Modeling framework for dynamic pricing in macroscopic urban road networks

G.D. Katsifaraki, C. Menelaou, S. Timotheou and C.G. Panayiotou

***Abstract*— We present a conceptual framework for applying pricing in macroscopic traffic networks, focusing on how traffic flows route themselves when path prices are utilized to manage demand at a macroscopic level. We define the minimum disutility mechanism that guides flows in macroscopic traffic networks, and illustrate the guiding process of flows under the effect of pricing, utilizing simple road network structures. The suggested mechanism forms the cornerstone of the conceptual framework for dynamic pricing in a macroscopic setting, as described herein. This framework highlights that, within a priced network, implementing a dynamic pricing policy to adjust path prices and equalize their disutilities can effectively distribute demand across both spatial and temporal dimensions. To ensure an efficient demand distribution, and thereby keep the network at the non-congested regime, an adequate availability of spatiotemporal paths is required however, both with and without perfect routing information.**

I. Introduction

Transportation research has witnessed the revival of the Macroscopic Fundamental Diagram (MFD) initially proposed in [1], [2], through the findings and empirical investigations of Geroliminis and Daganzo [3], [4]. Based on these foundational studies, transportation researchers have increasingly utilized the MFD model to devise congestion mitigation strategies that capitalize on macroscopic traffic dynamics. Interestingly, the work in [4] demonstrated that gridlocks can be effectively mitigated by regulating access through various traffic and demand management techniques, such as metering and pricing. This work paved the way for a series of research endeavours exploring traffic management (e.g., [5]) and demand management methodologies (e.g., [6]) based on a macroscopic traffic dynamics. In this context, significant studies on MFD-based pricing in large-scale networks have emerged, focusing primarily on cordon-based pricing (e.g., [7]) and relevant extensions (e.g., [8]). Dynamic pricing has also been used within MFD traffic networks [9], [10]. However, none of these works thoroughly address or establish the theoretical underpinnings regarding *the mechanism influencing a flow of drivers under the presence of pricing*, which is the focus of this work.

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Previous approaches have predominantly aimed to influence traffic patterns through restrictive measures, often overlooking how such strategies affect the utility (or satisfaction) of users, which is a critical factor driving their route choice decisions [11]. Furthermore, pricing is seen primarily as a means to deter price-sensitive users to traverse through specific regions in order to keep the network in desired traffic conditions, rather than a method to distribute traffic efficiently. On that account, and to the best of our knowledge, existing literature lacks a conceptual foundation to guide the application of congestion pricing within a macroscopic traffic framework to maintain desirable traffic conditions.

To address this adequately, it is critical to analyze and understand the underlying mechanism governing the routing process of users when pricing is utilized macroscopically. In such case, prices can be used in the form of control, providing a number of options to users. It is then up to users to distribute themselves among the provided routes on the basis of these options. In this context, the main contributions of this paper are: (i) defining the disutility mechanism that influences the movement of traffic flows; (ii) presenting a simple, yet fundamental conceptual framework for using pricing at a macroscopic level to efficiently distribute a traffic flow across a road network, maintaining desirable traffic conditions. This framework serves as a foundational basis for exploring more complex pricing strategies aimed at distributing efficiently heterogeneous traffic flows.

II. Foundational macroscopic framework with pricing

On a macroscopic level, we consider an urban area partitioned into R homogeneous regions, denoted by $\mathcal{R} = \{1, \dots, R\}$. The time horizon is divided into discrete time slots $k \in \mathcal{K}$, where $\mathcal{K} = \{1, \dots, K\}$. The key macroscopic parameters for each region $r \in \mathcal{R}$ include the *critical density* ρ_r^C, the *free-flow speed* u_r^f, and the *critical capacity* q_r^C. Each region is associated with the macroscopic variables $u_r(k)$, $q_r(k)$ and $\rho_r(k)$ representing the average speed, outflow, and density for each region $r \in \mathcal{R}$ at each time step $k \in \mathcal{K}$.

