L96 ANALOGS FOR THIS PROJECT

ANALOG FOR THE REAL WORLD

Lorenz, 1996, two time-scale equations, with accurate time-stepping (RK4 with sufficiently small Δt):

$$\frac{d}{dt}X_k = -X_{k-1}\left(X_{k-2} - X_{k+1}\right) - X_k + F - \left(\frac{hc}{b}\right)\sum_{j=0}^{J-1} Y_{j,k}$$

$$\frac{d}{dt}Y_{j,k} = -cbY_{j+1,k}\left(Y_{j+2,k} - X_{j-1,k}\right) - cY_{j,k} + \frac{hc}{b}X_k$$

We should agree and fix F, J and K for this purpose.

ANALOG FOR GCM

Lorenz, 1996, one time-scale equation, with inaccurate time-stepping (Euler-forward with only-just stable Δt) and an unknown parameterization of "unresolved processes), $P(X_k)$:

$$\frac{d}{dt}X_k = -X_{k-1}\left(X_{k-2} - X_{k+1}\right) - X_k + F - P\left(X_k\right)$$

Wilks, 2005, used $P(X_k) = b_0 + b_1 X_k + b_2 X_k^2 + b_3 X_k^3 + b_4 X_k^4 + e_k$ where e_k is a stochastic component. Arnold et al., 2013, used $P(X_k) = b_0 + b_1 X_k + b_2 X_k^2 + b_3 X_k^3 + e_k$.

THE REAL WORLD

Wilks, 2005 used

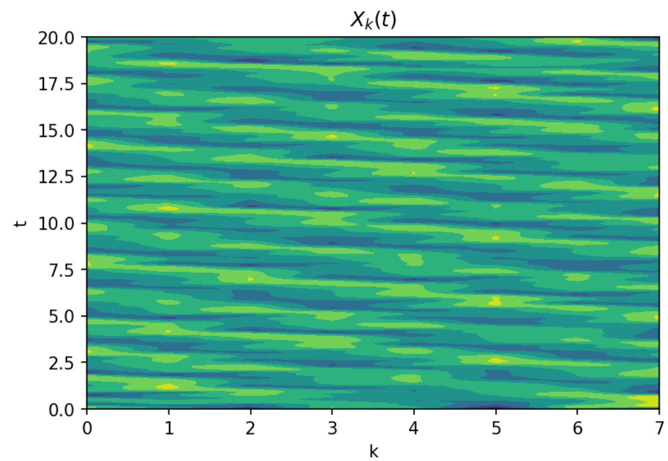
- $F = 18$ or 20
- $K = 8$
- $J = 32$

Traditional to use

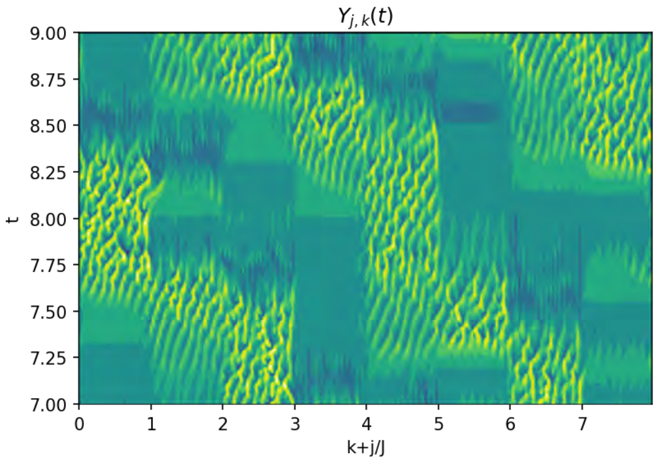
- $h = 1$
- $b = 10$
- $c = 10$

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from L96_model import L96

np.random.seed(23)
W = L96(8, 32)

%time X,Y,t = W.run(0.01, 20.)

