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Research paper

A generalized penalty method for a new class of differential inequality system☆

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ABSTRACT

The primary objective of this paper is to study a nonlinear system involving a parabolic variational inequality, a history-dependent hemivariational inequality and a differential equation constraints in a Banach space. First, we derive a unique solvability theorem for such problem under some mild hypotheses. Second, we construct a penalized problem for such nonlinear system, and show the existence and uniqueness of its solution to obtain an approximating sequence for the nonlinear system. Moreover, we prove the strong convergence of the sequence of approximate solution to the solution of the original system when the penalty parameter converges to zero. Finally, these results are applied to a quasistatic elastic frictional contact problem with heat equation with memory, and damage.

1. Introduction

As a powerful and practical mathematical tool, the theory of differential equations has been widely used to describe various physical laws, chemical reaction processes, and to solve a large number of industry-generated problems. In 1982, Grimmer first obtained some basic results for linear integral differential equations, see [1]. In 1983, Grimmer and Pritchard, presented their findings on the use of resolvent operators for mild solutions for integrodifferential systems, see [2]. The variational inequalities originated from the classical unilateral constraint model proposed by the famous Italian mathematician and physicist Signorini in the process of studying contact problems, that is, the Signorini problem. The variational inequalities deal with significant aspects of optimization theory and have shown a considerable progress over the past century. Relevant scholars have carried out many explorations and studies, which have enriched and developed the theory and algorithmics. In the process of study various complex system problems, we often need to use variational inequality analysis and the theory of differential equations to solve the problems. The combination of differential equations and variational inequalities is one of a basic generalization of differential equations. This intersection of disciplines not only expands the scope of research and applications of differential equations, but also offers a broader approach to modeling complex engineering systems in the real world. The use of this tool has been effective in the study of

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various theories, including physics, economic equilibrium, and optimization. The concept of a hemivariational inequality involves variational principles for nonconvex and nonsmooth functions, and it was initially proposed by Panagiotopoulos et al. in [3]. Later, hemivariational inequalities have been widely studied and applied. Many practical problems can be found in [4–6]. For other results on variational and hemivariational inequalities the reader may consult [7–21].

Contact problems are ubiquitous in the fields of physics, biology, and engineering, and they have been a major problem for applied mathematicians. Over the past one hundred years of research, a range of effective mathematical methods have been produced. Li et al. [22] consider a mathematical model which describes the frictional contact with subdifferential boundary conditions between a deformable body and a foundation. Djabi et al. [23] employ the thermo-elastic-viscoplastic with damage constitutive law for the material. Recently, Mesai Aoun et al. [24] have investigated a quasistatic problem describing the contact with friction and wear between a piezoelectric body and a moving foundation. Many models related to contact problems can be found in the books [25,26]. In particular, we mention that our model can be used to describe many real problems such as the dynamic Nash equilibrium problem with shared constraints, engineering operation research and others, see [27–30].

Let X and W be separable and reflexive Banach spaces. We consider two Gelfand triples of spaces $V \subset H \subset V^*$ and $Y \subset Y_1 \subset Y^*$ with continuous, dense embeddings, where V, Y are separable, reflexive Banach spaces, and H, Y_1 are separable Hilbert spaces. Let K and K_Y be nonempty, closed, and convex subsets of V and Y, respectively. Moreover, we assume that $\widehat{A}: D(\widehat{A}) \subset W \to W$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on W. Hence, $D(\widehat{A})$ endowed with graph norm $\|y\|_Z = \|y\| + \|\widehat{A}y\|$ is a Banach space, which will be denoted by $(Z, \|\cdot\|_Z)$. We suppose that $\widehat{A}$ and $C(t)$ are linear closed operators on W with domains $D(\widehat{A}) \subset D(C)$, R_0, R_1 are two history-dependent operators, and $M : V \to X$ is a compact operator. Let I denote a finite time interval $[0, T]$. We formulate the following differential inequality system.

Problem 1.1. Find $u : I \to K$, $\zeta : I \to K_Y$ and $w : I \to W$ such that, for all $t \in I$,

$$\begin{cases} \dot{w}(t) = \widehat{A}w(t) + F(t, (R_0 w)(t), u(t), (R_1\zeta)(t)) + \displaystyle\int_0^t C(t-s)w(s)\,ds, \\ \langle A(t, u(t)) + \displaystyle\int_0^t B(t-s, w(s), u(s), \zeta(s))\,ds, v - u(t)\rangle_{V^*\times V} \\ \quad + j^0(w(t), Mu(t), Mu(t); Mv - Mu(t)) \geq \langle f(t), v - u(t)\rangle_{V^*\times V} \quad \text{for all} \quad v \in K, \\ \langle \dot{\zeta}(t), \eta - \zeta(t)\rangle_{Y_1} + a(\zeta(t), \eta - \zeta(t)) \geq \langle \phi(t, w(t), u(t), \zeta(t)), \eta - \zeta(t)\rangle_{Y_1} \quad \text{for all} \quad \eta \in K_Y, \\ w(0) = w_0, \quad \zeta(0) = \zeta_0. \end{cases}$$

The present system was studied in the literature only in particular cases. We list below a few examples. If the infinitesimal generator $\widehat{A}$ of a semigroup and an integral term with a linear and closed operator C are omitted, and the hypotheses are more restrictive, namely, the operator B is linear and does not depend on w, the operator ϕ is independent of w, and the operator F in the ordinary differential equation does not involve h.-d. operators and is independent of the variable ζ, then the problem has been recently studied by Chen et al. [31]. Further, if $j(w, x, y) = j(w, y)$ and $K = V$, then Problem 1.1 has been treated by Xuan and Cheng [32]. If ζ is neglected, and the infinitesimal generator $\widehat{A}$ of a semigroup and an integral term with a linear and closed operator C are omitted, the operator F does not involve h.-d. operators, $A(t, v) = Av$, and $j(w, x, \cdot)$ is convex, then Problem 1.1 reduces to the following one

$$\begin{cases} \dot{w}(t) = F(t, w(t), u(t)) \\ \langle A(u(t)) + Su(t), v - u(t)\rangle_{V^*\times V} \\ \quad + j(u(t), w(t), v) - j(u(t), w(t), u(t)) \geq \langle f(t), v - u(t)\rangle_{V^*\times V} \quad \text{for all} \quad v \in K, \\ w(0) = w_0, \end{cases}$$

which is closely related to a problem studied by Chen et al. [33]. For several relevant particular cases of Problem 1.1, we refer to Migórski et al. [34], Migórski and Zeng [35], Han et al. [36], and Xuan and Cheng [32].

In this paper we construct the unconstrained differential variational–hemivariational inequalities governed by the set of constraints $K_n \supset K$ and a penalty parameter $\rho_n > 0$. Note that the method we use, the constraints imposed to the solution are not completely removed (since K_n is not supposed to be the whole space), but with a new unilateral constraint (since $K_n \supset K$). For this reason, we call this method as a generalized penalty method (cf. [37]).

Finally, we consider a new model of quasistatic elastic frictional contact problem involving damage, and a heat equation with memory. We will show that the weak formulation of the mechanical problem leads to a differential variational inequality to which our abstract results apply.

The remaining part of this paper is organized as follows. In Section 2, we recall a useful notation and some results which will be used in the rest of this paper. In Section 3, we present the existence and uniqueness result for Problem 1.1 by using a fixed point principle. In Section 4, we introduce the generalized penalty problem, establish its unique solvability, and prove our main convergence result, Theorem 4.3. In Section 5 we consider an example of a frictional contact problem which is transformed into Problem 1.1 and solved by the methods developed in the paper.

2. Preliminaries

In this section we recall a preliminary material which will be useful in the sequel. For more details, we refer to [29,38,39].

Let $(X, \|\cdot\|_X)$ be a Banach space, X^* be its dual space and $\langle\cdot,\cdot\rangle_{X^*\times X}$ denote the duality brackets between X^* and X. We write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{V^*\times V}$ everywhere in the paper. The symbols "$\to$" and "$\rightharpoonup$" stand for the strong and the weak convergences in different spaces, respectively. We denote by $C(I;X)$ the space of continuous functions defined on I with values in X which is equipped with the standard norm $\|v\|_{C(I;X)} = \max_{t\in I}\|v(t)\|_X$ for all $v \in C(I;X)$. It is well known that if X is a Banach space, then $C(I;X)$ is also a Banach space. A mapping $G : X \to X^*$ is called

(a) monotone, if $\langle Gu - Gv, u - v\rangle \geq 0$ for all u, $v \in X$,

(b) bounded, if G maps bounded sets of X into bounded sets of X^*,

(c) pseudomonotone, if G is bounded and for every sequence $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ in X such that $\limsup\langle Gu_n, u_n - u\rangle \leq 0$, we have $\langle Gu, u - v\rangle \leq \liminf\langle Gu_n, u_n - v\rangle$ for all $v \in X$,

(d) hemicontinuous, if the function $\lambda \mapsto \langle G(u + \lambda v), w\rangle$ is continuous on $[0, 1]$ for all u, v, $w \in X$,

(e) demicontinuous, if $u_n \to u$ in X implies $Gu_n \to Gu$ weakly in X^*,

(f) maximal monotone, if G is monotone and $\langle Gu - w, u - v\rangle \geq 0$ for any $u \in X$ entails $w = Gv$.

It is known that $G : X \to X^*$ is pseudomonotone (in the sense of the above definition) if and only if G is bounded and $x_n \to x$ in X with $\limsup\langle Gx_n, x_n - x\rangle \leq 0$ implies $\lim\langle Gx_n, x_n - x\rangle = 0$ and $Gx_n \rightharpoonup Gx$ in X^*.

A convex function $j : X \to \mathbb{R}\cup\{\infty\}$ is said to be proper, if $j > -\infty$ and there exists a point $u \in X$ such that $j(u) < +\infty$. It is well known that, if $j : X \to \mathbb{R}\cup\{\infty\}$ is a convex, proper and lower semicontinuous function, then the convex subdifferential $\partial_c j$ is a maximal monotone operator.

Next, we recall the notion of the Clarke generalized gradient, see [39].

Definition 2.1. The Clarke generalized directional derivative of a locally Lipschitz function $j : X \to \mathbb{R}$ at x in the direction v, denoted by $j^0(x; v)$, is defined by

$$j^0(x;v) = \limsup_{y\to x,\, \lambda\to 0^+} \frac{j(y+\lambda v) - j(y)}{\lambda} \quad \text{for all} \quad x, v \in X.$$

The generalized Clarke subdifferential of j at x is a subset of X^* given by

$$\partial j(x) = \left\{x^* \in X^* \mid j^0(x;v) \geq \langle x^*, v\rangle \quad \text{for all} \quad v \in X\right\}.$$

Definition 2.2 (*See [40]*)**.**Let K and $\{K_n\}$ be nonempty subsets of V. We say that the sequence K_n Mosco converges to K, and write $K_n \xrightarrow{M} K$, if

(a) for each $u \in K$, there exists a sequence $\{u_n\}$ such that for $n \in \mathbb{N}$, $u_n \in K_n$ and $u_n \to u$ in V,

(b) for each sequence $\{u_n\}$ such that for $n \in \mathbb{N}$, $u_n \in K_n$ and $u_n \rightharpoonup u$ in V, we have $u \in K$.

Definition 2.3. Let $\mathbb{X}$ and $\mathbb{Y}$ be normed spaces. An operator $\mathbb{S} : C(I;\mathbb{X}) \to C(I;\mathbb{Y})$ is called a history-dependent operator, (or h.-d. operator for short), if there is a constant $c > 0$ such that

$$\|(\mathbb{S}v_1)(t) - (\mathbb{S}v_2)(t)\|_{\mathbb{Y}} \leq c\int_0^t \|v_1(s) - v_2(s)\|_{\mathbb{X}}\, ds$$

for all v_1, $v_2 \in C(I;\mathbb{X})$, $t \in I$. Details on various classes of h.-d. operators, their properties, and applications, can be found in [21] and references therein.

Definition 2.4 (*See [2, Definition 2.1]*)**.** Let W be a Banach space and $\hat{A} : D(\hat{A}) \subset W \to W$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on W. Let $Z := D(\hat{A})$ denote a Banach space with the graph norm $\|y\|_Z = \|y\| + \|\hat{A}y\|$. A bounded linear operator-valued function $R(t) \in \mathcal{L}(W)$ for $t \geq 0$, is called a resolvent operator for the (1.1)(i):

$$\dot{w}(t) = \hat{A}w(t) + F(t, (R_0 w)(t), u(t), (R_1\zeta)(t)) + \int_0^t C(t-s)w(s)\, ds,$$

if it has the following properties:

(a) $R(0) = I$ (identity operator on W) and $\|R(t)\|_{\mathcal{L}(W)} \leq Ne^{\omega t}$ for some constants $N \geq 1$ and $\omega > 0$,

(b) for each $y \in W$, $t \mapsto R(t)y$ is continuous for $t \geq 0$,

(c) for all $y \in Z$, $R(\cdot)y \in C^1([0,+\infty), W) \cap C([0,+\infty), Z)$ and

$$\begin{aligned}\dot{R}(t)y &= \hat{A}R(t)y + \int_0^t C(t-s)R(s)y\, ds\\ &= R(t)\hat{A}y + \int_0^t R(t-s)C(s)y\, ds \text{ for all } t \geq 0.\end{aligned}$$

Lemma 2.5 (*See [21, Lemma 3, p.110]*). *Let X be a reflexive Banach space. If $A : X \to X^*$ is a bounded, hemicontinuous and monotone operator, then it is pseudomonotone. Moreover, if $A,\ B : X \to X^*$ are pseudomonotone operators, then $A + B : X \to X^*$ is pseudomonotone, too.*

3. Differential variational–hemivariational inequalities

In this section, we will study the existence and uniqueness of solutions to Problem 1.1.

First, we need the following assumptions.

H(A): The operator $A : I \times V \to V^*$ satisfies

(a) for all $v \in V$, $A(\cdot, v)$ is continuous on I,

(b) for all $t \in I$, $A(t, \cdot)$ is hemicontinuous and strongly monotone with $m_A > 0$, i.e.,

$$\left\langle A\left(t,u_1\right) - A\left(t,u_2\right), u_1 - u_2\right\rangle \geq m_A \left\|u_1 - u_2\right\|_V^2 \quad \text{for all} \quad u_1, u_2 \in V,$$

(c) for all $t \in I$, we have $A\left(t, 0_V\right) = 0_{V^*}$.