At time $t \in \mathcal{K}$, a flow F [1] of n users, who will be referred to as *particles* of the flow hereafter, would like to traverse the network from origin O to destination D. We consider that a flow emulates macroscopically the aggregated behaviour of a group of rational users with similar utility characteristics that would like to traverse from the same origin to the same destination, at the same desired departure time. There is

[1] As we deal with one flow, notation $F_{OD}(t)$ is simplified to F.

further a set $\check{\mathcal{P}} = \{p_1, \dots, p_Y\}$ of spatial regional paths[2] from region O to region D. Also, there exists a set $\mathcal{P}'$ of *spatiotemporal paths*, i.e., paths varying both in space and time. A spatiotemporal path $P' \in \mathcal{P}'$ is defined as the tuple (p', k'), where $p' \in \check{\mathcal{P}}$ and $k' \in \mathcal{K}$. To accommodate the preferences of individuals to depart at times later than the desired t, we define the set $\mathcal{P} = \{P_1, \dots, P_l\} \subseteq \mathcal{P}'$ consisting of spatiotemporal paths of the form $P_i = (p_y, t_d)$, $p_y \in \check{P}, t_d \in \mathcal{K}$, where $t_d = t + d$ represents the actual departure time such that $t_d \geq t$, with $d = 0, 1, \dots, K - t$.

In Figure 1, four spatial regional paths and twelve spatiotemporal paths exist. Note that each coloring denotes a future expansion of the spatial paths.

At the desired departure time t, the particles of the flow are provided with the prices and travel times of the P_i spatiotemporal OD paths. Given this input information, they will follow the spatiotemporal path that satisfies their needs. They will consider the associated costs, including the price and the time cost incurred by selecting the path, and via a particular routing mechanism they will be directed to specific regional paths at specific departure times. Prices thus can be utilized as controls so that particles may *distribute themselves* among desired splitting flows: n_1 particles of the flow will distribute themselves to path P_1, n_2 particles to path P_2, and generally, n_i particles will direct themselves as initial inflow to path P_i, $\forall P_i \in \mathcal{P}$, where $\sum_1^l n_i = n$.

Understanding therefore the routing mechanism of traffic flows through priced paths is essential to effectively apply pricing at a macroscopic level. Typically, utility models are used in microscopic transportation systems to model decision making and choice behavior of users. Utility represents an individual's satisfaction of a phenomenon, and an individual's utility of travel is their evaluation regarding the difficulty in making a trip [12]. Travel utility is therefore considered negative, and is usually referred to as *travel disutility* instead [13]. We lend from the microscopic disutility framework to propose a **macroscopic minimum disutility mechanism** as the routing mechanism governing the movement of traffic flows (Figure 1). This is the mechanism by which traffic flows distribute themselves in an initial inflow allocation when entering a priced road network. Given this initial inflow allocation, different macroscopic traffic models such as the accumulation-based MFD ([3], [4]), the trip-based MFD ([14]), etc. can then be utilized to simulate traffic dynamics on a macroscopic level. This constitutes future work.

III. Defining the Flow Disutility Mechanism

To define the disutility of a flow, let's now assume that paths' prices are such that the flow F distributes itself to l OD spatiotemporal paths. Then, n_i particles of F will follow the P_i path, where $i \in \{1, 2, \dots, l\}$, and $\sum_{i=1}^l n_i = n$.

A simplified version of the microscopic utility model in [15], [16] can be used to represent the disutility of a particle macroscopically, as a linear combination of the particle's

[2]A regional path may not correspond to a single road path composed of consecutive road links, but it rather includes multiple possible road paths.

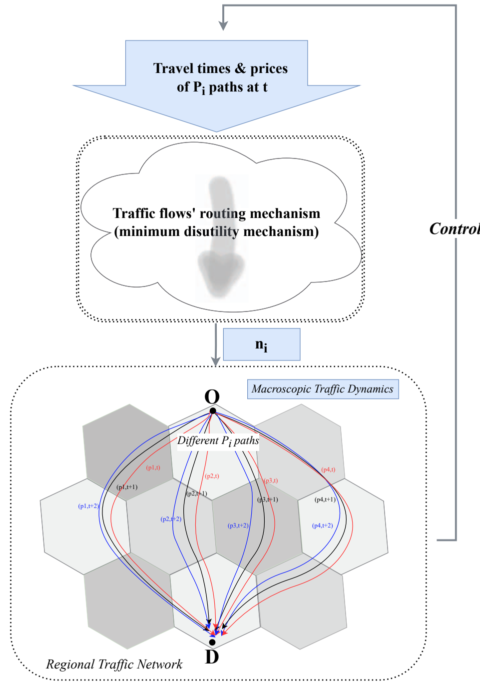

Fig. 1: The foundational macroscopic framework with pricing

disutilities with respect to the trip, the related activity at departure and the related activity at arrival. Then:

*Definition 1: The **incremental disutility** $v_a(P_i)$ of a particle of the F flow for the P_i spatiotemporal path is:*

$$v_a(P_i) = \alpha\tau(P_i) + \beta TT(P_i) + \gamma S^d(P_i) + \delta S^a(P_i), \quad (1)$$

where $\tau(P_i)$, $TT(P_i)$, $S^d(P_i)$, and $S^a(P_i)$ denote the price, travel time and absolute deviations from desired departure and arrival times (i.e., early/ late delays for departure and arrival) of the particle for the path P_i, respectively. Also, α, β, γ, and δ[3] are the disutility parameters of the flow.

Considering that n_i particles will follow the same P_i path, **the cumulative disutility** v_{F_i} of these n_i particles for P_i is:

$$\begin{aligned} v_{F_i}(P_i) &= n_i\Big(v_a(P_i)\Big) \quad (2) \\ &= n_i\Big(\alpha\tau(P_i) + \beta TT(P_i) + \gamma S^d(P_i) + \delta S^a(P_i)\Big). \end{aligned}$$

*Definition 2: The **total disutility of the flow** F can be expressed as follows:*

$$\begin{aligned} V_F &= \sum_{i=1}^{l} v_{F_i}(P_i) \\ &= \alpha\sum_{i=1}^{l} n_i\tau(P_i) + \beta\sum_{i=1}^{l} n_i TT(P_i) + \\ &+ \gamma\sum_{i=1}^{l} n_i S^d(P_i) + \delta\sum_{i=1}^{l} n_i S^a(P_i). \quad (3) \end{aligned}$$

Let further the *maximum incremental disutility* accepted by any particle of the flow in order to stay in the system be

[3]For simplicity, we hereby assume that the weights for early and late delays for departure are the same and equal to γ, whereas the weights for early and late delays for arrival are the same and equal to δ.

v_a^{Max}. After v_a^{Max} the particle exits the system, which means that it will not travel by car and may choose other means of transport, or not travel at all. Then:

Definition 3: The ***max disutility*** V_F^{Max} ***of the flow*** F *is:*

$$V_F^{Max} = nv_a^{Max} \tag{4}$$

In order for the n particles of the flow F to traverse any path P_i, their total flow disutility V_F should not surpass nv_a^{Max}:

$$V_F \leq nv_a^{Max} \tag{5}$$

Note that two different types of disutility are evident: the *perceived disutility* of a trip at departure, and the *experienced disutility* at arrival. These may differ on the basis of actual traffic conditions. As we will later see, having knowledge of the perceived disutility based on imperfect information, or the experienced disutility given perfect information leads the particles of the flow to different self-routing behavior.

A. *The minimum disutility principle*

At the microscopic level, rational individuals opt to travel the route that they perceive to be of the minimum disutility [16]. In a similar manner, we expect that the particles of the flow F will be self-routed towards the spatiotemporal path $P_i^* \in \mathcal{P}$ perceived to minimize their **incremental disutility**. Therefore, the **minimum disutility principle**, denoted as Π_1, states that at the macroscopic level, each particle will self-route through the perceived optimal spatiotemporal path $P_i^* = (p_y^*, t_d^*)$ of minimum disutility, such that:

$$(\Pi_1) \qquad P_i^* = \min_{p_y, t_d} v_a(P_i), \ \forall P_i \in \mathcal{P}, \tag{6}$$

where p_y^* and t_d^* represent the perceived optimum spatial path and departure time, respectively. In this setting, the **axiom of the minimum disutility for a flow** is deduced:

Axiom 1: All particles of a flow will select the same perceived minimum disutility path p_y^* at t_d^*, derived by (6).