plt.figure(dpi=150)
plt.contourf(W.k,t,X);
plt.xlabel('k'); plt.ylabel('t'); plt.title('$X_k(t)$');
```

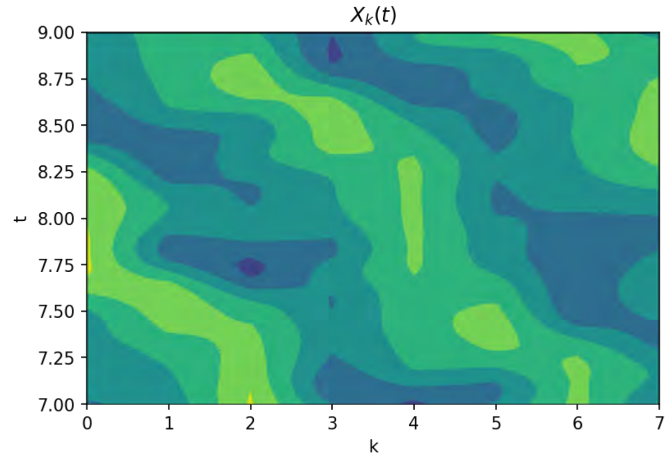
```
Wall time: 2.85 s
```



In [2]:
```
plt.figure(dpi=150)
plt.contourf(W.j/W.J, t, Y, levels=np.linspace(-1,1,10));
plt.xlabel('k+j/J'); plt.ylabel('t'); plt.title('$Y_{j,k}(t)$');
yl=plt.ylim(7,9);
```



In [3]:
```
plt.figure(dpi=150)
plt.contourf(W.k, t, X);
plt.xlabel('k'); plt.ylabel('t'); plt.title('$X_k(t)$');
plt.ylim(yl);
```



THE PARAMETIZATION $P(X_k)$

With the "real world" in hand, we can "observe" the sub-grid forcing on the large scale.

$$\frac{d}{dt}X_k = -X_{k-1}\left(X_{k-2} - X_{k+1}\right) - X_k + F - \underbrace{\left(\frac{hc}{b}\right)\sum_{j=0}^{J-1} Y_{j,k}}_{=U_k}$$

Need to model actual coupling, U_k, with function $P(X_k)$.

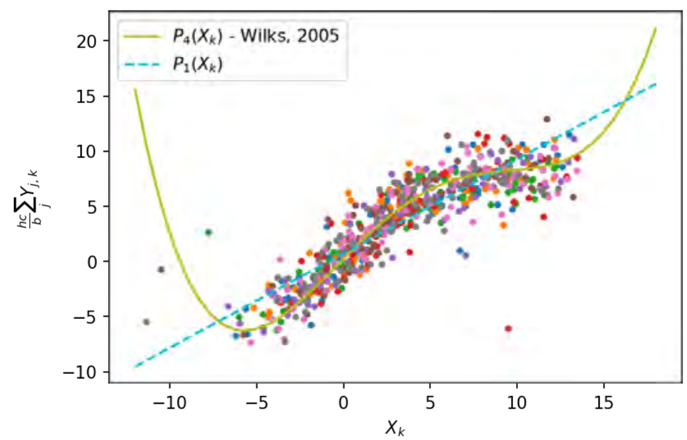
Note the sign of the slope of $P(X_k)$ determines sign of the feedback.