H(B): The operator $B : I \times W \times V \times Y \to V^*$ satisfies

(a) for any $w \in W$, $v \in V$ and $\zeta \in Y$, $B(\cdot, w, v, \zeta)$ is continuous on I,

(b) for any $t \in I$, $B(t, \cdot, \cdot, \cdot)$ is Lipschitz continuous with $L_B > 0$, i.e.,

$$\left\|B\left(t, w_1, u_1, \zeta_1\right) - B\left(t, w_2, u_2, \zeta_2\right)\right\|_{V^*} \leq L_B\left(\left\|w_1 - w_2\right\|_W + \left\|u_1 - u_2\right\|_V + \left\|\zeta_1 - \zeta_2\right\|_Y\right)$$

for all $w_1,\ w_2 \in W$, $u_1,\ u_2 \in V$ and $\zeta_1,\ \zeta_2 \in Y$.

H(C): A family of linear closed operators $\{C(t)\}_{t\in[0,T]}$ on W is continuous as a linear map from Z into W, and the map $t \to C(t)y$ is measurable for all $y \in Z$, $t \in I$, and belongs to $W^{1,1}(I; W)$. Moreover, there exists an integrable function $c : [0, +\infty) \to \mathbb{R}^+$ such that

$$\|C(t)y\|_W \leq c(t)\|y\|_Z \quad \text{and} \quad \left\|\frac{d}{dt}C(t)y\right\|_W \leq c(t)\|y\|_Z \quad \text{for all} \quad y \in \mathrm{Z}.$$

H(j): The functional $j : W \times X \times X \to \mathbb{R}$ satisfies

(a) for any $w \in W$ and $x \in X$, $j(w, x, \cdot)$ is locally Lipschitz on X,

(b) there exist constants $c_0,\ c_1 > 0$ such that

$$\|\partial j(w, x, y)\|_{X^*} \leq c_0\|y\|_X + c_1\left(1 + \|w\|_W + \|x\|_X\right), \quad \text{for all} \quad (x, y, w) \in X \times X \times W,$$

(c) there exist constants $\alpha_0,\ \alpha_1 \geq 0$ such that

$$\begin{aligned} &j^0\left(w_1, Mu_1, Mv_1; Mv_2 - Mv_1\right) + j^0\left(w_2, Mu_2, Mv_2; Mv_1 - Mv_2\right) \\ &\quad \leq\ \alpha_0\left\|w_1 - w_2\right\|_W\left\|v_1 - v_2\right\|_V + \alpha_1\left\|u_1 - u_2\right\|_V\left\|v_1 - v_2\right\|_V \end{aligned}$$

for all $w_1,\ w_2 \in W$ and $u_1,\ u_2,\ v_1,\ v_2 \in V$.

H(F): The operator $F : I \times W \times V \times Y \to W$ satisfies

(a) for any $w \in W$, $v \in V$ and $\zeta \in Y$, $F(\cdot, w, v, \zeta)$ is continuous on I,

(b) for any $t \in I$, $F(t, \cdot, \cdot, \cdot)$ is Lipschitz continuous with $L_F > 0$, i.e.,

$$\left\|F\left(t, w_1, u_1, \zeta_1\right) - F\left(t, w_2, u_2, \zeta_2\right)\right\|_W \leq L_F\left(\left\|w_1 - w_2\right\|_W + \left\|u_1 - u_2\right\|_V + \left\|\zeta_1 - \zeta_2\right\|_Y\right)$$

for all $w_1,\ w_2 \in W$, $u_1,\ u_2 \in V$ and $\zeta_1,\ \zeta_2 \in Y$.

H(ϕ): The operator $\phi : I \times W \times V \times Y \to Y_1$ satisfies

(a) for any $t \in I$, $\phi(t, \cdot, \cdot, \cdot)$ is Lipschitz continuous with $L_\phi > 0$, i.e.,

$$\|\phi\left(t, w_1, u_1, \zeta_1\right) - \phi\left(t, w_2, u_2, \zeta_2\right)\|_{Y_1} \leq L_\phi\left(\left\|w_1 - w_2\right\|_W + \left\|u_1 - u_2\right\|_V + \left\|\zeta_1 - \zeta_2\right\|_Y\right)$$

for all $w_1,\ w_2 \in W$, $u_1,\ u_2 \in V$ and $\zeta_1,\ \zeta_2 \in Y$,

(b) $\phi\left(\cdot, 0_W, 0_V, 0_Y\right) \in L^2\left(I; Y_1\right)$.

H(R): $R_0 : L^2(0,T;W) \to L^2(0,T;W)$, $R_1 : L^2(0,T;Y) \to L^2(0,T;Y)$ are such that

(a) $\left\|\left(R_0 v_1\right)(t) - \left(R_0 v_2\right)(t)\right\|_W \leq r_0 \int_0^t \left\|v_1(s) - v_2(s)\right\|_W\, ds$
for all $v_1,\ v_2 \in L^2(0,T;W)$, all $t \in I$ with constant $r_0 > 0$,

(b) $\left\|\left(R_1 v_1\right)(t) - \left(R_1 v_2\right)(t)\right\|_Y \leq r_1 \int_0^t \left\|v_1(s) - v_2(s)\right\|_Y\, ds$
for all $v_1,\ v_2 \in L^2(0,T;Y)$, all $t \in I$ with constant $r_1 > 0$.

H(a): The form $a : Y \times Y \to \mathbb{R}$ is bilinear, continuous, symmetric and coercive in the following sense: there exist $a_1 \in \mathbb{R}$ and $a_2 > 0$ such that

$$a(\eta, \eta) + a_1\|\eta\|_Y^2 \geq a_2\|\eta\|_Y^2 \quad \text{for all} \quad \eta \in Y.$$

(H_0): The initial data are such that $w_0 \in W$ and $\zeta_0 \in (0, 1)$.

Lemma 3.1 (*See [31, Lemma 3.1]*). *Assume $H(A)$ and $H(j)$. Let K with $0_V \in K$ be closed, and convex subsets of V. If $m_A > \max\{c_0\|M\|^2_{\mathcal{L}(V;X)}, \alpha_1\}$, then for any fixed $w \in C(I;W)$ and $f \in C(I;V^*)$, there exists a unique solution $u_w \in C(I;K)$ to the following problem:*

$$\begin{aligned}&\left\langle A\left(t,u_w(t)\right), v-u_w(t)\right\rangle + j^0\left(w(t), Mu_w(t), Mu_w(t), Mv - Mu_w(t)\right)\\ &\quad \geq \langle f(t), v-u_w(t)\rangle \quad \text{for all} \quad v\in K,\ t\in I.\end{aligned}$$

Lemma 3.2 (*See [31, Lemma 3.4]*). *Assume that $H(a)$ holds and K_Y be a nonempty, closed, and convex subset of Y. Then for any fixed $g\in L^2\left(I;Y_1\right)$, there exists a unique function $\zeta \in H^1\left(I;Y_1\right)\cap L^2(I;Y)$ such that*

$$\langle\dot\zeta(t), \eta-\zeta(t)\rangle_{Y_1} + a(\zeta(t),\eta-\zeta(t)) \geq \langle g(t),\eta-\zeta(t)\rangle_{Y_1} \quad \text{for all} \quad \eta\in K_Y,\ \text{a.e.} \quad t\in I$$

with $\zeta(0)=\zeta_0\in K_Y$. Moreover, if ζ_i is the unique solution to above inequality for $g=g_i\in L^2\left(I;Y_1\right)$, $i=1,\,2$, respectively, then

$$\|\zeta_1(t)-\zeta_2(t)\|^2_{Y_1} \leq d_1\int_0^t \|g_1(s)-g_2(s)\|^2_{Y_1}\,ds \quad \text{for a.e.} \quad t\in(0,T)$$

with a constant $d_1>0$.

Next, we have the following existence result.

Theorem 3.3. *Assume that $H(A)$, $H(B)$, $H(C)$, $H(j)$, $H(F)$, $H(\phi)$, $H(R)$, $H(a)$ and (H_0) are fulfilled. Let K and K_Y be nonempty, closed, and convex subsets of V and Y, respectively and $0_V\in K$. If $m_A > \max\{c_0\|M\|^2_{\mathcal{L}(V;X)},\alpha_1\}$, then Problem 1.1 has a unique solution*

$$(\zeta,u,w)\in\left(H^1\left(I;Y_1\right)\cap L^2(I;Y)\right)\times C(I;K)\times C(I;W).$$

Proof. We denote the three formulas in Problem 1.1 by (1.1)(i), (1.1)(ii), (1.1)(iii), respectively. Fix $\eta\in C(I;K)$, $\zeta\in H^1\left(I;Y_1\right)\cap L^2(I;Y)$, $w\in C(I;W)$ and define f_η by $f_\eta(t) := f(t)-\int_0^t B(t-s,w(s),\eta(s),\zeta(s))\,ds$ for $t\in I$. We consider the following reduced form of inequality (1.1)(ii):

$$\langle A(t,u(t)), v-u(t)\rangle + j^0(w(t), Mu(t), Mu(t); Mv-Mu(t)) \geq \langle f_\eta(t), v-u(t)\rangle \tag{1}$$

for all $v\in K$ and $t\in I$.

First, we will show that $f_\eta\in C(I;V^*)$. Let $t_1,\,t_2\in I$ with $t_1<t_2$. From the definition of f_η, we derive

$$\begin{aligned}&\left\|f_\eta\left(t_2\right)-f_\eta\left(t_1\right)\right\|_{V^*}\\ &\leq \left\|f\left(t_1\right)-f\left(t_2\right)\right\|_{V^*} + \left\|\int_0^{t_2} B\left(t_2-s,w(s),\eta(s),\zeta(s)\right)\,ds - \int_0^{t_1} B\left(t_1-s,w(s),\eta(s),\zeta(s)\right)\,ds\right\|_{V^*}\\ &\leq \left\|f\left(t_1\right)-f\left(t_2\right)\right\|_{V^*} + \int_{t_1}^{t_2}\left\|B\left(t_2-s,w(s),\eta(s),\zeta(s)\right)\right\|_{V^*}\,ds\\ &\quad + \int_0^{t_1}\left\|B\left(t_2-s,w(s),\eta(s),\zeta(s)\right)-B\left(t_1-s,w(s),\eta(s),\zeta(s)\right)\right\|_{V^*}\,ds.\end{aligned}$$

The hypothesis H(B) allows us to invoke the Lebesgue-dominated convergence theorem to obtain

$$\lim_{t_1\to t_2}\int_0^{t_1}\left\|B\left(t_2-s,w(s),\eta(s),\zeta(s)\right)\,ds - B\left(t_1-s,w(s),\eta(s),\zeta(s)\right)\right\|_{V^*}\,ds = 0$$

and get

$$\int_{t_1}^{t_2}\left\|B\left(t_2-s,w(s),\eta(s),\zeta(s)\right)\right\|_{V^*}\,ds\to 0, \quad \text{as} \quad \left|t_1-t_2\right|\to 0.$$

Thus, $\lim_{|t_1-t_2|\to 0}\left\|f_\eta\left(t_2\right)-f_\eta\left(t_1\right)\right\|_{V^*}=0$, i.e., $f_\eta\in C(I;V^*)$. We apply Lemma 3.1, we can conclude that inequality (1) has a unique solution $u\in C(I;K)$ for any fixed $\zeta\in H^1\left(I;Y_1\right)\cap L^2(I;Y)$, $w\in C(I;W)$ and $f\in C(I;V^*)$

Further, we prove that for any fixed $\zeta\in H^1\left(I;Y_1\right)\cap L^2(I;Y)$, $w\in C(I;W)$ and $f\in C(I;V^*)$, the inequality (1.1)(ii) has a unique solution $u\in C(I;K)$. To this end, we consider the operator $\tau : C(I;V)\to C(I;K)$ defined by $\eta\mapsto u_\eta$, where $\eta\in C(I;V)$ and $u_\eta(t)$ is the unique solution of the inequality (1). Then we can show that the operator τ has a unique fixed point in $C(I;V)$. In fact, let $u_0,\,u_1\in C(I;V)$ be two unique solutions of (1) with respect to $\eta_0,\,\eta_1\in C(I;V)$, i.e.,

$$\left\langle A\left(t,u_i(t)\right), v-u_i(t)\right\rangle + j^0\left(w, Mu_i(t), Mu_i(t); Mv-Mu_i(t)\right) \geq \left\langle f_{\eta_i}(t), v-u_i(t)\right\rangle \tag{2}$$

for all $v\in K$ and $t\in I$ with $i=0,\,1$. Putting $v=u_{1-i}(t)$ for $i=0,\,1$, in the inequality (2), we get

$$\begin{aligned}&\left\langle A\left(t,u_0(t)\right)-A\left(t,u_1(t)\right), u_0(t)-u_1(t)\right\rangle \leq \left\langle f_{\eta_1}(t)-f_{\eta_0}(t), u_0(t)-u_1(t)\right\rangle\\ &\quad + j^0\left(w, Mu_1(t), Mu_1(t); Mu_0(t)-Mu_1(t)\right) + j^0\left(w, Mu_0(t), Mu_0(t); Mu_1(t)-Mu_0(t)\right)\end{aligned} \tag{3}$$

for all $t \in I$. By the definition of f_η and hypotheses H(A)(b), H(B)(b), H(j)(c), the inequality (3) yields

$$\begin{aligned} m_A \|u_0(t)-u_1(t)\|_V^2 &\le \left\langle A\left(t,u_0(t)\right)-A\left(t,u_1(t)\right), u_0(t)-u_1(t)\right\rangle \\ &\le \left\|f_{\eta_1}(t)-f_{\eta_0}(t)\right\|_{V^*}\|u_0(t)-u_1(t)\|_V+\alpha_1\|u_0(t)-u_1(t)\|_V^2 \\ &\le L_B\|u_0(t)-u_1(t)\|_V\int_0^t\|\eta_0(s)-\eta_1(s)\|_V\,ds+\alpha_1\|u_0(t)-u_1(t)\|_V^2 \end{aligned}$$

for all $t \in I$, and hence due to $m_A > \max\{c_0\|M\|^2_{\mathcal{L}(V;X)}, \alpha_1\}$,

$$\|u_0(t)-u_1(t)\|_V \le \frac{L_B}{m_A-\alpha_1}\int_0^t \|\eta_0(s)-\eta_1(s)\|_V\,ds$$

for all $t \in I$. Invoking [21, Theorem 25], τ has a unique fixed point $u_{w\zeta} \in C(I;K)$, as claimed.

