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Description of world GDP rate changes by using discrete dynamic model

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Abstract: The rate of world GDP is changing periodically. A discrete dynamic model (DDM) describes this process. The model is based on the assumption that the global economy has certain "inertia". This allows us to describe the rate of change of global GDP in the subsequent year as a function of its change in the preceding year. This function can be approximated by using a finite number of terms of its Taylor series. A methodologically more rigorous approach is proposed for approximating the rate of world GDP change on non-overlapping time intervals. Radii of convergence were determined for approximating polynomials for these time ranges. Studies have shown the dependence of the shape of the radius of convergence from the nature of the convergence. DDM has a practical significance because it allows identifying the change in a character of economic dynamics without prior assumptions about the factors driving this trend.

JEL Classifications: C51, C62, E32

Keywords: Economic cycles, attractive fixed points, periodic points, strange attractor, radii of convergence

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1. Introduction

The world economy is experiencing periodic changes in GDP growth rate. The well-known economic cycles are Kitchin (1923) and Juglar (1862) cycles, Kuznets (1930) swings and Kondratieff (1922) waves. It is generally accepted that they have different economic nature (Abramovitz, 1961; Akaev, 2009; Forrester, 1977; Juglar, 1862; Kitchin, 1923; Kondratieff, 1922; Kondratieff, 1984; Kondratieff, Yakovets & Abalkin, 2002; Korotayev & Tsirel, 2010a; Korotayev & Tsirel, 2010b). A model describing all these processes was proposed by Chaldaeva & Kilyachkov (2012). In the proposed model the dependence of the rate of the GDP relative annual change in ($n+1$) period of time (X_{n+1}) from its value during the previous period (X_n), had the form $X_{n+1}=\lambda X_n(1-X_n)$. However, the proposed model described all known cycles uniformly as the result of doubling the period of the basic cycle ($T=3$ years) with increase the value of λ. In addition, the proposed model allowed explaining the existence of a cycle with a period of $T=35\pm5$ years. The presence of this cycle was shown by Korotayev & Tsirel (2010a, 2010b) by a spectral analysis of the world annual GDP changes rates from 1871 to 2007. The proposed model was called the bifurcation model.

However, in this model a sustainable development is possible only for positive growth rates of GDP. To describe the oscillation between negative and positive economic growth, it is necessary to use a slightly more complex model having the following form $X_{n+1} = \lambda X_n(1 - X_n^2)$ (Kilyachkov & Chaldaeva, 2013). A little more complicated model was proposed by Chaldaeva & Kilyachkov (2014) to describe the change in the world GDP on long time intervals. The dependence of X_{n+1} on X_n in this model contains all the components of approximating polynomial of degree 3 that has the form: $X_{n+1} = a_0 + a_1 X_n + a_2 X_n^2 + a_3 X_n^3$. Along with this, in this model it is possible the appearance of cycles with other periods, which allows to explain the complicated spectrum of the rates of world GDP changes (Korotayev & Tsirel, 2010a; Korotayev & Tsirel, 2010b).

Approximation of the function X_{n+1} by polynomial of degree 3 allows for the description of the rate of world GDP change at short time intervals and determine the qualitative characteristics of such set of polynomials. This work was done by Kilyachkov, Chaldaeva & Kilyachkov (2015) by using a "moving" approximation of the rates of global GDP change by polynomials. These characteristics include the basins of attraction, attractive fixed points, periodic points and local stable space.

Initially, the proposed model was called the bifurcation model. However, it was shown that the model allows us to describe other effects. Therefore, it is more appropriately called the discrete dynamic model (DDM).

The proposed model has a disadvantage as using a "moving" approximation leads to the mutual influence of crisis and prosperous years on the value of coefficients of approximating polynomials. In addition, there is no satisfactory methodological justification for the transition from a polynomial of the second degree to a polynomial of the degree 3.

In the article the results obtained in studies, mentioned above, will be presented using a common methodological approach and complemented by results of recent research.

2. Common methodological approach

The basic assumption of the model is that the economy has some "inertia". In this case, the factors that determine the value of the world GDP change over time quite smoothly. Of course, this assumption is true in the absence of crises. This basic assumption allows us to describe the rate of change of global GDP in the subsequent year as some function of this rate in the previous year. In the general case, this dependence has the form:

$$X_{n+1} = F(t_{n+1}, X_n), \tag{1}$$

where, X_{n+1} - GDP growth rate in the $(n+1)^{-th}$ year; X_n - GDP growth rate in the previous, n^{-th} year; t_{n+1} - the time value of the $(n+1)$ year.