```
In [4]: %time X, Y, t = W.run(0.05, 200.)
```

Wall time: 7.9 s

```
In [5]: Xsamp = X
Usamp = (W.h*W.c/W.b)*Y.reshape((Y.shape[0],W.K,W.J)).sum(axis=-1)
p = np.polyfit(Xsamp.flatten(), Usamp.flatten(), 1)
print('Poly coeffs:',p)
```

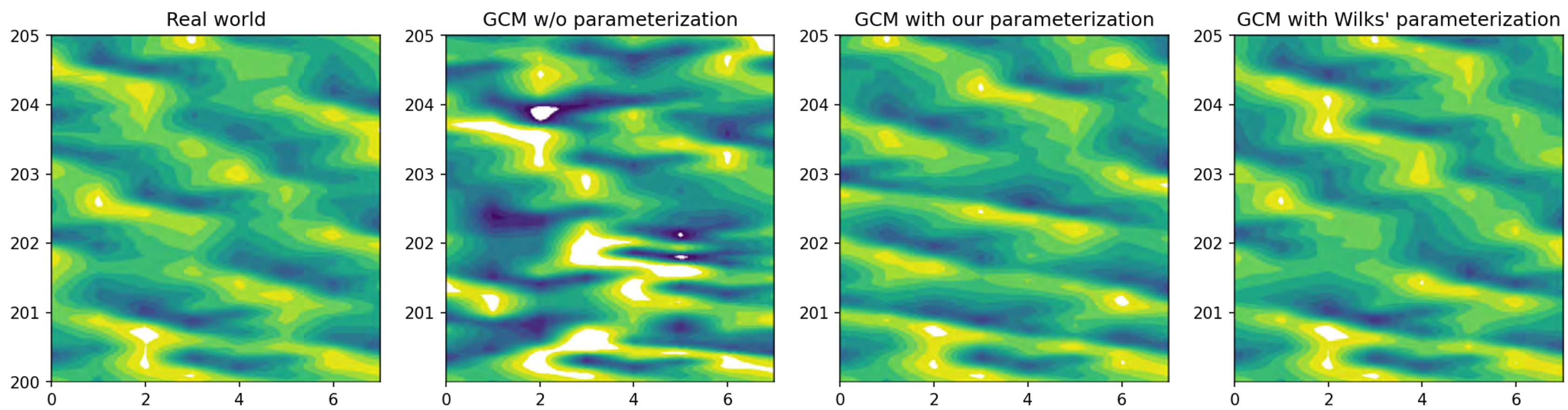
Poly coeffs: [0.85439536 0.75218026]

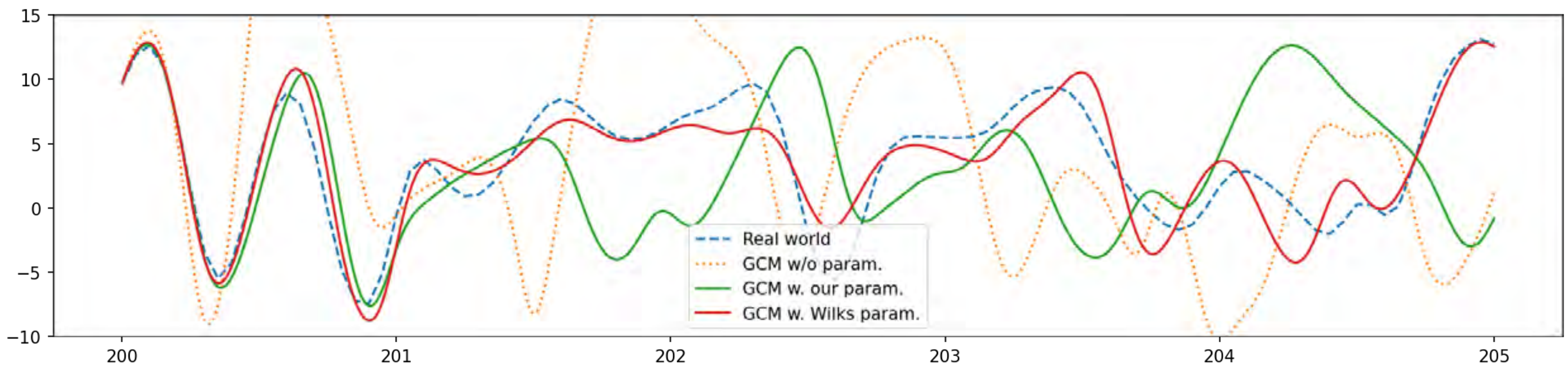


THE MODEL "GCM"

$$\frac{d}{dt}X_k = \underbrace{-X_{k-1}\left(X_{k-2} - X_{k+1}\right) - X_k + F}_{\dot{X}\text{ from eq. (1) of Lorenz '96}} - P(X_k)$$

```
In [8]: from L96_model import L96_eq1_xdot

def GCM(X0, F, dt, nt, param=[0]):
    time, hist, X = dt*np.arange(nt), np.zeros((nt,len(X0)))*np.nan, X0.copy()

    for n in range(nt):
        X = X + dt * ( L96_eq1_xdot(X, F) - np.polyval(param, X) )
        if np.abs(X).max()>1e3:
            break
        hist[n], time[n] = X, dt*(n+1)
    return hist, time

np.random.seed(13); T=5
Xtrue,Ytrue,Ttrue = W.randomize_IC().run(0.05, T)
Xinit, dt, Fmod = Xtrue[0] + 0.0*np.random.randn(W.K), 0.002, W.F+0.0
Xgcm1,Tgcm1 = GCM(Xinit, Fmod, dt, int(T/dt))
Xgcm2,Tgcm2 = GCM(Xinit, Fmod, dt, int(T/dt), param=p)
Xgcm3,Tgcm3 = GCM(Xinit, Fmod, dt, int(T/dt), param=p18)
```





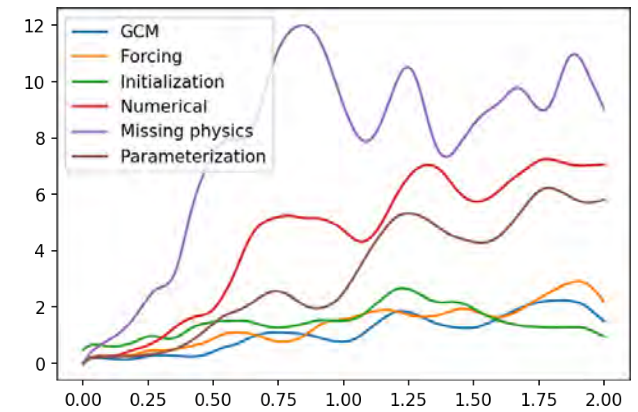
15
10
5
0
−5
−10
200
201
202
203
204
205
Real world
GCM w/o param.
GCM w. our param.
GCM w. Wilks param.

3.2

MODEL ERROR

- Missing physics, or poorly parameterized unresolved physics
  - $P_4 \to P_1$
- Unknown forcing
  - $F \to F + error$
- Numerical errors
  - $\Delta t \to 10\Delta t$
- Initialization error
  - $X(t=0) \to X(t=0) + error$

In [11]:
```python
def err(X, Xtrue):
    return np.sqrt( ((X-Xtrue[1:,:])**2) .mean(axis=1) )