We claim that the solution $u_{w\zeta}$ depends continuously on w and ζ. To show this, let $w_1, w_2 \in C(I;W)$ and $\zeta_1, \zeta_2 \in H^1\left(I;Y_1\right) \cap L^2(I;Y)$. Insert $w=w_i$ and $\zeta=\zeta_i$ with $i=1$, $v=u_{w_2\zeta_2}$ and $i=2$, $v=u_{w_1\zeta_1}$ in (1.1)(ii), respectively. Summing up the resulting inequalities, we get

$$\begin{aligned} &\left(m_A-\alpha_1\right)\left\|u_{w_1\zeta_1}(t)-u_{w_2\zeta_2}(t)\right\|_V \\ &\quad\le L_B\int_0^t\left\|u_{w_1\zeta_1}(s)-u_{w_2\zeta_2}(s)\right\|_V ds+L_B\int_0^t\|\zeta_1(s)-\zeta_2(s)\|_{Y_1}+\|w_1(s)-w_2(s)\|_W\,ds \\ &\quad\quad+\alpha_0\|w_1(t)-w_2(t)\|_W \end{aligned}$$

and so

$$\begin{aligned} \left\|u_{w_1\zeta_1}(t)-u_{w_2\zeta_2}(t)\right\|_V \le{}& \frac{L_B}{m_A-\alpha_1}\int_0^t\left\|u_{w_1\zeta_1}(s)-u_{w_2\zeta_2}(s)\right\|_V ds+\frac{L_BT}{m_A-\alpha_1}\|\zeta_1-\zeta_2\|_{L^2(I;Y_1)} \\ &+\frac{\alpha_0}{m_A-\alpha_1}\|w_1(t)-w_2(t)\|_W+\frac{L_B}{m_A-\alpha_1}\int_0^t\|w_1(s)-w_2(s)\|_W\,ds. \end{aligned} \tag{4}$$

Now Gronwall's inequality yields

$$\begin{aligned} &\left\|u_{w_1\zeta_1}(t)-u_{w_2\zeta_2}(t)\right\|_V \\ &\quad\le\left(\left(\frac{L_B\alpha_0}{\left(m_A-\alpha_1\right)^2}+\frac{L_B^2+L_B}{(m_A-\alpha_1)^2}\right)e^{\frac{L_BT}{m_A-\alpha_1}}+\frac{L_B}{m_A-\alpha_1}\right)\int_0^t\|w_1(s)-w_2(s)\|_W\,ds \\ &\quad\quad+\frac{\alpha_0}{m_A-\alpha_1}\|w_1(t)-w_2(t)\|_W+\left(\frac{L_BT^2L_B}{\left(m_A-\alpha_1\right)^2}e^{\frac{L_BT}{m_A-\alpha_1}}+\frac{L_BT}{m_A-\alpha_1}\right)\|\zeta_1-\zeta_2\|_{L^2(I;Y_1)}. \end{aligned} \tag{5}$$

In what follows, we shall prove that for any fixed $\zeta \in H^1\left(I;Y_1\right) \cap L^2(I;Y)$, the two inequalities (1.1)(i) and (1.1)(ii) have a unique solution $(u_\zeta, w_\zeta) \in C(I;K)\times C^1(I;W)$. We define the operator $Q: C^1(I;W) \to C(I;K)$ by setting $Q(w_\zeta)(t) := u_{w\zeta}(t)$, where $u_{w\zeta}$ is the solution of inequality (1.1)(ii). Thus, we only need to show that there exists a unique $w \in C^1(I;W)$ such that

$$\dot{w}_\zeta(t)=\widehat{A}w_\zeta(t)+F(t,(R_0w_\zeta)(t),Q(w_\zeta)(t),(R_1\zeta)(t))+\int_0^t C(t-s)w_\zeta(s)\,ds$$

for all $\zeta \in H^1\left(I;Y_1\right)\cap L^2(I;Y)$. Therefore, $\left(Q\left(w_\zeta\right), w_\zeta\right) \in C(I;K)\times C^1(I;W)$ is the unique solution of inequality (1.1). Note that a continuous function $w: I \to W$ is called a mild solution of (1.1)(i) with the initial condition $w(0)=w_0$, see [2, Theorem 2.4], if $H(F)$, $H(C)$ are fulfilled and $w \in C(I;W)$ satisfies

$$w(t)=T(t)w_0+\int_0^t T(t-s)F(t,(R_0w)(s),u(s),(R_1\zeta)(s))\,ds \quad \text{for all} \quad t\in I.$$

We consider the operator $\Lambda: C(I;W) \to C^1(I;W)$ defined by

$$\Lambda w_\zeta(t)=T(t)w_0+\int_0^t T(t-s)F(t,(R_0w_\zeta)(s),Q(w_\zeta)(s),(R_1\zeta)(s))\,ds \quad \text{for all} \quad t\in I.$$

From conditions H(F)(a) and (b), and the definition of operator Q, we get $\Lambda w \in C^1(I;W)$ when $w \in C(I;W)$. Hence, we only need to prove that Λ has a unique fixed point in $C(I;W)$. Let $w_1, w_2 \in C(I;W)$. We conclude from the definitions of Q and Λ, and the hypothesis H(F)(b) that

$$\begin{aligned}
&\left\|\Lambda w_{\zeta_1}(t)-\Lambda w_{\zeta_2}(t)\right\|_W\\
&=\Big\|\int_0^t T(t-s)F\left(s,(R_0w_{\zeta_1})(s),Q\left(w_{\zeta_1}\right)(s),(R_1\zeta)(s)\right)ds\\
&\quad-\int_0^t T(t-s)F\left(s,(R_0w_{\zeta_2})(s),Q\left(w_{\zeta_2}\right)(s),(R_1\zeta)(s)\right)ds\Big\|_W\\
&\le M_1\int_0^t\left\|F\left(s,(R_0w_{\zeta_1})(s),Q\left(w_{\zeta_1}\right)(s),(R_1\zeta)(s)\right)ds-F\left(s,(R_0w_{\zeta_2})(s),Q\left(w_{\zeta_2}\right)(s),(R_1\zeta)(s)\right)\right\|_W ds\\
&\le M_1L_F\int_0^t(\left\|\left(R_0w_{\zeta_1}\right)(s)-(R_0w_{\zeta_2})(s)\right\|_W+\left\|Q\left(w_{\zeta_1}\right)(s)-Q\left(w_{\zeta_2}\right)(s)\right\|_V)ds\\
&\le M_1L_F\int_0^t(r_0T\left\|w_{\zeta_1}(s)-w_{\zeta_2}(s)\right\|_W+\left\|Q\left(w_{\zeta_1}\right)(s)-Q\left(w_{\zeta_2}\right)(s)\right\|_V)ds,
\end{aligned}$$

where $M_1=\sup_{t\in I}\|T(t)\|$. Let

$$k_0=\Big(\Big(\frac{(L_B+L_R)\alpha_0}{\left(m_A-\alpha_1\right)^2}+\frac{L_B^2+L_BL_R}{(m_A-\alpha_1)^2}\Big)e^{\frac{(L_B+L_R)T}{m_A-\alpha_1}}+\frac{L_B}{m_A-\alpha_1}\Big),\quad k_1=\frac{\alpha_0}{m_A-\alpha_1}.$$

Then it follows from inequality (5) that

$$\left\|Q\left(w_{\zeta_1}\right)(s)-Q\left(w_{\zeta_2}\right)(s)\right\|_V\le k_0\int_0^s\left\|w_{\zeta_1}(r)-w_{\zeta_2}(r)\right\|_W dr+k_1\left\|w_{\zeta_1}(s)-w_{\zeta_2}(s)\right\|_W.$$

We consider the equivalent norm $\|\cdot\|_L$ on $C(I;W)$ given by

$$\|z\|_L=\max_{t\in I}e^{-Lt}\|z(t)\|_W\quad\text{for all}\quad z\in C(I;W).$$

Therefore, we get

$$\begin{aligned}
\left\|\Lambda w_{\zeta_1}(t)-\Lambda w_{\zeta_2}(t)\right\|&\le\left(M_1L_Fr_0T+M_1L_Fk_1\right)\int_0^t\left\|w_{\zeta_1}(s)-w_{\zeta_2}(s)\right\|ds\\
&\quad+M_1L_Fk_0\int_0^t\int_0^s\left\|w_{\zeta_1}(r)-w_{\zeta_2}(r)\right\|drds
\end{aligned}$$

and so

$$\left\|\Lambda w_{\zeta_1}(t)-\Lambda w_{\zeta_2}(t)\right\|\le\Big(\frac{M_1L_Fr_0T+M_1L_Fk_1}{L}\Big)\left\|w_{\zeta_1}-w_{\zeta_2}\right\|_L+\frac{M_1L_Fk_0}{L^2}\left\|w_{\zeta_1}-w_{\zeta_2}\right\|_L \tag{6}$$

for all $t\in I$ with $L>0$ and L is large enough, which yields that the operator Λ is a contraction on $C(I;W)$ endowed with the norm $\|\cdot\|_L$. This shows that Λ has a unique fixed point $w_\zeta\in C(I;W)$, i.e., $\left(w_\zeta,u_\zeta\right):=\left(w_\zeta,Q\left(w_\zeta\right)\right)$.

Let us fix ζ_1, $\zeta_2\in H^1\left(I;Y_1\right)\cap L^2(I;Y)$, and $\left(u_{\zeta_i},w_{\zeta_i}\right)\in C(I;V)\times C^1(I;W)$ be the solutions of the two inequalities (1.1)(i) and (1.1)(ii) with $\zeta=\zeta_i$ for $i=1$, 2. Then it follows from the proof of (6) that

$$\begin{aligned}
\left\|w_{\zeta_1}(t)-w_{\zeta_2}(t)\right\|_W&\le\left(M_1L_Fr_0T+M_1L_Fk_1\right)\int_0^t\left\|w_{\zeta_1}(s)-w_{\zeta_2}(s)\right\|_W ds\\
&\quad+M_1L_Fk_0\int_0^t\int_0^s\left\|w_{\zeta_1}(r)-w_{\zeta_2}(r)\right\|_W drds+M_1k_2L_FT\left\|\zeta_1-\zeta_2\right\|_{L^2(I;Y_1)}\\
&\le\left(M_1L_Fr_0T+M_1L_Fk_1+M_1L_Fk_0T\right)\int_0^t\left\|w_{\zeta_1}(s)-w_{\zeta_2}(s)\right\|_W ds\\
&\quad+M_1k_2L_FT\left\|\zeta_1-\zeta_2\right\|_{L^2(I;Y_1)}
\end{aligned}$$

where $k_2=\frac{L_BT^2(L_B+L_R)}{\left(m_A-\alpha_1\right)^2}e^{\frac{(L_B+L_R)T}{m_A-\alpha_1}}+\frac{L_BT}{m_A-\alpha_1}$. Applying Gronwall's inequality, we are led to

$$\left\|w_{\zeta_1}(t)-w_{\zeta_2}(t)\right\|_W\le\left(M_1k_2L_FT\left\|\zeta_1-\zeta_2\right\|_{L^2(I;Y)}\right)e^{(M_1L_Fr_0T+M_1L_Fk_1+M_1L_Fk_0T)T} \tag{7}$$

for all $t\in I$. We have

$$\begin{aligned}
\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V&\le k_0\int_0^t\left\|w_{\zeta_1}(s)-w_{\zeta_2}(s)\right\|_W ds+k_1\left\|w_{\zeta_1}(t)-w_{\zeta_2}(t)\right\|_W+k_2\left\|\zeta_1-\zeta_2\right\|_{L^2(I;Y_1)}\\
&\le\left(k_0T+k_1\right)\left\|w_{\zeta_1}(t)-w_{\zeta_2}(t)\right\|_W+k_2\left\|\zeta_1-\zeta_2\right\|_{L^2(I;Y_1)}
\end{aligned}$$

for all $t\in I$. Hence and by (7), it follows that

$$\begin{aligned}
&\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V\\
&\le\left(\left(M_1k_0k_2L_FT^2+M_1k_1k_2L_FT\right)e^{(M_1L_Fr_0T+M_1L_Fk_1+M_1L_Fk_0T)T}+k_2\right)\left\|\zeta_1-\zeta_2\right\|_{L^2(I;Y_1)}
\end{aligned} \tag{8}$$

for all $t \in I$.