B. *The principle of indifference*

At the microscopic level, when some prospects are *evidentially symmetrical*, i.e., evidence no more supports one prospect over the others, the **principle of indifference** posits that individuals become indifferent to these prospects [17], i.e., they will choose among the prospects fairly evenly. Similarly, at the macroscopic level, for spatiotemporal paths of the same perceived disutility, no single path is more suitable for particles than the others. Particles are indifferent to these paths, thus routing themselves evenly among them.

Axiom 2: When the particles of a flow F have the same perceived minimum disutility for the paths P_i and $P_j \in \mathcal{P}$, considering $v_a(P_i) = v_a(P_j)$ $\forall i \neq j$, and $v_a(P_i) \leq v_a^{Max}$, they will distribute themselves evenly among P_i and P_j.

IV. Efficient allocations

For a spatiotemporal path $P_i = (p_y, t_d) \in \mathcal{P}$, each spatial regional path $p_y \in \check{\mathcal{P}}$ is a simple path [18] that passes through a set of regions. Hence, p_y can be defined as a vector of regional path's parts $p_y = (p_{y_O}, \ldots, p_{y_r}, \ldots, p_{y_D})$, where p_{y_r} represents the part of p_y within region $r \in \mathcal{R}$. Let further

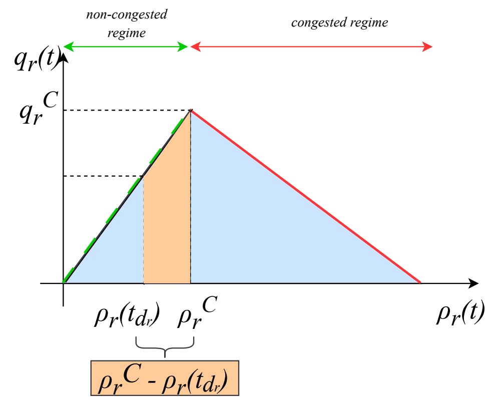

Fig. 2: The remaining density to critical density when particles traverse region r at t_{d_r}. A simple, triangular MFD is used.

t_{d_r} denote the starting time that a particle enters region r when traversing p_y at t_d, with $t_{d_O} = t_d$.

Definition 4: The ***Availability*** A_{P_i} *of a spatiotemporal path $P_i = (p_y, t_d)$ is defined as the number of particles that can traverse p_y departing at t_d without creating congestion.*

Therefore, the availability of a path is determined by the minimum "remaining density to critical density" (as depicted in Figure 2), among all parts of the path within each passing region, at the time the particle is expected to traverse them.

Then, a Path $P_i = (p_y, t_d)$ is **available**, when a vehicle can traverse the p_y path starting at t_d with:

$$A_{P_i} > 0 \tag{7}$$

From the above definitions it is derived that when more particles than A_{P_i} enter p_y at t_d, the path will become congested. We then define an efficient allocation as follows:

Definition 5: Let n particles of a flow F be allocated at t to $\hat{P} = \{P_1, ..., P_l\}$ available spatiotemporal paths with availabilities $A_1, ..., A_l$, respectively, in an $\mathcal{N} = \{n_1, ..., n_l\}$ allocation, where $\sum_{i=1}^{l} n_i = n$. Then, $\mathcal{N}$ is an ***efficient allocation*** *at t, if it preserves the network in a non-congested regime, i.e., if it applies:*

$$n_i \leq A_i, \ \forall i \in 1, ..., l \tag{8}$$

Note that an efficient allocation may exist or not, and may not be unique, as there may exist a number of efficient allocations of the flow of n users to the l available paths.

V. Flow self-distribution models

As discussed, when paths have equal disutilities, particles distribute themselves according to the principle of indifference. Based on the available routing information, different flow self-distribution models apply. We present two such models for imperfect and perfect traffic state information.

i) **When imperfect information is provided, i.e., only the current state of each region is known while the paths' availaibilities are unknown**. This is a realistic scenario where the flow knows only the current traffic state, similar to using a navigation app. Then, all particles will distribute themselves near evenly among the paths of equal disutilities.

ii) **When perfect information is available, i.e., the current and future states of each region are perfectly known, including paths' availabilities**. This is the ideal case when the perceived disutility is similar to the experienced one[4]. Now particles know when not to take a path, to avoid increase travel times due to congestion. The self-distribution process is as follows: particles evenly distribute based on the principle of indifference until the path of lowest availability is filled; then, exclude this path (since its disutility will increase once its availability is surpassed) and distribute to the remaining paths until the next lowest availability path is filled. This process continues until all particles are distributed.

