t_n$, obtained by standard arguments, is as follows.

Problem $\mathcal{P}^t$. *Find u such that*

$$u \in K, \int_\Omega \nabla u \cdot (\nabla v - \nabla u)\,\mathrm{d}x + \int_{\Gamma_2} q(v-u)\,\mathrm{d}a = \int_\Omega g(v-u)\,\mathrm{d}x \quad \forall\, v \in K.$$

Problem $\mathcal{P}^t_n$. *Find u_n such that*

$$\begin{aligned} u_n \in V, &\int_\Omega \nabla u_n \cdot (\nabla v - \nabla u_n)\,\mathrm{d}x + \int_{\Gamma_2} q(v-u_n)\,\mathrm{d}a \\ &+ \frac{1}{\lambda_n}\int_{\Gamma_3} (u_n - b)(v-u_n)\,\mathrm{d}a \geq \int_\Omega g(v-u_n)\,\mathrm{d}x \quad \forall\, v \in V. \end{aligned}$$

Our main result in this section is the following.

Theorem 6.1: *Assume* (88) *and* (89). *Then, Problem $\mathcal{P}^t$ has a unique solution and, for each $n \in \mathbb{N}$, Problem $\mathcal{P}^t_n$ has a unique solution. Moreover, if $\lambda_n \to 0$, then the solution of Problem $\mathcal{P}^t_n$ converges to the solution of Problem $\mathcal{P}^t$, i.e.*

$$u_n \to u \quad \text{in } V \text{ as } n \to \infty. \tag{91}$$

Proof: We consider the operators $A : V \to V$, $G : V \to V$ and the element $f \in V$ defined as follows:

$$(Au, v)_V = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x \quad \forall\, u, v \in V, \tag{92}$$

$$(Gu, v)_V = \int_{\Gamma_3} (u-b)v\,\mathrm{d}a \quad \forall\, u, v \in V, \tag{93}$$

$$(f, v)_V = \int_\Omega gv\,\mathrm{d}x - \int_{\Gamma_2} qv\,\mathrm{d}a \quad \forall\, v \in V. \tag{94}$$

Then, since the set $\{ v - v_0 : v \in K\}$ is a linear subspace on V, it is easy to see that

$$\begin{cases} u \text{ is a solution of Problem } \mathcal{P}^t \text{ if and only if} \\ u \in K, \quad (Au, v-u)_V \geq (f, v-u)_V \;\forall\, v \in K. \end{cases} \tag{95}$$

Moreover, for each $n \in \mathbb{N}$,

$$\begin{cases} u_n \text{ is a solution of Problem } \mathcal{P}_n^t \text{ if and only if} \\ u_n \in V, \quad (Au_n, v - u_n)_V + \frac{1}{\lambda_n}(Gu_n, v - u_n)_V \\ \quad \geq (f, v - u_n)_V \ \forall\, v \in V. \end{cases} \tag{96}$$

We use the abstract results in Sections 2 and 3 with $X = V$, K defined by (90), A defined by (92), G defined by (93), f given by (94), and $j \equiv 0$. It is easy to see that in this case conditions (5), (8), (11), (13), (17) and (18) are satisfied. Therefore, we are in a position to apply Theorem 2.6 to deduce the existence of a unique solution of the variational inequalities in (95) and (96), respectively. Moreover, using Corollary 4.2 we deduce the convergence (91). These results combined with (95) and (96) allow us to conclude the proof. ■

We end this section with the following physical interpretation of Theorem 6.1. First, the solutions of Problems $\mathcal{P}^t$ and $\mathcal{P}_n^t$ represent weak solutions of the heat transfer Problems $\mathcal{C}^t$ and $\mathcal{C}_n^t$, respectively. Therefore, Theorem 6.1 provides the unique weak solvability of these problems. Second, the convergence result (91) shows that the weak solution of Problem $\mathcal{C}^t$ with prescribed temperature on Γ_3 can be approached by the solution of Problem $\mathcal{C}_n^t$ with heat transfer on Γ_3, for a large heat transfer coefficient.

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