Similarly as we have proved the above, for any fixed $\theta \in L^2(I; Y_1)$, the inequalities (1.1)(i) and (1.1)(ii) have a unique solution (w_θ, u_θ). Let $\phi_\theta(t) := \phi(t, w_\theta, u_\theta(t), \theta(t))$. Then, we claim that $\phi_\theta \in L^2(I; Y_1)$. In fact, hypothesis H($\phi$)(a) leads to the following inequality

$$\begin{aligned}\|\phi_\theta(t)\|_{Y_1}^2 &\le 2\left\|\phi(t, w_\theta, u_\theta(t), \theta(t)) - \phi\left(t, 0_W, 0_V, 0_{Y_1}\right)\right\|_{Y_1}^2 + 2\left\|\phi\left(t, 0_W, 0_V, 0_{Y_1}\right)\right\|_{Y_1}^2 \\ &\le 2\left(L_\phi\left(\|w_\theta(t)\|_W + \|u_\theta(t)\|_V + \|\theta(t)\|_{Y_1}\right)\right)^2 + 2\left\|\phi\left(t, 0_W, 0_V, 0_{Y_1}\right)\right\|_{Y_1}^2 \\ &\le 6L_\phi^2\left(\|w_\theta(t)\|_W^2 + \|u_\theta(t)\|_V^2 + \|\theta(t)\|_{Y_1}^2\right) + 2\left\|\phi\left(t, 0_W, 0_V, 0_{Y_1}\right)\right\|_{Y_1}^2.\end{aligned}$$

From assumption H(ϕ)(b), we have

$$\begin{aligned}\|\phi_\theta\|_{L^2(I;Y_1)} \le{}& \sqrt{6L_\phi^2 T}\left(\|u_\theta\|_{C(I;V)}^2 + \|w_\theta\|_{C(I;W)}^2\right) + \sqrt{6L_\phi^2 T}\|\theta\|_{L^2(I;Y_1)} \\ &+ \sqrt{2}\left\|\phi\left(t, 0_W, 0_V, 0_{Y_1}\right)\right\|_{L^2(I;Y_1)},\end{aligned}$$

and so $\phi_\theta \in L^2(I; Y_1)$. Now taking $g = \phi_\theta$ in Lemma 3.2, we deduce that there exists a unique $\zeta_\theta \in H^1(I; Y_1) \cap L^2(I; Y)$. We define an operator $\gamma : L^2(I; Y_1) \to H^1(I; Y_1) \cap L^2(I; Y)$ by putting $\gamma(\theta)(t) := \zeta_\theta(t)$. Since (Y, Y_1, Y^*) is a Gelfand triple with dense embeddings, one has $H^1(I; Y_1) \cap L^2(I; Y) \subset L^2(I; Y_1)$. Eventually, we prove that γ has a unique fixed point in $L^2(I; Y_1)$. Letting $\theta_i \in L^2(I; Y_1)$ with $i = 1, 2$, it follows from Lemma 3.2 that

$$\begin{aligned}\left\|\gamma(\theta_1)(t) - \gamma(\theta_2)(t)\right\|_{Y_1}^2 &= \left\|\zeta_{\theta_1}(t_1) - \zeta_{\theta_2}(t_2)\right\|_{Y_1}^2 \\ &\le d_1 \int_0^t \left\|\phi\left(s, w_{\theta_1}(s), u_{\theta_1}(s), \theta_1(s)\right) - \phi\left(s, w_{\theta_2}(s), u_{\theta_2}(s), \theta_2(s)\right)\right\|_{Y_1}^2 ds\end{aligned}$$

for all $t \in I$. From assumption H(ϕ)(a), we get

$$\begin{aligned}&\left\|\gamma(\theta_1)(t) - \gamma(\theta_2)(t)\right\|_{Y_1}^2 \\ &\le d_1 \int_0^t \left\|\phi\left(s, w_{\theta_1}(s), u_{\theta_1}(s), \theta_1(s)\right) - \phi\left(s, w_{\theta_2}(s), u_{\theta_2}(s), \theta_2(s)\right)\right\|_{Y_1}^2 ds \\ &\le 3d_1 L_\phi^2 \int_0^t \left\|w_{\theta_1}(s) - w_{\theta_2}(s)\right\|_W^2 + \left\|u_{\theta_1}(s) - u_{\theta_2}(s)\right\|_V^2 + \|\theta_1(s) - \theta_2(s)\|_{Y_1}^2\, ds.\end{aligned} \tag{9}$$

Now we combine inequalities (7) and (8) to see that we can rewrite (9) as

$$\begin{aligned}&\left\|\gamma(\theta_1)(t) - \gamma(\theta_2)(t)\right\|_{Y_1}^2 \\ &\le 3d_1 L_\phi^2 \int_0^t \left\|w_{\theta_1}(s) - w_{\theta_2}(s)\right\|_W^2 + \left\|u_{\theta_1}(s) - u_{\theta_2}(s)\right\|_V^2 + \|\theta_1(s) - \theta_2(s)\|_{Y_1}^2\, ds \\ &\le c\left\|\theta_1 - \theta_2\right\|_{L^2(I;Y_1)}^2\end{aligned} \tag{10}$$

for all $t \in I$ with

$$\begin{aligned}c :={}& 3d_1 L_\phi^2\left(\left(k_0 k_2 T^2 + k_1 k_2 T\right) e^{(L_F + L_F c_p + L_F c_q T)T} + k_2\right)^2 T \\ &+ 3d_1 L_\phi^2\left(k_2 L_F T e^{(M_1 L_F r_0 T + M_1 L_F k_1 + M_1 L_F k_0 T)T}\right)^2 T + 3d_1 L_\phi^2.\end{aligned}$$

From (10) we deduce

$$\|\gamma(\theta_1)(t) - \gamma(\theta_2)(t)\|_{Y_1}^2 \le c\int_0^t \|\theta_1(s) - \theta_2(s)\|_{Y_1}^2\, ds \quad \text{for all} \quad t \in I \text{ with } c > 0.$$

This implies that the operator $\gamma : L^2(I; Y_1) \to L^2(I; Y_1)$ satisfies

$$\|\gamma(\theta_1)(t) - \gamma(\theta_2)(t)\|_{Y_1} \le c\int_0^t \|\theta_1(s) - \theta_2(s)\|_{Y_1}\, ds \quad \text{for all} \quad t \in I \text{ with } c > 0.$$

We apply [21, Theorem 67, p. 118] to get a unique point $\theta^* \in L^2(I, Y_1)$ such that $\theta^* = \gamma(\theta^*)$. So inequality (1.1)(iii) has a unique solution ζ which implies that Problem 1.1 has a unique solution $(\zeta, u_\zeta, w_\zeta) \in \left(H^1(I; Y_1) \cap L^2(I; Y)\right) \times C(I; K) \times C(I; W)$. Finally, we choose $(\zeta, u, w) = (\zeta, u_\zeta, w_\zeta)$, which concludes the proof. □

4. Generalized penalty method

In this section we keep the hypotheses of Section 3 and introduce the generalized penalty method in the study of Problem 1.1. We define a sequence of penalized problems, see Problem 4.2 below, prove their unique solvability and establish the convergence of the sequence of their solutions to the unique solution of Problem 1.1, obtained in Theorem 3.3.

Definition 4.1. An operator $P : V \to V^*$ is said to be a penalty operator of a set $K \subset V$ if P is bounded, demicontinuous, monotone and $K = \{u \in V \mid Pu = 0_{V^*}\}$.

We need the following assumptions.

H(P): $P : V \to V^*$ is a bounded, demicontinuous and monotone operator.

H(K): For each $n \in \mathbb{N}$, K_n is a nonempty closed convex subset of V with $K_n \supset K$. There exists a set $\widetilde{K}$ such that

(a) $K_n \subset \widetilde{K} \subset V$ for each $n \in \mathbb{N}$,

(b) $K_n \xrightarrow{M} \widetilde{K}$ as $n \to \infty$,

(c) $\langle Pu, v - u\rangle \le 0$ for all $u \in \widetilde{K}$ and $v \in K$,

(d) if $u \in \widetilde{K}$ and $\langle Pu, v - u\rangle = 0$ for all $v \in K$, then $u \in K$.

H(ρ_n): for each $n \in \mathbb{N}$, $\rho_n > 0$, and $\rho_n \to 0$ as $n \to \infty$.

Problem 4.2. Find $u_n : I \to V$, $\zeta_n : I \to K_Y$ and $w_n : I \to W$ such that, for all $t \in I$,

$$\begin{cases} \dot{w}_n(t) = \widehat{A} w_n(t) + F(t, (R_0 w_n)(t), u_n(t), (R_1\zeta_n)(t)) + \displaystyle\int_0^t C(t-s) w_n(s)\, ds, \\ \langle A(t, u_n(t)) + \displaystyle\int_0^t B(t-s, w_n(t), u_n(s), \zeta_n(s))\, ds, v - u_n(t)\rangle + \dfrac{1}{\rho_n}\langle P u_n(t), v - u_n(t)\rangle \\ \qquad + j^0(w_n(t), Mu_n(t), Mu_n(t); Mv - Mu_n(t)) \ge \langle f(t), v - u_n(t)\rangle \quad \text{for all} \quad v \in K_n, \\ \langle \dot{\zeta}_n(t), \eta - \zeta_n(t)\rangle_{Y_1} + a(\zeta_n(t), \eta - \zeta_n(t)) \ge \langle \phi(t, w_n(t), u_n(t), \zeta_n(t)), \eta - \zeta_n(t)\rangle_{Y_1} \\ \qquad \text{for all } \eta \in K_Y, \\ w_n(0) = w_0, \quad \zeta_n(0) = \zeta_0. \end{cases}$$

The main result of this section is the following.

Theorem 4.3. *Suppose that the assumptions $H(A)$, $H(B)$, $H(C)$, $H(j)$, $H(F)$, $H(\phi)$, $H(R)$, $H(P)$, $H(K)$, $H(a)$, (H_0) and $H(\rho_n)$ hold, and $m_A > \max\{c_0\|M\|^2_{L(V;X)}, \alpha_1\}$. Let K and K_Y be nonempty, closed, and convex subsets of V and Y, respectively and $0_V \in K$. Then*

(i) *for any fixed $n > 0$, Problem 4.2 has a unique solution*

$$(\zeta_n, u_n, w_n) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C(I; V) \times C(I; W),$$

(ii) *$(\zeta_n(t), u_n(t), w_n(t)) \to (\zeta(t), u(t), w(t))$ in $Y_1 \times V \times W$ for all $t \in I$, as $n \to \infty$, where (ζ, u, w) is the unique solution of Problem 1.1.*

Proof. (i) Consider the operator $A_n : I \times V \to V^*$, $n \in \mathbb{N}$, defined by

$$A_n(t, u) = A(t, u) + \frac{1}{\rho_n} Pu \quad \text{for} \quad (t, u) \in I \times V.$$

Since P is bounded, demicontinuous, monotone, and $K = \{u \in V \mid Pu = 0_{V^*}\}$, it follows from H($A$) that $A_n(\cdot, v)$ is continuous for any fixed $v \in V$ and $A_n(t, \cdot)$ is hemicontinuous, pseudomonotone, strongly monotone, and $A_n(t, 0_V) = 0_{V^*}$ for all $t \in I$. This shows that the mapping A_n satisfies hypothesis $H(A)$. Therefore we can use Theorem 3.3 with $K_n = V$ and $A = A_n$ rendering that Problem 4.2 has a unique solution $(\zeta_n, u_{n\zeta}, w_{n\zeta}) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C(I; V) \times C(I; W)$.

(ii) Let $(\zeta, u, w) = (\zeta, u_\zeta, w_\zeta) \in (H^1(I; Y_1) \cap L^2(I; Y)) \times C(I; V) \times C(I; W)$ be the unique solution of Problem 4.2. We consider the following auxiliary problem: find a map $\tilde{u}_n(t) : I \to V$ such that $\tilde{u}_n(t) \in K_n$ for all $t \in I$, and

$$\begin{aligned} &\langle A(t, \tilde{u}_n(t)), v - \tilde{u}_n(t)\rangle + \frac{1}{\rho_n}\langle P\tilde{u}_n(t), v - \tilde{u}_n(t)\rangle \\ &\quad + j^0(w(t), M\tilde{u}_n(t), M\tilde{u}_n(t); Mv - M\tilde{u}_n(t)) \ge \langle \overline{f}(t), v - \tilde{u}_n(t)\rangle, \end{aligned} \tag{11}$$

for all $t \in I$ and $v \in K_n$, where $\overline{f}$ is defined by

$$\overline{f}(t) := f(t) - \int_0^t B(t-s, w(s), u(s), \zeta(s))\, ds \quad \text{for} \quad t \in I.$$

Carrying out the same arguments as in the proof of Theorem 3.3, we know that the problem (11) is uniquely solvable for each $n \in \mathbb{N}$.

Claim 1. The sequence $\{\widetilde{u}_n(t)\}_{n\in\mathbb{N}, t\in I}$ is uniformly bounded in V. Let $n \in \mathbb{N}$, $t \in I$ and $u_0 \in K$. We take $v = u_0$ in (11) to deduce that

$$\begin{aligned} &\langle A(t, \tilde{u}_n(t)), u_0 - \tilde{u}_n(t)\rangle + \frac{1}{\rho_n}\langle P\tilde{u}_n(t), u_0 - \tilde{u}_n(t)\rangle \\ &\quad + j^0(w(t), M\tilde{u}_n(t), M\tilde{u}_n(t); Mu_0 - M\tilde{u}_n(t)) \ge \langle \overline{f}(t), u_0 - \tilde{u}_n(t)\rangle. \end{aligned}$$

It follows from the strong monotonicity of $A(t,\cdot)$ that

$$\begin{aligned} & m_A \left\|u_0-\tilde{u}_n(t)\right\|_V^2 \\ & \quad \leq \left\langle A\left(t, \tilde{u}_n(t)\right)-A\left(t, u_0\right), \tilde{u}_n(t)-u_0\right\rangle \\ & \quad \leq \left\langle A\left(t, u_0\right), u_0-\tilde{u}_n(t)\right\rangle+\frac{1}{\rho_n}\left\langle P \tilde{u}_n(t), u_0-\tilde{u}_n(t)\right\rangle \\ & \qquad +j^0\left(w(t), M \tilde{u}_n(t), M \tilde{u}_n(t) ; M u_0-M \tilde{u}_n(t)\right)-\left\langle\overline{f}(t), u_0-\tilde{u}_n(t)\right\rangle \end{aligned}$$

for all $t \in I$. As P is monotone, $Pv=0$ for all $v \in K$ and $u_0 \in K$, one has

$$\begin{aligned} & m_A\left\|u_0-\tilde{u}_n(t)\right\|_V^2 \\ & \leq\left\langle A\left(t, u_0\right), u_0-\tilde{u}_n(t)\right\rangle+\frac{1}{\rho_n}\left\langle P \tilde{u}_n(t)-P u_0, u_0-\tilde{u}_n(t)\right\rangle \\ & \quad +j^0\left(w(t), M \tilde{u}_n(t), M \tilde{u}_n(t) ; M u_0-M \tilde{u}_n(t)\right)-\left\langle\overline{f}(t), u_0-\tilde{u}_n(t)\right\rangle \\ & \leq\left\langle A\left(t, u_0\right), u_0-\tilde{u}_n(t)\right\rangle-\left\langle\overline{f}(t), u_0-\tilde{u}_n(t)\right\rangle \\ & \quad +j^0\left(w(t), M \tilde{u}_n(t), M \tilde{u}_n(t) ; M u_0-M \tilde{u}_n(t)\right) \end{aligned} \tag{12}$$

for all $t \in I$. The hypothesis H(j)(c) implies that

$$\begin{aligned} & j^0\left(w(t), M u_0, M u_0 ; M u_0-M \tilde{u}_n(t)\right)+j^0\left(w(t), M \tilde{u}_n(t), M \tilde{u}_n(t) ; M \tilde{u}_n(t)-M u_0\right) \\ & \quad \leq \alpha_1\left\|u_0-\tilde{u}_n(t)\right\|_V^2 . \end{aligned} \tag{13}$$