VI. Routing of a Flow in Basic Priced Road Network Structures

We now analyze how a traffic flow distributes to simple road network structures under the effect of pricing. We assume that there is a flow F of n particles with v_a^{Max} that want to traverse from O to D at t through the simple road structures of Figure 3. Imperfect traffic information is provided.

1) Scenario I: We assume there is a single path p, i.e., $P_i = p$, with a fixed price τ, and the path's availability is A. Particles will choose to leave at t rather than delaying departure anticipating better prices, since τ is fixed and there is a disutility penalty due to the delay. Therefore:

- if $v_a(P_i) \leq v_a^{Max}$, all n particles traverse p at t.
  i) If $n \leq A$, the path's availability is not surpassed and particles travel the path at free-flow speed.
  ii) If $n > A$, availability is surpassed, particles' travel time increases and the path gets congested.
- if $v_a(P_i) > v_a^{Max}$, particles abstain from travelling.

2) Scenario II: Now the same path p can be traversed i) either at t with a price of τ_1 or, ii) at $t+1$ with a price of τ_2, assuming same $A > 0$ for the two time-steps [5], hence $P_i = P_1, P_2$ where $P_1 = (p, t)$ and $P_2 = (p, t+1)$. Then, particles assume they will travel in free-flow due to imperfect information. The path's travel time is TT in free-flow and the desired starting time is at t. Then, the perceived disutilities for P_1 and P_2 would be: $v_a(P_1) = \alpha\tau_1 + \beta TT$ (no delays); $v_a(P_2) = \alpha\tau_2 + \beta TT + \gamma + \delta$ (a departure and arrival delay of 1 should be considered at $t+1$).

- If prices $\tau 1, \tau 2$ are such that $v_a(P_1), v_a(P_2) > v_a^{Max}$, all particles will not travel.
- If $\tau_1 < \frac{\alpha\tau_2+\gamma+\delta}{\alpha} \implies v_a(P_1) < v_a(P_2)$ and $v_a(P_2) \leq v_a^{Max}$, all particles choose to travel p at time t. i) If $n \leq A$, particles travel in free-flow. ii) If $n > A$, the path will get congested.
- If $\tau_1 > \frac{\alpha\tau_2+\gamma+\delta}{\alpha} \implies v_a(P_1) > v_a(P_2)$ and $v_a(P_1) \leq v_a^{Max}$, all particles choose to travel p at time $t+1$. Then, i) if $n \leq A$, particles travel in free-flow. ii) If $n > A$, the path gets congested.
- If now $\tau_1 = \frac{\alpha\tau_2+\gamma+\delta}{\alpha} \implies v_a(P_1) = v_a(P_2)$ and $v_a(P_1) \leq v_a^{Max}$, particles become indifferent, and will distribute themselves between the two time-steps.

[4]Although generally not applicable, this can be approximated with technical modifications like a reservations and queuing system.

[5]If availability for p at t is A_1, and for p at $t+1$ is A_2, then this scenario resembles Scenario III.

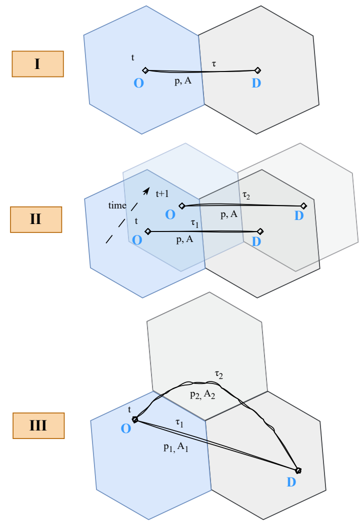

Fig. 3: Three scenarios of a flow traversing simple road structures.

  i) If $n \leq 2A$, particles split near evenly to (p, t) and $(p, t+1)$, travelling at (near) free-flow.
  ii) If $n > 2A$, the path gets congested after availability is surpassed.