np.random.seed(13); T, dt = 2, 0.001
Xtr,_,_ = W.randomize_IC().set_param(0.0001).run(dt, T)
Xgcm,Tc = GCM(W.X, W.F, dt, int(T/dt), param=p18)
Xfrc,Tc = GCM(W.X, W.F+1.0, dt, int(T/dt), param=p18)
Xic,Tc = GCM(W.X+0.5, W.F, dt, int(T/dt), param=p18)
Xdt,Tdt = GCM(W.X, W.F, 10*dt, int(T/dt/10), param=p18)
Xphys,_ = GCM(W.X, W.F, dt, int(T/dt))
Xprm,_ = GCM(W.X, W.F, dt, int(T/dt), param=p)

plt.figure(dpi=150)
plt.plot(Tc, err(Xgcm,Xtr), label='GCM');
plt.plot(Tc, err(Xfrc,Xtr), label='Forcing');
plt.plot(Tc, err(Xic,Xtr), label='Initialization');
plt.plot(Tdt, err(Xdt,Xtr[::10]), label='Numerical');
plt.plot(Tc, err(Xphys,Xtr), label='Missing physics');
plt.plot(Tc, err(Xprm,Xtr), label='Parameterization');
plt.legend();
```



In [12]:

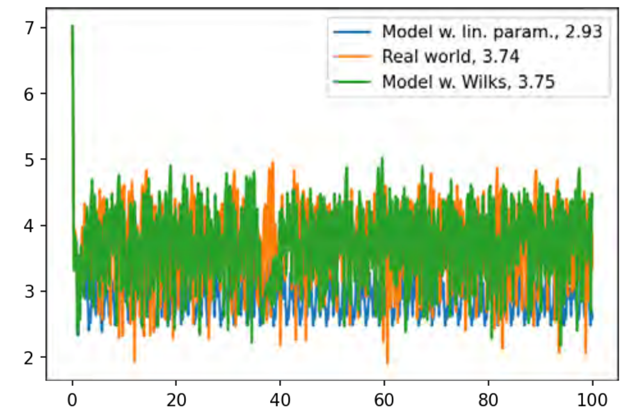
```
# Build a 100-"day" climatology

T,dt = 100., 0.001
%time Xclim,Yclim,Tclim = W.run(0.1, T)
%time X1,t1 = GCM(Xinit, Fmod, dt, int(T/dt), param=p)
%time X2,t2 = GCM(Xinit, Fmod, dt, int(T/dt), param=p18)
```

```
Wall time: 41.9 s
Wall time: 2.37 s
Wall time: 2.92 s
```

In [13]:

```
plt.figure(dpi=150)
plt.plot(t1, X1.mean(axis=1), label='Model w. lin. param., %.2f'%(X1.mean()));
plt.plot(Tclim, Xclim.mean(axis=1), label='Real world, %.2f'%(Xtrue.mean()));
plt.plot(t2, X2.mean(axis=1), label='Model w. Wilks, %.2f'%(X2.mean()));
plt.legend();
print('\n\n       Truth      P1   Wilks')
print('mean:  %.3f   %.3f   %.3f'%(
    Xtrue.mean(), X1.mean(), X2.mean()))
print('std:   %.3f   %.3f   %.3f'%(
    Xtrue.std(), X1.std(), X2.std()))
```

```
       Truth      P1   Wilks
mean:  3.741   2.926   3.750
std:   4.679   4.375   4.524
```



SUMMARY

- Used L96 two time-scale model to generate a real world, or "truth", dataset
- Build a "GCM" with a rudimentary parameterization of coupling to unresolved processes ($\frac{hc}{b} \sum_{j=0}^{J-1} Y_{j,k}$)
    - Deliberately using low-order integration and longer time-step for non-trivial numerical model errors

SOFTWARE QUESTIONS

- `numba` package needed for efficiency but can be temperamental
- Should we make this L96 model package? Would that make it easier/harder to build subsequent exercises?
- We could store data to files to exercise packages such as `xarray` - in practice we will do most training via file...