From (12), (13), and H(j)(b), we have

$$\begin{aligned} & \left(m_A-\alpha_1\right)\left\|u_0-\tilde{u}_n(t)\right\|_V^2 \\ & \quad \leq m_A\left\|u_0-\tilde{u}_n(t)\right\|_V^2+j^0\left(w(t), M u_0, M u_0 ; M u_0-M \tilde{u}_n(t)\right) \\ & \qquad +j^0\left(w(t), M \tilde{u}_n(t), M \tilde{u}_n(t) ; M \tilde{u}_n(t)-M u_0\right) \\ & \quad \leq\left(\left\|A\left(t, u_0\right)\right\|_{V^*}+\left\|\overline{f}(t)\right\|_{V^*}\right)\left\|u_0-\tilde{u}_n(t)\right\|_V \\ & \qquad +\left\|M^*\right\|\left(c_1\left(1+\|w\|_{C(I ; W)}+\|M\|\left\|u_0\right\|_V\right)+c_0\|M\|\left\|u_0\right\|_V\right)\left\|u_0-\tilde{u}_n(t)\right\|_V \end{aligned}$$

and so

$$\begin{aligned} \left(m_A-\alpha_1\right)\left\|u_0-\tilde{u}_n(t)\right\|_V \leq & \left(\left\|A\left(t, u_0\right)\right\|_{V^*}+\left\|\overline{f}(t)\right\|_{V^*}\right) \\ & +\|M\|^*\left(c_1\left(1+\|w\|_{C(I ; W)}+\|M\|\left\|u_0\right\|_V\right)+c_0\|M\|\left\|u_0\right\|_V\right) . \end{aligned} \tag{14}$$

Moreover, since $A(t,\cdot)$ is pseudomonotone, it is bounded, so there exists a constant $N_1>0$, such that

$$\left\|A\left(t, u_0\right)\right\|_{V^*} \leq N_1 \tag{15}$$

and from H(B) there exists a constant $N_2>0$ such that

$$\left\|\overline{f}(t)\right\|_{V^*} \leq\|f(t)\|_{V^*}+\int_0^t\|B(t-s, w(s), u(s), \zeta(s))\|_{V^*} d s \leq\|f\|_{C\left(I, V^*\right)}+N_2 . \tag{16}$$

Combining (14), (15), and (16), one has

$$\begin{aligned} & \left(m_A-\alpha_1\right)\left\|u_0-\tilde{u}_n(t)\right\|_V \\ & \quad \leq N_1+\|f\|_{C\left(I, V^*\right)}+N_2+\left\|M^*\right\|\left(c_1\left(1+\|w\|_{C(I ; W)}+\|M\|\left\|u_0\right\|_V\right)+c_0\|M\|\left\|u_0\right\|_V\right) . \end{aligned}$$

We conclude that $\left\{\tilde{u}_n(t)\right\}_{n \in \mathbb{N}, t \in I}$ is uniformly bounded in V, which proves the claim.

We may assume that for each $t \in I$, along a relabeled subsequence depending on t, one has

$$\tilde{u}_n(t) \rightharpoonup \tilde{u}(t) \text { in } V, \text { as } n \rightarrow \infty$$

with some $\tilde{u}(t) \in V$. Since $\widetilde{u}_n(t) \in K_n$, from H$(K)$(b) combined with Definition 2.2(b), it follows that $\tilde{u}(t) \in \widetilde{K}$.

Claim 2. For all $t \in I$, we have $u(t)=\tilde{u}(t) \in K$. Let $t \in I$ and $v \in \tilde{K}$. Then, according to Definition 2.2(a), there exists a sequence $\left\{v_n\right\}$ such that $v_n \in K_n$ for $n \in \mathbb{N}$ and $v_n \rightarrow v \in V$, as $n \rightarrow \infty$. From (11), we have

$$\begin{aligned} & \frac{1}{\rho_n}\left\langle P \tilde{u}_n(t), \tilde{u}_n(t)-v_n\right\rangle \\ & \quad \leq\left\langle A\left(t, \tilde{u}_n(t)\right), v_n-\tilde{u}_n(t)\right\rangle \\ & \qquad +j^0\left(w(t), M \tilde{u}_n(t), M \tilde{u}_n(t) ; M v_n-M \tilde{u}_n(t)\right)+\left\langle\overline{f}(t), \tilde{u}_n(t)-v_n\right\rangle \\ & \quad \leq\left\|A\left(t, \tilde{u}_n(t)\right)\right\|_{V^*}\left\|v_n-\tilde{u}_n(t)\right\|_V+\|\overline{f}(t)\|_{V^*}\left\|v_n-\tilde{u}_n(t)\right\|_V \\ & \qquad +\|M\|^*\left(c_1\left(1+\|w\|_{C(I ; W)}+\|M\|\left\|\tilde{u}_n(t)\right\|_V\right)+c_0\|M\|\left\|\tilde{u}_n(t)\right\|_V\right)\left\|v_n-\tilde{u}_n(t)\right\|_V . \end{aligned} \tag{17}$$

Thus, since $\{v_n\}$ and $\{\widetilde{u}_n(t)\}$ are bounded sequences in V, and $A(t,\cdot)$ is a bounded operator, we deduce that there exists a constant $\widetilde{C}_0 > 0$, which does not depend on n, such that

$$\frac{1}{\rho_n}\langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle \le \widetilde{C}_0.$$

Therefore

$$\limsup \langle P\tilde{u}_n(t), \tilde{u}_n(t) - v_n\rangle \le 0. \tag{18}$$

Now, since the sequence $\{P\tilde{u}_n(t)\}$ is bounded in V^* and $v_n \to v$ in V, we have

$$\begin{aligned}\limsup \langle P\tilde{u}_n(t), \widetilde{u}_n(t) - v\rangle &\le \limsup \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle + \limsup \langle P\widetilde{u}_n(t), v_n - v\rangle \\ &= \limsup \langle P\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n\rangle\end{aligned}$$

and, therefore, (18) yields

$$\limsup \langle P\tilde{u}_n(t), \tilde{u}_n(t) - v\rangle \le 0 \quad \text{for all} \quad v \in \widetilde{K}. \tag{19}$$

Moreover, the regularity $\tilde{u}(t) \in \widetilde{K}$ allows us to take $v = \widetilde{u}(t)$ in (19) to obtain

$$\limsup \langle P\tilde{u}_n(t), \tilde{u}_n(t) - \tilde{u}(t)\rangle \le 0. \tag{20}$$

On the other hand, assumption $H(P)$ combined with [4, Definition 3.69] guarantees that P is a pseudomonotone operator. Thus, inequality (20), together with the pseudomonotonicity of P, implies that

$$\langle P\tilde{u}(t), \tilde{u}(t) - v\rangle \le \liminf \langle P\widetilde{u}_n(t), \tilde{u}_n(t) - v\rangle \le \limsup \langle P\widetilde{u}_n(t), \tilde{u}_n(t) - v\rangle$$

for all $v \in V$ and, next, (20) yields

$$\langle P\widetilde{u}(t), \widetilde{u}(t) - v\rangle \le 0 \quad \text{for all} \quad v \in \widetilde{K}. \tag{21}$$

Next, since $K \subset \widetilde{K}$, we use (21) to deduce that

$$\langle P\tilde{u}(t), \tilde{u}(t) - v\rangle \le 0 \quad \text{for all} \quad v \in K. \tag{22}$$

We now combine inequality (22) with assumption $H(K)$(c) to find

$$\langle P\tilde{u}(t), \tilde{u}(t) - v\rangle = 0 \quad \text{for all} \quad v \in K,$$

which by $H(K)$(d) implies $\widetilde{u}(t) \in K$. Moreover, according to (11) and $Pv = 0$ for all $v \in K$, one has

$$\begin{aligned}&\left\langle A\left(t, \widetilde{u}_n(t)\right), \widetilde{u}_n(t) - v\right\rangle \\ &\quad \le -\frac{1}{\rho_n}\langle Pv - P\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + \left\langle \overline{f}(t), \widetilde{u}_n(t) - v\right\rangle \\ &\qquad + j^0\left(w(t), M\widetilde{u}_n(t), M\widetilde{u}_n(t); Mv - M\widetilde{u}_n(t)\right) \\ &\quad \le j^0\left(w(t), M\widetilde{u}_n(t), M\widetilde{u}_n(t); Mv - M\widetilde{u}_n(t)\right) + \left\langle \overline{f}(t), \widetilde{u}_n(t) - v\right\rangle\end{aligned} \tag{23}$$

for all $v \in K$. Taking $v = \tilde{u}(t)$ in (23) and passing to the upper limit as $n \to \infty$, we have

$$\limsup \left\langle A\left(t, \widetilde{u}_n(t)\right), \widetilde{u}_n(t) - \widetilde{u}(t)\right\rangle_{V^*\times V} \le 0.$$

Moreover, the pseudomonotonicity of $A(t,\cdot)$ implies

$$\langle A(t, \tilde{u}(t)), \tilde{u}(t) - v\rangle \le \liminf \left\langle A\left(t, \widetilde{u}_n(t)\right), \widetilde{u}_n(t) - v\right\rangle. \tag{24}$$

Passing to the upper limit as $n \to \infty$ in (23), we obtain

$$\begin{aligned}&\limsup \left\langle A\left(t, \widetilde{u}_n(t)\right), \widetilde{u}_n(t) - v\right\rangle \\ &\quad \le j^0(w(t), M\tilde{u}(t), M\tilde{u}(t); Mv - M\tilde{u}(t)) + \left\langle \overline{f}(t), \tilde{u}(t) - v\right\rangle.\end{aligned} \tag{25}$$

Combining (24) and (25), we get

$$\langle A(t, \tilde{u}(t)), v - \tilde{u}(t)\rangle + j^0(w(t), M\tilde{u}(t), M\tilde{u}(t); Mv - M\tilde{u}(t)) + \left\langle \overline{f}(t), \tilde{u}(t) - v\right\rangle \ge 0.$$

Since (1) has a unique solution, we know that $\tilde{u}(t) = u(t)$ and so $\tilde{u} \in C(I, K)$.

Claim 3. We now prove that $\widetilde{u}_n(t) \to u(t)$ in V for all $t \in I$. Indeed, because $\{\widetilde{u}_n(t)\}$ is bounded in V and for any weakly convergent subsequence of $\{\widetilde{u}_n(t)\}$ converges weakly in V to the same limit $u(t)$, we know that the whole sequence $\{\widetilde{u}_n(t)\}$ converges weakly in V to $u(t)$, for any $t \in I$. On the other hand, using the monotonicity of $A(t,\cdot)$, one has

$$\langle A(t, v), \widetilde{u}_n(t) - v\rangle \le \left\langle A\left(t, \widetilde{u}_n(t)\right), \widetilde{u}_n(t) - v\right\rangle. \tag{26}$$

Similarly to the proof of (24), we have

$$\begin{aligned}&\left\langle A\left(t,\widetilde{u}_n(t)\right),\widetilde{u}_n(t)-v\right\rangle\\&\quad\le j^0\left(w(t),M\widetilde{u}_n(t),M\widetilde{u}_n(t);Mv-M\widetilde{u}_n(t)\right)+\left\langle\overline{f}(t),\widetilde{u}_n(t)-v\right\rangle.\end{aligned}\tag{27}$$

Taking $v=u(t)$ in (26) and (27), and then passing to the limit as $n\to\infty$, one has

$$\lim\left\langle A\left(t,\widetilde{u}_n(t)\right),\widetilde{u}_n(t)-u(t)\right\rangle=0.$$

Using $\widetilde{u}_n(t)\to u(t)$ in V for all $t\in I$, it follows from the strong monotonicity of $A(t,\cdot)$ that

$$\lim m_A\left\|\widetilde{u}_n(t)-u(t)\right\|_V^2\le\lim\left\langle A(t,u(t))-A\left(t,\widetilde{u}_n(t)\right),u(t)-\widetilde{u}_n(t)\right\rangle=0$$

for all $t\in I$. Consequently, we conclude for each $t\in I$, $\widetilde{u}_n(t)\to u(t)$ in V, as $n\to\infty$.

Let $t\in I$ and $n\in\mathbb{N}$. We write (11) with $v=u_n(t)$. Then, we take (1.1)(ii) with $v=\widetilde{u}_n(t)$ and add the resulting inequalities to see that

$$\begin{aligned}&\langle A(t,\tilde{u}_n(t))-A(t,u_n(t)),u_n(t)-\widetilde{u}_n(t)\rangle+\frac{1}{\rho_n}\langle Pu_n(t)-P\widetilde{u}_n(t),\widetilde{u}_n(t)-u_n(t)\rangle\\&\quad+j^0(w(t),M\tilde{u}_n(t),M\tilde{u}_n(t);Mu_n(t)-M\tilde{u}_n(t))\\&\qquad+j^0(w_n(t),M\tilde{u}_n(t),M\tilde{u}_n(t);M\tilde{u}_n(t)-M\tilde{u}_n(t))\ge 0.\end{aligned}$$

Therefore, under hypotheses H(j), H(A), and the monotonicity of the operator P, we get

$$\left\|\tilde{u}_n(t)-u_n(t)\right\|_V\le\frac{\alpha_0}{m_A-\alpha_1}\left\|w_n(t)-w(t)\right\|_V.$$

We now write

$$\left\|u_n(t)-u(t)\right\|_V\le\left\|u_n(t)-\widetilde{u}_n(t)\right\|_V+\left\|\widetilde{u}_n(t)-u(t)\right\|_V,$$

which implies that there exists a constant $D_0>0$ such that

$$\left\|u_n(t)-u(t)\right\|_V\le D_0\left\|w_n(t)-w(t)\right\|_W+\left\|\widetilde{u}_n(t)-u(t)\right\|_V.\tag{28}$$

Subsequently, we consider the previously introduced operator $\Lambda:C(I;W)\to C^1(I;K)$ defined by

$$\Lambda w(t)=T(t)w_0+\int_0^t T(t-s)F(t,(R_0w)(t),u(t),(R_1\zeta)(t))\,ds\quad\text{for}\quad t\in I.$$

Then, by the proof of Theorem 3.3, we know that Λ has a unique fixed point. Under the assumption H(F) we see that

$$\begin{aligned}&\left\|w_n(t)-w(t)\right\|=\left\|\Lambda w_n(t)-\Lambda w(t)\right\|\\&\le M_\lambda\int_0^t\left\|F\left(s,(R_0w_n)(s),u_n(s),(R_1\zeta)(s)\right)ds-F\left(s,(R_0w)(s),u(s),(R_1\zeta)(s)\right)\right\|ds\\&\le M_\lambda L_F\int_0^t(\left\|(R_0w_n)(s)-(R_0w)(s)\right\|+\left\|u_n(s)-u(s)\right\|)\,ds\\&\le\int_0^t M_\lambda L_F\left(r_0T\left\|w_n(s)-w(s)\right\|+\left\|u_n(s)-u(s)\right\|\right)ds.\end{aligned}$$