3) Scenario III: Particles can now take paths p_1, p_2, where p_1 is the shortest path with TT_1 travel time, and $TT_2 > TT_1$ denotes p_2's travel time, assuming free-flow speed conditions. Hence $P_i = P_1, P_2$ where $P_1 = (p_1, t)$ and $P_2 = (p_2, t)$. The two paths have availabilities $A_1, A_2 > 0$ and they can be traversed with a fixed price of τ_1, τ_2, respectively. Given fixed prices, particles won't choose to depart later than t. The perceived disutilities are: $v_a(P_1) = \alpha\tau_1 + \beta TT_1$ (no delays); $v_a(P_2) = \alpha\tau_2 + \beta TT_2 + \delta(TT_2 - TT_1)$ (no delay at departure, a $TT_2 - TT_1$ delay at arrival from shortest path).

- if prices $\tau 1, \tau 2$ are such that $v_a(P_1), v_a(P_2) > v_a^{Max}$, all particles will abstain from travelling.
- if $\tau_1 < \frac{\alpha\tau_2+(\beta+\delta)(TT_2-TT_1)}{\alpha} \implies v_a(P_1) < v_a(P_2)$ and $v_a(P_2) \leq v_a^{Max}$, all particles choose p_1 at t.
  - if $n \leq A_1$, particles travel in free-flow.
  - if $n > A_1$, the path gets congested.
- if $\tau_1 > \frac{\alpha\tau_2+(\beta+\delta)(TT_2-TT_1)}{\alpha} \implies v_a(P_1) > v_A(P_2)$ and $v_a(P_1) \leq v_a^{Max}$, all particles choose p_2 at t.
  - if $n \leq A_2$, particles travel in free-flow.
  - if $n > A_2$, the path gets congested.
- if $\tau_1 = \frac{\alpha\tau_2+(\beta+\delta)(TT_2-TT_1)}{\alpha} \implies v_a(P_1) = v_a(P_2)$ and $v_a(P_1) \leq v_a^{Max}$, particles become indifferent and distribute among the two paths in fairly even $\frac{n}{2}$ parts:
  i) if $\frac{n}{2} \leq \min(A_1, A_2)$ particles will travel both paths in (near) free-flow.
  ii) if $\frac{n}{2} > \min(A_1, A_2)$ at least a path gets congested.

Similarly, these basic scenarios can be analyzed with perfect information. This is omitted for the sake of brevity. From the above scenarios, we can extract significant insights regarding the behavior of traffic flows when pricing is used, which could be applicable to broader road network configurations. More specifically, the fact that all particles of a flow are prone to follow the same path of minimum disutility at the same time may eventually lead to congestion, especially for large flows at peak hours. Also, when high prices are used on specific paths and associated regions to control the respective heterogeneous inflows, whole flows of price-sensitive drivers may be discouraged from choosing those routes. This is common practice, yet it is criticized as excessively unfair to price-sensitive social groups. Furthermore, the only time a flow distributes itself among paths is when the paths' prices result in the particles perceiving equal disutilities. Importantly, when the size of a flow is greater than paths' availabilities, maintaining the network non-congested may be achieved only in the case of equal disutilities.

VII. Theorems on efficient allocation of traffic flows using pricing

Driven by these scenarios, several questions arise: in order for flows to distribute to more than two paths, do all paths need to be of equal disutility? How many such paths are needed to prevent congestion? How does perfect or imperfect information affect flow distribution? The subsequent theorems elucidate these matters.

Theorem 1: Let a flow F consist of n particles that would like to traverse a priced road network at t (with desired arrival time t^a), given perfect or imperfect traffic information.

i) Then, the flow distributes itself to $\mathcal{P} = \{P_1, ..., P_l\}$ spatiotemporal paths, in an allocation $\mathcal{L} = \{n_1, ..., n_l\}$, where $\sum_{i=1}^{l} n_i = n$, $n_1, ..., n_l > 0$, only in the case where all l paths are priced such that they are of the same disutility.

ii) Let further the *"longest" path* $P_f \in \mathcal{P}$ be the path of the largest incremental disutility in $\mathcal{P}$ with no prices, i.e., $v_a(P_f) \geq v_a(P_i), \tau_f = \tau_i = 0 \; \forall i \in \{1, ..., l\}$.