We now apply Gronwall's inequality to get

$$\begin{aligned}&\left\|w_n(t)-w(t)\right\|\\&\le M_\lambda L_F\int_0^t\left\|u_n(s)-u(s)\right\|ds+M_\lambda^2L_F^2e^{M_\lambda L_FT}\int_0^t\int_0^t\left\|u_n(s)-u(s)\right\|dsdr\\&\le D_1\int_0^t\left\|u_n(s)-u(s)\right\|ds\end{aligned}\tag{29}$$

for all $t\in I$, where $D_1=M_\lambda L_F+TM_\lambda^2L_F^2e^{M_\lambda L_Fr_0T^2}$. We insert (29) into (28) to find

$$\left\|u_n(t)-u(t)\right\|_V\le D_0D_1\int_0^t\left\|u_n(s)-u(s)\right\|ds+\left\|\widetilde{u}_n(t)-u(t)\right\|_V.$$

We use Gronwall's inequality again to derive

$$\left\|u_n(t)-u(t)\right\|_V\le\left\|\widetilde{u}_n(t)-u(t)\right\|_V+D_0D_1\int_0^t\left\|\widetilde{u}_n(s)-u(s)\right\|e^{\int_0^t D_0D_1}ds$$

Since for each $s\in I$, $\widetilde{u}_n(s)\to u(s)$ in V as $n\to\infty$, and $\widetilde{u}_n\in C(I;V)$, one has $u_n(t)\to u(t)$ in V as $n\to\infty$ for each $t\in I$. From (29), we can conclude that $\left(u_n(t),w_n(t)\right)\to(u(t),w(t))$ as $n\to\infty$ for each $t\in I$. Since $u(t)=u_\zeta(t)$ and $w(t)=w_\zeta(t)$, we have

$$\left(u_{n\zeta}(t),w_{n\zeta}(t)\right)\to\left(u_\zeta(t),w_\zeta(t)\right)\quad\text{in}\quad V\times W,\quad\text{as}\quad n\to\infty$$

for each $t \in I$. Let $\phi_\zeta(t) := \phi\left(t, u_\zeta(t), \zeta(t), w_\zeta(t)\right)$. Then, by taking $g = \phi_\zeta$ in Lemma 3.2 and using H(ϕ), we have

$$\begin{aligned}\|\zeta(t) - \zeta_n(t)\|_{Y_1}^2 &\le d_1 \int_0^t \left\|\phi\left(s, u_{n\zeta}(s), \zeta_n(s), w_{n\zeta}(s)\right) - \phi\left(s, u_\zeta(s), \zeta(s), w_\zeta(t)\right)\right\|_{Y_1}^2 ds \\ &\le 3d_1 L_\phi^2 \int_0^t \left\|u_{n\zeta}(s) - u_\zeta(s)\right\|_V^2 + \|\zeta(s) - \zeta_n(s)\|_{Y_1}^2 + \left\|w_{n\zeta}(s) - w_\zeta(s)\right\|_V^2 ds.\end{aligned}$$

Now, Gronwall's inequality yields

$$\begin{aligned}\|\zeta(t) - \zeta_n(t)\|_{Y_1}^2 &\le 3d_1 L_\phi^2 \int_0^t \left\|u_{n\zeta}(s) - u_\zeta(s)\right\|_V^2 + \left\|w_{n\zeta}(s) - w_\zeta(s)\right\|_V^2 ds \\ &\quad + 9d_1^2 L_\phi^4 e^{3d_1 L_\phi^2 T} \int_0^t \int_0^t \left\|u_{n\zeta}(s) - u_\zeta(s)\right\|_V^2 + \left\|w_{n\zeta}(s) - w_\zeta(s)\right\|_V^2 dsdr \\ &\le (3d_1 L_\phi^2 + 9d_1^2 L_\phi^4 T e^{3d_1 L_\phi^2 T}) \int_0^t \left\|u_{n\zeta}(s) - u_\zeta(s)\right\|_V^2 + \left\|w_{n\zeta}(s) - w_\zeta(s)\right\|_V^2 ds.\end{aligned}$$

Since for each $s \in I$, $u_{n\zeta}(s) \to u_\zeta(s)$ in V and $w_{n\zeta}(s) \to w_\zeta(s)$ in W, as $n \to \infty$, and $u_{n\zeta} \in C(I; V)$, $w_{n\zeta} \in C(I; W)$, $u_\zeta \in C(I; K)$, $w_\zeta \in C(I; W)$, one has

$$\zeta_n(t) \to \zeta(t) \ \text{ in } \ Y_1, \ \text{ as } \ n \to \infty$$

for each $t \in I$. The proof of the theorem is complete. □

5. Application to a problem in contact mechanics

In this section, we will illustrate the applicability of the results obtained in Sections 3 and 4 in the study of the quasistatic elastic frictional contact problem with heat equation with memory, and damage. We will show that the weak solution of the mechanical problem leads to a differential variational inequality.

First, we recall some notations needed in this section. Let Ω be a bounded domain in $\mathbb{R}^d$, ($d = 2, 3$) with a Lipschitz continuous boundary $\Gamma := \partial\Omega$ occupied by a deformable body. The points in Ω are denoted by $\boldsymbol{x} = \left(x_1, x_2, \ldots, x_d\right)$. We are interested in the evolution of the body on a finite time interval $I := [0, T]$ with $T > 0$. We denote the space of second order symmetric tensors on $\mathbb{R}^d$ by $\mathbb{S}^d$. Moreover, $\mathbb{R}^d$ and $\mathbb{S}^d$ are equipped with the standard inner products $\boldsymbol{u} \cdot \boldsymbol{v} := u_i v_i$, $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} := \sigma_{ij}\tau_{ij}$ (where the summation convention over repeated indices is used), and the associated norms $\|\boldsymbol{v}\|_{\mathbb{R}^d} = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$, $\|\boldsymbol{\tau}\|_{\mathbb{S}^d} = \sqrt{\boldsymbol{\tau} \cdot \boldsymbol{\tau}}$ for $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^d$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau} \in \mathbb{S}^d$. We use the notation $\boldsymbol{u} = \left(u_i\right)$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$, and $\boldsymbol{\sigma} = \left(\sigma_{ij}\right)$ to denote the displacement vector, the linearized strain tensor and the stress tensor, respectively, where $u_{i,j} := \frac{\partial u_i}{\partial x_j}$, $i, j = 1, 2, \ldots, d$. Here and below, the spatial derivative is defined in the sense of distribution. The normal and tangential components of stress field $\boldsymbol{\sigma}$ and displacement field $\boldsymbol{u}$ on Γ are denoted by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$, $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively, where $\boldsymbol{\nu}$ stands for the normal outward vector which is defined a.e. on Γ. We split the boundary Γ into three disjoint measurable parts Γ_1, Γ_2, and Γ_3 with $|\Gamma_1| > 0$. We also use the notation $Q = \Omega \times I$, $\Sigma = \Gamma \times I$, $\Sigma_i = \Gamma_i \times I$, $i = 1, 2, 3$. The time partial derivative for a function is denoted by a dot. We usually do not explicitly point out the dependence of the functions on $\boldsymbol{x}$. The classical quasistatic frictional contact problem reads as follows.

Problem 5.1. Find a displacement field $\boldsymbol{u} : Q \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : Q \to \mathbb{S}^d$, a damage field $\zeta : Q \to [0, 1]$, and a temperature $w : Q \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathcal{B}(t - s, w(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))\, ds \quad \text{in } Q, \tag{30}$$

$$\dot{\zeta} - \kappa\, \Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \phi(t, w(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta) \quad \text{in } Q, \tag{31}$$

$$\frac{\partial\zeta}{\partial\nu} = 0, \quad \frac{\partial w}{\partial\nu} = 0 \quad \text{on } \Sigma, \tag{32}$$

$$-\operatorname{Div} \boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \quad \text{on } Q, \tag{33}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_1, \tag{34}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Sigma_2, \tag{35}$$

$$u_\nu(t) \le g, \quad \sigma_\nu(t) + \xi_\nu(t) \le 0, \quad (u_\nu(t) - g)(\sigma_\nu(t) + \xi_\nu(t)) = 0 \quad \text{on } \Sigma_3, \tag{36}$$

$$\xi_\nu(t) \in \partial j_\nu(u_\nu(t), u_\nu(t)) \quad \text{on } \Sigma_3, \tag{37}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(u_\nu(t), \boldsymbol{u}_\tau(t)) \quad \text{on } \Sigma_3, \tag{38}$$

$$w_t(t) - \Delta w(t) = e(t, (R_0 w)(t), \boldsymbol{u}(t), (R_1\zeta)(t)) + \int_0^t b(t - s)\Delta w(s)\, ds \quad \text{on } Q, \tag{39}$$

$$w(0) = w_0, \quad \zeta(0) = \zeta_0 \quad \text{on } \Omega. \tag{40}$$

We provide short comments on equations and conditions in Problem 5.1. In the elastic constitutive relation (30), $\mathcal{A}$ denotes the time dependent elasticity operator and $\mathcal{B}$ is the relaxation operator which depends on the damage function and temperature. The inclusion (31) describes the evolution of the damage field ζ of the system, see, for example, [22,41,42]. If $\zeta = 1$, then there is no damage in the material, when $\zeta = 0$ the material is completely damaged, and if $0 < \zeta < 1$ means that the material is partially damaged. Here, ϕ represents the mechanical source of damage which is a general function of the strain, temperature and damage, and κ is a positive constant, the so-called microcrack diffusion coefficient. The term $\partial I_{[0,1]}$ stands for the subdifferential of the indicator function of the interval $[0,1]$ and it guarantees that ζ remains within the interval $[0,1]$. Eqs. (32) represent the boundary condition for the damage function and the no heat flux condition for the temperature, (33) is the equilibrium equation for the stress, where $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$ denotes the divergence operator for tensor valued functions. The boundary conditions (34) and (35) are the displacement and the traction conditions, respectively. The relations (36) and (37) represent the frictional Signorini unilateral contact condition for the normal displacement in which ∂j_ν is the Clarke generalized subgradient of a function j_ν. This condition models the contact with a rigid foundation which is covered by a layer of deformable material of the thickness $g > 0$, see [21, Sections 8.4, 8.5]. Condition (38) is a friction condition modeled by the Clarke subgradient of a nonconvex potential j_τ, see [4,21].

The heat equation with memory (39) represents the law of conservation of energy, where the function e describes the influence on the heat sources of the displacement field and of the history operators R_0 and R_1 of the temperature and the damage. The incorporation of the parabolic Eq. (39) in the model is motivated by the studies of Maxwell, Boltzmann, Volterra, see details and discussions in recent papers [43,44] and references therein. Moreover, the initial temperature and damage w_0 and ζ_0 are specified in (40). For more details on the mathematical modeling of contact problems, we refer to [20,42].

We introduce two evolution triples of spaces $V \subset H \subset V^*$ and $Y \subset Y_1 \subset Y^*$ with continuous and dense embeddings, where

$$V = \left\{\boldsymbol{v} \in H^1\left(\Omega;\mathbb{R}^d\right) \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1\right\}, \quad H = L^2(\Omega;\mathbb{R}^d), \quad Y = H^1(\Omega), \quad Y_1 = L^2(\Omega).$$

We also need the spaces

$$\mathcal{H} = L^2(\Omega;\mathbb{S}^d), \quad \mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \mid \mathrm{Div}\,\boldsymbol{\tau} \in H\}, \quad X = L^2(\Gamma_3;\mathbb{R}^d), \quad W = L^2(\Omega).$$

It is well known that all aforementioned spaces are Hilbert spaces equipped with their standard inner products and norms. Let $c_V > 0$ be an embedding constant such that $\|v\|_H \le c_V \|v\|_V$ for all $v \in V$. We define the following sets

$$K = \{\boldsymbol{v} \in V \mid v_\nu \le g \text{ a.e. on } \Gamma_3\} \text{ and } K_Y = \{\zeta \in Y \mid 0 \le \zeta \le 1 \text{ a.e. in } \Omega\}.$$

The first set is needed to model the Signorini condition (36) and the second one is the set of admissible damage functions. We introduce the linear and closed operator $\widehat{A} : D(\widehat{A}) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\widehat{A}w = \Delta w \quad \text{for all} \quad w \in D(\widehat{A}),$$

with the domain $D(\widehat{A})$ defined by

$$D(\widehat{A}) := \left\{ v \in H^2(\Omega) \mid \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma \right\}.$$

It is well-known that $\widehat{A}$ is the generator of a C_0-semigroup $\{T(t)\}_{t>0}$ of contractions on $W = L^2(\Omega)$, that is, $\sup_{t\in[0,+\infty)} \|T(t)\| \le 1$. Let $\gamma : V \to L^2(\Gamma;\mathbb{R}^d)$ be the trace operator. We define $\boldsymbol{f} : I \to V^*$, $a : Y \times Y \to \mathbb{R}$ and $j : X \times X \to \mathbb{R}$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle = \langle \boldsymbol{f}_2(t), \gamma\boldsymbol{v}\rangle_{L^2(\Gamma_2;\mathbb{R}^d)} + \langle \boldsymbol{f}_0(t), \boldsymbol{v}\rangle_H \quad \text{for} \quad \boldsymbol{v} \in V,\ t \in I,$$
$$a(\zeta,\eta) = \kappa \int_\Omega \nabla\zeta \cdot \nabla\eta\, dx \quad \text{for} \quad \zeta,\eta \in Y,$$

and

$$j(\boldsymbol{z}_1, \boldsymbol{z}_2) = \int_{\Gamma_3} j_\nu(z_{1\nu}, z_{2\nu}) + j_\tau(\boldsymbol{z}_{1\tau}, \boldsymbol{z}_{2\tau})\, d\Gamma \quad \text{for} \quad \boldsymbol{z}_1, \boldsymbol{z}_2 \in X. \tag{41}$$

Note that we suppose here that j is independent of the variable w. Moreover, let R_0 and R_1 be two history-dependent operators that satisfy $H(R)$.

We use the standard procedure, see [4,21,31], to derive the following the variational formulation of Problem 5.1.

Problem 5.2. Find $\boldsymbol{u} : I \to K$, $\zeta : I \to K_Y$ and $w : I \to W$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathcal{B}(t-s, w(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \zeta(s))\, ds \quad \text{in} \quad Q,$$
$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\rangle_{\mathcal{H}} + j^0(\gamma\boldsymbol{u}, \gamma\boldsymbol{u}; \gamma\boldsymbol{v} - \gamma\boldsymbol{u}) \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle \text{ for all } \boldsymbol{v} \in K,$$
$$\langle \dot\zeta(t), \eta - \zeta(t)\rangle_{Y_1} + a(\zeta, \eta - \zeta) \ge \langle \phi(w(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \zeta(t)), \eta - \zeta(t)\rangle_{Y_1} \quad \text{for all} \quad \eta \in K_Y,$$
$$w_t(t) - \Delta w(t) = e(t, (R_0 w)(t), \boldsymbol{u}(t), (R_1\zeta)(t)) + \int_0^t b(t-s)\Delta w(s)\, ds \quad \text{in} \quad Q,$$
$$w(0) = w_0, \quad \zeta(0) = \zeta_0$$

for a.e. $t \in I$.

In order to solve Problem 5.2, we need the following hypotheses.

$H(1)$: The elasticity operator $\mathcal{A}: \Omega \times I \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

$$\begin{cases} \text{(a) } \mathcal{A}(\boldsymbol{x},\cdot,\cdot) \text{ is continuous on } I \times \mathbb{S}^d \text{ for a.e. } \boldsymbol{x} \in \Omega, \\ \text{(b) } \mathcal{A}(\cdot,t,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } (t,\boldsymbol{\varepsilon}) \in I \times \mathbb{S}^d, \\ \text{(c) } \mathcal{A}(\boldsymbol{x},t,\cdot) \text{ is Lipschitz continuous with } L_{\mathcal{A}} > 0 \text{ for all } t \in I, \text{ i.e.,} \\ \qquad \left\| \mathcal{A}\left(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1\right) - \mathcal{A}\left(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2\right) \right\| \le L_{\mathcal{A}} \left\| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \right\| \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{(d) for all } t \in I \text{ and a.e. } \boldsymbol{x} \in \Omega, \mathcal{A}\left(\boldsymbol{x},t,0_{\mathbb{S}^d}\right) = 0_{\mathbb{S}^d}, \\ \text{(e) } \mathcal{A}(\boldsymbol{x},t,\cdot) \text{ is strongly monotone with } m_{\mathcal{A}} > 0 \text{ for all } t \in I, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ i.e.,} \\ \qquad \left(\mathcal{A}\left(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1\right) - \mathcal{A}\left(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2\right)\right) \cdot \left(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\right) \ge m_{\mathcal{A}} \left\| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \right\|^2 \text{ for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d. \end{cases}$$

$H(2)$: The relaxation operator $\mathcal{B}: \Omega \times I \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ is such that

$$\begin{cases} \text{(a) } \mathcal{B}(\cdot,t,w,\boldsymbol{\varepsilon},\zeta) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ t \in I, w, \zeta \in \mathbb{R}, \\ \text{(b) } \mathcal{B}(\boldsymbol{x},\cdot,w,\boldsymbol{\varepsilon},\zeta) \text{ is continuous on } I \text{ for a.e. } \boldsymbol{x} \in \Omega \text{ and all } (w,\boldsymbol{\varepsilon},\zeta) \in \mathbb{R} \times \mathbb{S}^d \times \mathbb{R}, \\ \text{(c) } \mathcal{B}(\boldsymbol{x},t,\cdot,\cdot,\cdot) \text{ is Lipschitz continuous with } L_{\mathcal{B}} > 0, \text{ all } t \in I \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ i.e.,} \\ \qquad \left\| \mathcal{B}\left(\boldsymbol{x},t,w_1,\boldsymbol{\varepsilon}_1,\zeta_1\right) - \mathcal{B}\left(\boldsymbol{x},t,w_2,\boldsymbol{\varepsilon}_2,\zeta_2\right) \right\| \le m_{\mathcal{B}} \left(\left\| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \right\| + \left| w_1 - w_2 \right| + \left| \zeta_1 - \zeta_2 \right| \right) \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ all } w_1, w_2, \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega, \\ \text{(d) there exists } \varrho_{\mathcal{B}} \in L^2(I) \text{ such that } \|\mathcal{B}(\boldsymbol{x},t,w,\boldsymbol{\varepsilon},\zeta)\| \le \varrho_{\mathcal{B}}(t)(|w| + |\zeta| + \|\boldsymbol{\varepsilon}\|) \\ \qquad \text{for all } (t,w,\boldsymbol{\varepsilon},\zeta) \in I \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega; \end{cases}$$

$H(3)$: The damage source function $\phi: \Omega \times I \times \mathbb{R} \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{cases} \text{(a) } \phi(\cdot,t,w,\boldsymbol{\varepsilon},\zeta) \text{ is measurable on } \Omega, \text{ for all } t \in I, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } w, \zeta \in \mathbb{R}, \\ \text{(b) } \phi(\boldsymbol{x},t,\cdot,\cdot,\cdot) \text{ is Lipschitz continuous with } \tilde{L}_{\phi} > 0 \text{ for all } t \in I \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ i.e.,} \\ \qquad \left\| \phi\left(\boldsymbol{x},w_1,\boldsymbol{\varepsilon}_1,\zeta_1\right) - \phi\left(\boldsymbol{x},w_2,\boldsymbol{\varepsilon}_2,\zeta_2\right) \right\| \le \tilde{L}_{\phi} \left(\left\| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \right\| + \left| w_1 - w_2 \right| + \left| \zeta_1 - \zeta_2 \right| \right) \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ all } w_1, w_2, \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) } \phi\left(\cdot,\cdot,0,0_{\mathbb{S}^d},0\right) \in L^2\left(I; L^2(\Omega)\right). \end{cases}$$

$H(4)$: The potential function $j_\nu: \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{cases} \text{(a) } j_\nu(\cdot,r_1,r_2) \text{ is measurable on } \Gamma_3 \text{ for all } r_1, r_2 \in \mathbb{R} \text{ and there exists} \\ \qquad e \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,r,e(\cdot)) \in L^1(\Gamma_3) \text{ for all } r \in \mathbb{R}, \\ \text{(b) } j_\nu(\boldsymbol{x},r,\cdot) \text{ is regular, locally Lipschitz on } \mathbb{R} \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(c) there are constants } \tilde{c}_1, \tilde{c}_2 > 0 \text{ such that for a.e. } \boldsymbol{x} \in \Gamma_3 \\ \qquad \left| \partial j_\nu\left(\boldsymbol{x},r_1,r_2\right) \right| \le \tilde{c}_2\left(1 + \left|r_1\right|\right) + \tilde{c}_1 \left|r_2\right| \text{ for all } r_1, r_2 \in \mathbb{R}, \\ \text{(d) there is a constant } \tilde{a}_1 > 0 \text{ such that} \\ \qquad j_\nu^0\left(\boldsymbol{x},s_1,r_1;r_2 - r_1\right) + j_\nu^0\left(\boldsymbol{x},s_2,r_2;r_1 - r_2\right) \le \tilde{a}_1 |s_1 - s_2| \, |r_1 - r_2| \\ \qquad \text{for all } r_1, r_2, s_1, s_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

$H(5)$: The potential function $j_\tau: \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is such that

$$\begin{cases} \text{(a) } j_\tau(\cdot,r,\boldsymbol{\xi}) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R},\ \boldsymbol{\xi} \in \mathbb{R}^d \text{ and there exists} \\ \qquad \boldsymbol{e} \in L^2(\Gamma_3;\mathbb{R}^d) \text{ such that } j_\tau(\cdot,r,\boldsymbol{e}(\cdot)) \in L^1(\Gamma_3) \text{ for all } r \in \mathbb{R}, \\ \text{(b) } j_\tau(\boldsymbol{x},r,\cdot) \text{ is regular, locally Lipschitz on } \mathbb{R}^d \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \\ \text{(c) there exist constants } \overline{c}_1, \overline{c}_2 > 0 \text{ such that for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \qquad \left| \partial j_\tau(\boldsymbol{x},r,\boldsymbol{\xi}) \right| \le \overline{c}_2(1 + |r|) + \overline{c}_1 \|\boldsymbol{\xi}\| \text{ for all } r \in \mathbb{R},\ \boldsymbol{\xi} \in \mathbb{R}^d, \\ \text{(d) there is a constant } \overline{a}_1 > 0 \text{ such that} \\ \qquad j_\tau^0\left(\boldsymbol{x},s_1,\boldsymbol{\xi}_1;\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\right) + j_\tau^0\left(\boldsymbol{x},s_2,\boldsymbol{\xi}_2;\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\right) \le \overline{a}_1 |s_1 - s_2| \, \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\ \qquad \text{for all } s_1, s_2 \in \mathbb{R},\ \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d. \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \end{cases}$$

$H(6)$: The function $e: Q \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{cases} \text{(a) } e(\cdot,\cdot,s,\boldsymbol{\xi},l) \text{ is continuous on } Q \text{ for all } s, l \in \mathbb{R}, \boldsymbol{\xi} \in \mathbb{R}^d. \\ \text{(b) there exists function } \psi \in L^2(0,T) \text{ such that} \\ \qquad \left| e\left(z,t,s_1,\boldsymbol{\xi}_1,l_1\right) - e\left(z,t,s_2,\boldsymbol{\xi}_2,l_2\right) \right| \le \psi(t)\left(\left|s_1 - s_2\right| + \left\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\right\| + \left|l_1 - l_2\right| \right) \\ \qquad \text{for a.e. } (z,t) \in Q, \text{ all } s_1, s_2, l_1, l_2 \in \mathbb{R}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d. \end{cases}$$

$H(7)$: $\boldsymbol{f}_0 \in C(I;H)$, $\boldsymbol{f}_2 \in C(I;L^2(\Gamma_2;\mathbb{R}^d))$, $b \in W^{1,\infty}(0,T)$, $b \ge 0$, $w_0 \in W$, $\zeta_0 \in K_Y$, $g \ge 0$, and $\kappa > 0$.

Theorem 5.3. *Under the hypotheses $H(1)$-$H(7)$ and $H(R)$, Problem 5.2 is equivalent to Problem 1.1. Moreover, if $m_{\mathcal{A}} > \|\gamma\|^2 \max\left\{\tilde{a}_1 + \overline{a}_1, \overline{c}_1 + \tilde{c}_1\right\}$, then Problem 5.2 has a unique solution $(\zeta, u, w) \in \left(H^1\left(I;Y_1\right) \cap L^2(I;Y)\right) \times C\left(I;K\right) \times C\left(I;W\right)$.*

Proof. For any $t \in I$, we define the operators $A(t,\cdot): V \to V^*$, $B(t,\cdot,\cdot,\cdot): W \times V \times Y \to V^*$, $F(t,\cdot,\cdot,\cdot): W \times V \times Y \to W$, and $C(t): Z \to W$ by

$$\begin{cases} \langle A(t,\boldsymbol{u}),\boldsymbol{v}\rangle = \int_\Omega \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \\ \langle B(t,w,\boldsymbol{u},\zeta),\boldsymbol{v}\rangle = \int_\Omega \mathcal{B}(\boldsymbol{x},t,w,\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \\ F(t,w,\boldsymbol{u},\zeta)(\boldsymbol{x}) = e(\boldsymbol{x},t,w(\boldsymbol{x}),\boldsymbol{u}(\boldsymbol{x}),\zeta(\boldsymbol{x})), \\ (C(t)w)(\boldsymbol{x}) = b(t)\,\Delta w(\boldsymbol{x}) \end{cases}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, $w \in W$, $\zeta \in Y$, a.e. $\boldsymbol{x} \in \Omega$. Let $M: V \to X$ be given by $M\boldsymbol{v} = \gamma\boldsymbol{v}$ for $\boldsymbol{v} \in X$. Under this notation, Problem 5.2 can be reformulated as follows

$$\begin{cases} \dot{w}(t) = \widehat{A}w(t) + F(t,(R_0w)(t),\boldsymbol{u}(t),(R_1\zeta)(t)) + \int_0^t C(t-s)w(s)\,ds, \\ \langle A(t,\boldsymbol{u}(t)) + \int_0^t B(t-s,w(s),\boldsymbol{u}(s),\zeta(s))\,ds, \boldsymbol{v}-\boldsymbol{u}(t)\rangle \\ \qquad\qquad + j^0(Mu(t),M\boldsymbol{u}(t);M\boldsymbol{v}-M\boldsymbol{u}(t)) \geq \langle \boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t)\rangle \text{ for all } \boldsymbol{v}\in K, \\ \langle\dot{\zeta}(t),\eta-\zeta(t)\rangle_{Y_1} + a(\zeta(t),\eta-\zeta(t)) \geq \langle\phi(t,w(t),\boldsymbol{u}(t),\zeta(t)),\eta-\zeta(t)\rangle_{Y_1} \text{ for all } \eta\in K_Y, \\ w(0) = w_0, \quad \zeta(0) = \zeta_0. \end{cases}$$