If P_f is priced with $\tau_f \geq 0$, there is only one price τ_i for all other paths $P_i \in \mathcal{P}, i \neq f$, which does not distort the $\mathcal{L}$ allocation (i.e., it does not lead to an allocation of less or greater than l paths):

$$\tau_i = \tau_f + \frac{(\beta+\delta)(TT_f - TT_i) + (\gamma+\delta)(t_d - t_f)}{\alpha}, \quad \forall i \in \{1, 2, \ldots, l\}, \tag{9}$$

where $\alpha, \beta, \gamma, \delta$ are the flow's disutility parameters, TT_i, TT_f the P_i, P_f paths' travel times, and t_d, t_f the actual P_i, P_f paths' departure times, respectively.

Proof: Let's first assume that the l paths of the $\mathcal{L}$ allocation are priced such that they are perceived as of equal disutility for all particles, thus, $v_a(P_1) = ... = v_a(P_l)$.

Among the different paths of the $\mathcal{L}$ allocation, particles will be willing to pay the minimum price for the "farthest" path $P_f, f \in \{1, \ldots, l\}$, considering $v_a(P_f) \leq v_a^{Max}$. If P_f is priced with $\tau_f \geq 0$, for the paths of equal disutilities it applies:

$$\begin{aligned}
&v_a(P_i) = v_a(P_f), \; \forall i \in \{1, 2, ..., l\} \implies \\
&\alpha\tau_i + \beta TT_i + \gamma(t_d - t) + \delta((t_d + TT_i) - t^a) = \\
&= \alpha\tau_f + \beta TT_f + \gamma(t_f - t) + \delta((t_f + TT_f) - t^a) \implies \\
&\implies [\textit{solving the linear equation system}] \implies \\
&\tau_i = \tau_f + \frac{(\beta+\delta)(TT_f - TT_i) + (\gamma+\delta)(t_d - t_f)}{\alpha}, \\
&\forall i \in \{1, 2, \ldots, l\}
\end{aligned} \tag{10}$$

If all paths are priced as in (10), then, based on the principle of indifference, particles will distribute themselves to the l paths as discussed in Section V.

Let's now assume that paths are priced so that they all are of equal disutility $v_a(P_l)$ (following (10)), except one path P_j, which is priced so that $v_a(P_j) < v_a(P_l)$. Then, all particles of the flow will move towards the P_j of minimum disutility, probably leading to congestion as previously discussed. If on the other hand P_j is priced so that $v_a(P_j) > v_a(P_l)$, no particle will opt for this path, leading to the exclusion of P_j from the $\mathcal{P}$ options provided to particles, and thus distorting the $\mathcal{L}$ allocation. The above apply also when more than one paths are priced so that their disutility differs from the equal disutility paths in $\mathcal{P}$, and obviously when all path prices do not follow (10).

We therefore conclude that the flow distributes itself into an $\mathcal{L}$ allocation of $\mathcal{P} = \{P_1, ..., P_l\}$ spatiotemporal paths only in the case where the l paths are priced such that they are of the same disutility, according to (10). These prices are not unique, since τ_f may vary. However, given τ_f, diverging from these prices results to $\mathcal{L}$ being distorted. ∎

Theorem 2: Let $\mathcal{P} = \{P_1, ..., P_l\}$ spatiotemporal paths of $\hat{A} = \{A_1, ..., A_l\}$ availabilities at t be priced for the n particles of a flow F according to Theorem 1, to result in equal disutilities. Then, the particles distribute themselves to the l spatiotemporal paths in an allocation $\mathcal{L} = \{n_1, ..., n_l\}$, where $\sum_{i=1}^{l} n_i = n$ and $n_1, ..., n_l > 0$. Let further $\grave{P} \in \mathcal{P}$ be the path of the minimum availability $\grave{A}$ among all paths at t, with $\grave{A} > 0$. Then, the $\mathcal{L}$ allocation is *efficient*:

a) with perfect information, if it applies:

$$\sum_{i=1}^{l} A_i \geq n \tag{11}$$

b) with imperfect information, if it applies:

$$l \geq \left\lceil \frac{n}{\grave{A}} \right\rceil \geq \frac{n}{\grave{A}}, \tag{12}$$

$\left\lceil g \right\rceil$ is g rounded-up to the nearest larger integer.