By hypotheses $H(R)$, the operators R_0 and R_1 are history-dependent operators. Condition (H_0) is a consequence of $H(7)$. Since $g \geq 0$, we assert that $0_V \in K$. To conclude the proof of Theorem 5.3, it remains to verify the conditions $H(A)$, $H(B)$, $H(C)$ $H(j)$, $H(F)$, $H(\phi)$ and $H(a)$. For a fixed $t \in I$, it follows from $H(1)$(a), (b) and (e) and the continuity of Nemytsky operator that $A(t,\cdot)$ is continuous, so it is hemicontinuous on V. The condition $H(1)$(c) shows that

$$\begin{aligned} \langle A(t,\boldsymbol{u}) - A(t,\boldsymbol{v}),\boldsymbol{u}-\boldsymbol{v}\rangle &= \int_\Omega (\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}(\boldsymbol{u})) - \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}(\boldsymbol{v})))\cdot(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\boldsymbol{v}))\,dx \\ &\geq m_{\mathcal{A}}\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2\,dx = m_{\mathcal{A}}\|\boldsymbol{u}-\boldsymbol{v}\|_V^2 \end{aligned}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, $t \in I$ Moreover, the condition $H(1)$(e) implies that $A(t,\cdot)$ is bounded in V and so $A(t,\cdot)$ is a monotone and pseudomonotone operator for all $t \in I$. The condition H(B)(a) comes directly from $H(2)$(b). Moreover, by $H(2)$(c), and Hölder's inequality, we have

$$\begin{aligned} &\left\langle B\left(t,w_1,\boldsymbol{u}_1,\zeta\right) - B\left(t,w_2,\boldsymbol{u}_2,\eta\right),\boldsymbol{v}\right\rangle \\ &\leq \left(\int_\Omega \left\|\mathcal{B}\left(\boldsymbol{x},t,w_1,\boldsymbol{\varepsilon}\left(\boldsymbol{u}_1\right),\zeta\right) - \mathcal{B}\left(\boldsymbol{x},t,w_2,\boldsymbol{\varepsilon}\left(\boldsymbol{u}_2\right),\eta\right)\right\|^2\,dx\right)^{\frac{1}{2}}\|\boldsymbol{v}\|_V \\ &\leq \sqrt{3}\,m_{\mathcal{B}}\left(\int_\Omega \left\|\boldsymbol{\varepsilon}\left(\boldsymbol{u}_1\right)-\boldsymbol{\varepsilon}\left(\boldsymbol{u}_2\right)\right\|^2 + \|\zeta-\eta\|_Y^2 + \|w_1-w_2\|_W^2\,dx\right)^{\frac{1}{2}}\|\boldsymbol{v}\|_V \end{aligned}$$

and

$$\begin{aligned} \langle B(t,w,\boldsymbol{u},\zeta),\boldsymbol{v}\rangle &\leq \left(\int_\Omega \|\mathcal{B}(\boldsymbol{x},t,w,\boldsymbol{\varepsilon}(\boldsymbol{u}),\zeta)\|^2\,dx\right)^{\frac{1}{2}}\|\boldsymbol{v}\|_V \\ &\leq \sqrt{3}\,\varrho_{\mathcal{B}}(t)\left(\int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2 + \|w\|_W^2 + \|\zeta\|_Y^2\,dx\right)^{\frac{1}{2}}\|\boldsymbol{v}\|_V \end{aligned}$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u} \in V$, w_1, $w_2 \in W$, ζ, $\eta \in Y$ with $\varrho_{\mathcal{B}} \in L^2(I)$. This shows that

$$\left\|B\left(t,w_1,\boldsymbol{u}_1,\zeta\right) - B\left(t,w_2,\boldsymbol{u}_2,\eta\right)\right\|_{V^*} \leq \sqrt{3}\,m_{\mathcal{B}}\left(\|w_1-w_2\|_W + \|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V + \|\zeta-\eta\|_{Y_1}\right)$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$, w_1, $w_2 \in$ W and ζ, $\eta \in Y$. Hence $H(B)$(b) holds with $L_B = \sqrt{3}\,m_{\mathcal{B}}$. Moreover, since $b \in W^{1,\infty}(0,T)$, the family of operators $\{C(t)\}_{t\in I}$ satisfies $H(C)$. We consider the function $j: X \times X \longrightarrow \mathbb{R}$ defined by (41). From the regularity hypotheses $H(4)$(b), $H(5)$(b), by [4, Proposition 3.35 and Theorem 3.47], we get

$$j^0(\boldsymbol{z}_1,\boldsymbol{z}_2;\boldsymbol{v}) = \int_{\Gamma_3} j_\nu^0(z_{1\nu},z_{2\nu};v_\nu) + j_\tau^0(\boldsymbol{z}_{1\tau},\boldsymbol{z}_{2\tau};\boldsymbol{v}_\tau)\,d\Gamma \quad \text{for all} \quad \boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{v}\in X.$$

Then, from the proof of [31, Theorem 4.1], we conclude that H(j)(c) holds with $\alpha_0 = 0$ and $\alpha_1 = \left(\tilde{a}_1 + \bar{a}_1\right)\|\gamma\|^2$. The condition $H(F)$(a) follows directly from hypothesis $H(6)$(a). Using hypothesis $H(6)$(b) and the Poincaré inequality, we derive

$$
\begin{aligned}
&\| F\left(t, w_1, \boldsymbol{u}_1, \zeta_1\right) - F\left(t, w_2, \boldsymbol{u}_2, \zeta_2\right) \|_W \\
&= \left(\int_\Omega \left| e\left(z, t, w_1(z), \boldsymbol{u}_1(z), \zeta_1(z)\right) - e\left(z, t, w_2(z), \boldsymbol{u}_2(z), \zeta_2(z)\right) \right|^2 dz \right)^{\frac{1}{2}} \\
&\le \psi(t) \left(\int_\Omega 3\left(\left|w_1(z) - w_2(z)\right|^2 + \left\|\boldsymbol{u}_1(z) - \boldsymbol{u}_2(z)\right\|^2 + \left|\zeta_1(z) - \zeta_2(z)\right|^2 \right) dz \right)^{\frac{1}{2}} \\
&\le \psi(t)\sqrt{3}\max\left\{1, c_V\right\} \left(\|w_1 - w_2\|_W + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\zeta_1 - \zeta_2\|_Y \right)
\end{aligned}
$$

for all w_1, $w_2 \in W$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$, ζ_1, $\zeta_2 \in Y$ and a.e. $t \in I$, which ensures that $H(F)$(b) is valid with $L_F(t) = \psi(t)\sqrt{3}\max\left\{1, c_V\right\}$ for $t \in I$. According to $H(3)$, we conclude that $H(\phi)$ holds with $L_\phi = \sqrt{3\, L_\phi}$. Finally, by the definition

$$
\|\zeta\|_Y^2 = \|\zeta\|_{Y_1}^2 + \int_\Omega \nabla\zeta \cdot \nabla\zeta \, dx \quad \text{for all} \quad \zeta \in Y,
$$

we know that $a(\zeta, \zeta) + \kappa\|\zeta\|_{Y_1}^2 = \kappa\|\zeta\|_Y^2$, and so H(a) holds with $a_1 = a_2 = \kappa$. The smallness condition $m_A > \max\{c_0\|M\|^2_{\mathcal{L}(V;X)}, \alpha_1\}$ of Theorem 3.3 follows from our hypothesis. By the above discussion, we infer that Problem 5.2 is equivalent to Problem 1.1. Therefore, Theorem 5.3 holds as a consequence of Theorem 3.3. □

Let $\rho_n > 0$ for $n \in \mathbb{N}$, and $\rho_n \to 0$ as $n \to \infty$. We consider the following penalized problem associated to Problem 5.1.

Problem 5.4. Find a displacement field $\boldsymbol{u}_n : Q \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}_n : Q \to \mathbb{S}^d$, a damage field $\zeta_n : Q \to [0, 1]$, and a temperature $w_n : Q \to \mathbb{R}$ such that

$$
\begin{aligned}
\boldsymbol{\sigma}_n(t) &= \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t))) + \int_0^t \mathcal{B}(t - s, w_n(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(s)), \zeta_n(s))\, ds && \text{in } Q, \\
\dot{\zeta}_n - \kappa\, \Delta\zeta_n + \partial I_{[0,1]}(\zeta_n) &\ni \phi(t, w_n(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t)), \zeta_n) && \text{in } Q, \\
\frac{\partial\zeta_n}{\partial\nu} = 0, \quad \frac{\partial w_n}{\partial\nu} &= 0 && \text{on } \Sigma, \\
-\operatorname{Div} \boldsymbol{\sigma}_n(t) &= \boldsymbol{f}_0(t) && \text{on } Q, \\
\boldsymbol{u}_n(t) &= \boldsymbol{0} && \text{on } \Sigma_1, \\
\boldsymbol{\sigma}_n(t)\boldsymbol{\nu} &= \boldsymbol{f}_2(t) && \text{on } \Sigma_2, \\
u_{n\nu}(t) \le g,\ \sigma_{n\nu}(t) + \xi_{n\nu}(t) \le 0,\ & \frac{1}{\rho_n}(u_{n\nu}(t) - g)^+ + \sigma_{n\nu}(t) + \xi_{n\nu}(t) = 0 && \text{on } \Sigma_3, \\
\xi_{n\nu}(t) &\in \partial j_{n\nu}\left(u_{n\nu}(t), u_{n\nu}(t)\right) && \text{on } \Sigma_3, \\
-\boldsymbol{\sigma}_{n\tau}(t) &\in \partial j_{n\tau}\left(u_{n\nu}(t), \boldsymbol{u}_{n\tau}(t)\right) && \text{on } \Sigma_3, \\
(w_n)_t(t) - \Delta w_n(t) &= e(t, (R_0 w_n)(t), \boldsymbol{u}_n(t), (R_1\zeta_n)(t)) + \int_0^t b(t - s)\Delta w_n(s)\, ds && \text{on } Q, \\
w_n(0) = w_0, \quad \zeta_n(0) &= \zeta_0 && \text{on } \Omega
\end{aligned}
$$

where r^+ stands for the positive part of $r \in \mathbb{R}$. Compared with Problem 5.1, the contact conditions (36) are replaced by the term with the penalty parameter ρ_n. Now, we define the operator $P : V \to V^*$ and the sequence of sets K_n by

$$
\langle P\boldsymbol{u}, \boldsymbol{v}\rangle = \int_{\Gamma_3} \left(u_\nu - g\right)^+ v_\nu \, d\Gamma \quad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in V, \tag{42}
$$

$$
K_n = \{\boldsymbol{v} \in V \mid v_\nu \le g_n \text{ a.e. on } \Gamma_3\} \quad \text{for all} \quad n \in \mathbb{N},
$$

respectively. Then, the variational formulation of Problem 5.4 can be stated as follows.

Problem 5.5. Find $\boldsymbol{u}_n : I \to V$, $\zeta_n : I \to K_Y$ and $w_n : I \to L^2(\Omega)$ such that

$$
\begin{aligned}
&\boldsymbol{\sigma}_n(t) = \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t))) + \int_0^t \mathcal{B}(t - s, w_n(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_n(s)), \zeta_n(s))\, ds \quad \text{in} \quad Q, \\
&\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n(t))\rangle + j^0(\gamma\boldsymbol{u}_n, \gamma\boldsymbol{u}_n; \gamma\boldsymbol{v} - \gamma\boldsymbol{u}_n) \\
&\qquad + \frac{1}{\rho_n}\langle P\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle_{V^*\times V} \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t)\rangle \quad \text{for all} \quad \boldsymbol{v} \in K_n, \\
&\langle \dot{\zeta}(t), \eta - \zeta(t)\rangle_{Y_1} + a(\zeta_n, \eta - \zeta_n) \ge \langle \phi(w_n(t), \varepsilon(\boldsymbol{u}_n(t)), \zeta_n(t)), \eta - \zeta_n(t)\rangle_{Y_1} \quad \text{for all} \quad \eta \in K_Y, \\
&w_{nt}(t) - \Delta w_n(t) = e(t, (R_0 w_n)(t), \boldsymbol{u}_n(t), (R_1\zeta_n)(t)) + \int_0^t b(t - s)\Delta w_n(s)\, ds \quad \text{on} \quad Q, \\
&w_n(0) = w_0, \quad \zeta_n(0) = \zeta_0
\end{aligned}
$$

for all $t \in I$.

Theorem 5.6. *Under the hypotheses $H(1)$-$H(7)$ and $H(R)$, Problem 5.5 is equivalent to Problem 4.2. Moreover, if $m_{\mathcal{A}} > \|\gamma\|^2 \max\left\{\tilde{a}_1 + \bar{a}_1, \bar{c}_1 + \tilde{c}_1\right\}$, then*

(i) *for any fixed $n > 0$, Problem 5.5 has a unique solution*

$$(\zeta_n, u_n, w_n) \in (H^1\left(I; Y_1\right) \cap L^2(I; Y)) \times C(I; V) \times C(I; W),$$

(ii) $\left(\zeta_n(t), u_n(t), w_n(t)\right) \to (\zeta(t), u(t), w(t))$ *in $Y_1 \times V \times W$ for all $t \in I$, as $n \to \infty$, where (ζ, u, w) is the unique solution of Problem 5.2.*

Proof. It follows from (42), that the penalty operator P is a monotone operator. By the trace theorem, we know that there exists a constant $\kappa_1 > 0$ such that $\|\boldsymbol{v}\|_{L^2(\Gamma,\mathbb{R}^d)} \le \kappa_1 \|\boldsymbol{v}\|_V$ for all $\boldsymbol{v} \in V$. From Hölder's inequality, we have

$$\langle P\boldsymbol{u}_1 - P\boldsymbol{u}_2, \boldsymbol{v}\rangle = \int_{\Gamma_3} \left(\left(u_{1\nu} - g\right)^+ - \left(u_{2\nu} - g\right)^+\right) v_\nu \, d\Gamma$$

$$\le \left(\int_{\Gamma_3} \left(\left(u_{1\nu} - g\right)^+ - \left(u_{2\nu} - g\right)^+\right)^2 d\Gamma\right)^{\frac{1}{2}} \left(\int_{\Gamma_3} v_\nu^2 \, d\Gamma\right)^{\frac{1}{2}} \le \kappa_1^2 \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V \|\boldsymbol{v}\|_V$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in V$, and so $\left\|P\boldsymbol{u}_1 - P\boldsymbol{u}_2\right\|_{V^*} \le \kappa_1^2 \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V$. This shows that P is a bounded and continuous operator. Furthermore, we have

$$K = \left\{\boldsymbol{u} \in V \mid P\boldsymbol{u} = \boldsymbol{0}_{V^*}\right\} = \left\{\boldsymbol{u} \in V \mid u_\nu \le g \text{ a.e. on } \Gamma_3\right\}$$

and so P is a penalty operator of the set K. Thus, we can conclude that Problem 5.5 is equivalent to Problem 4.2. We apply Theorem 4.3 to conclude the proof. □

CRediT authorship contribution statement

Ze Yuan: Conceptualization, Methodology, Writing – original draft, Data curation, Writing – review & editing. **Zijia Peng:** Conceptualization, Methodology, Writing – original draft, Data curation, Writing – review & editing, Supervision. **Zhenhai Liu:** Conceptualization, Methodology, Writing – original draft, Data curation, Writing – review & editing, Supervision. **Stanislaw Migórski:** Conceptualization, Methodology, Writing – original draft, Data curation, Writing – review & editing, Supervision.

Declaration of competing interest

On behalf of all the authors, I want to state that we have no conflict of interest to disclose.

Data availability

No data was used for the research described in the article.

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