Proof: All l paths are priced such that they are perceived of equal disutility for the n particles, thus $v_a(P_1) = ... = v_a(P_l)$, and particles are indifferent to the paths.

a) In the case perfect information is assumed, as discussed in Section V, particles distribute themselves to the l paths evenly on the basis of the principle of indifference, up to the point the path of the lowest availability $\grave{A}$ is filled; then, particles exclude this path and distribute themselves to the rest, until the path of the next lowest availability is filled, etc. This is continued until all particles are distributed to the l paths. Through this process, it is easily derived that in order for the allocation to stay non-congested and therefore effective, the sum of the l paths' availabilities should be at least equal to the total number of particles, $\sum_{i=1}^{l} A_i \geq n$.

b) In the case imperfect information of only current state travel times is assumed, the flow divides itself roughly equally to the l paths on the basis of the principle of indifference, specifically in parts consisting of x particles, and let's assume in an efficient allocation as follows:

$$l = \left\lceil \frac{n}{x} \right\rceil \geq \frac{n}{x}, \tag{13}$$

If $l < \left\lceil \frac{n}{\grave{A}} \right\rceil$, from equation (13) it applies:

$$\left\lceil \frac{n}{x} \right\rceil < \left\lceil \frac{n}{\grave{A}} \right\rceil \implies \frac{n}{x} < \frac{n}{\grave{A}} \implies x > \grave{A}. \tag{14}$$

In this allocation then, since $x > \grave{A}$, particles are allocated to each path, the path $\grave{P} \in \mathcal{P}$ of the minimum availability $\grave{A}$ will become congested, which means that the allocation is not efficient.

In the case now $l \geq \left\lceil \frac{n}{\grave{A}} \right\rceil$, then $x \leq \grave{A}$. Particles split so that the paths' minimum availability is not surpassed, thus the network stays non-congested and $\mathcal{L}$ is efficient. ■

Theorems 1 and 2 imply that when static pricing is utilized, only a limited number of regional paths to which traffic can distribute itself is available, since all particles will leave at the desired time (otherwise they experience a delay penalty in their disutility). Therefore, in order for the network to stay non-congested especially in peak-hours, the toll should be high enough to control the demand volume, by deterring price-sensitive flows from traversing. When on the other hand dynamic pricing is used, i.e., different prices per regional path per time on the basis of demand, appropriate pricing can ensure that parts of the flow may distribute themselves to many more available spatiotemporal paths, so that demand is spread in time and congestion is avoided.

As a last remark, it is important to note that, following the price strategy in Theorems 1, 2, all identical particles eventually allocate in a path of the same disutility. Therefore, the allocation is considered *fair* for all particles. Thus:

Axiom 3: The allocation accruing from the pricing strategy of Theorems 1 and 2, is a fair and efficient allocation.

VIII. Conclusions

In this work, a novel modeling framework for applying pricing in macroscopic traffic networks is proposed, and the particles' self-routing mechanism is introduced on the basis of the minimum disutility principle. We demonstrate that dynamic pricing can substantially improve the distribution of traffic across spatial and temporal dimensions, and may therefore keep road networks' operation non-congested. Our research fills a significant gap in the current literature by establishing a theoretical foundation regarding the mechanism guiding particles macroscopically under the effect of pricing. At the same time, it paves the way for future investigations into complex pricing strategies aimed at optimizing traffic flow and reducing congestion. Ultimately, the proposed conceptual framework not only clarifies the role of pricing in macroscopic traffic management, but also highlights the critical need for a systematic approach to integrating spatiotemporal path availability with pricing policies, in order to attain both efficient and equitable traffic distribution.

Future research should extend this work to address the management of diverse, heterogeneous flows of different disutility profiles, ODs, and desired departure times. Moreover, the proposed modeling framework is path-based; developing a region-based macroscopic framework for pricing along the same lines is another prominent research strand. Finally, future research may develop different control schemes on the basis of the proposed modeling framework, utilizing a number of different macroscopic traffic models.

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