Suppose that the function $X_{n+1} = F(t_{n+1}, X_n)$ can be represented as a Taylor series (e.g., Korn & Korn, 1968) as:

$$X_{n+1} = F(t_{n+1}, X_n) = F(t_{n+1}, 0) + \frac{F'(t_{n+1}, 0)}{1!} X_n + \frac{F''(t_{n+1}, 0)}{2!} X_n^2 + \dots$$
$$\dots + \frac{F^{(m)}(t_{n+1}, 0)}{\text{m!}} X_n^m + P_m(t_{n+1}, X_n), \qquad (2)$$

or

$$X_{n+1} = F(t_{n+1}, X_n) = a_0(t_{n+1}) + a_1(t_{n+1}) X_n + a_2(t_{n+1}) X_n^2 + \dots$$
$$\dots + a_m(t_{n+1}) X_n^m + P_m(t_{n+1}, X_n) \qquad (3)$$

where, $m!$ - the factorial of m; $F(t_{n+1}, 0)$ - the m-th derivative of $F(t_{n+1}, X_n)$ evaluated at the point $X_n = 0$; $a_m(t_{n+1}) = \frac{F^{(m)}(t_{n+1}, 0)}{\text{m!}}$; function $P_m(t_{n+1}, X_n)$ describes the influence of factors leading to non-polynomial dependence.

The coefficients $\{a_m(t_{n+1})\}$ from formula (3) reflect the influence of all factors that determine the rate of world GDP change. These factors are of a different nature (resource, technological, financial, etc.) and in the general case are functions of time. However, for small time intervals, not accompanied by economic shocks, these coefficients can be considered as constant $\{a_m(t_{n+1}) = a_m\}$. Next, we will consider that the function $P_m(t_{n+1}, X_n)$ is insignificant, and the use of the model showed that this assumption is justified.

We used the approximation of function $F(t_{n+1}, X_n)$ by polynomials of various degrees to describe the observed economic processes. Note that we did not consider the cases of polynomials of degree zero and one, because they are trivial. Approximation of the function $F(t_{n+1}, X_n)$ by a polynomial of degree two was used in (Chaldaeva & Kilyachkov, 2012; Kilyachkov & Chaldaeva, 2013) for the description of economic cycles.

Approximation of the function $F(t_{n+1}, X_n)$ by polynomial of degree 3

$$X_{n+1} = F(t_{n+1}, X_n) = a_0(t_{n+1}) + a_1(t_{n+1}) X_n + a_2(t_{n+1}) X_n^2 + a_3(t_{n+1}) X_n^3 \qquad (4)$$

allows for the description of the rate of world GDP change at short time intervals, (Kilyachkov, Chaldaeva & Kilyachkov, 2015) determine the values of coefficients of approximating polynomials and their qualitative characteristics (basins of attraction, attractive fixed points, periodic points and local stable space). These mathematical concepts have been studied in various papers (Arnold, Afraymovich, Ilyashenko & Shilnikov, 1985; Arnold, Vasilyev, Goryunov & Lyashko, 1988; Arnold, Vasilyev, Goryunov, & Lyashko, 1989; Arnold, 2009; Arnold, 2012; Arrowsmith & Place, 1982;

Bezruchko, Koronovskiy, Trubetskov & Khramov, 2010; Danilov, 2010; Haken, 1978; Haken, 2004; Hassard, Kazarinoff, & Wan, 1981; Magnitsky, & Sidorov, 2016; Malinetsky, 2009; Nemytsky, & Stepanov, 2004; Sekovanov, 2013; Trubetskov, 2010). Using current terminology the attractive fixed points and periodic points will be called as attractors, and local complicated sets - as the strange attractors.

3. Qualitative characteristics of approximating polynomials at non-overlapping time ranges

In works (Kilyachkov, Chaldaeva & Kilyachkov, 2015) were used a "moving" approximation of the rate of global GDP change by polynomials. However, methodologically it is more correct to identify non-overlapping time intervals, in which the accuracy of approximation by polynomials of degree 3 would give the best result. This approach allows the elimination of the mutual influence of crisis and prosperous years to each other, which is inevitable when calculating the moving approximation, and therefore gives a more reliable result. In the time interval from 1962 to 2015 we determined nine non-overlapping time ranges.

The important characteristic of a DDM is the radius of convergence of the approximating polynomials. The radius of convergence refers to the border region of values of the argument X_n, the membership of which ensures the convergence of the model towards the attractor. In other words, if X_n belongs to a region bounded by a radius of convergence, then at constant terms of economic development it will not leave this area. For the one-dimensional case when X_n takes real values, the radius of convergence consists of two numbers, the minimum and maximum value between which X_n exists. However, more informative is the two-dimensional radius of convergence, which is defined for complex values of X_n. In a theory of fractal sets it is called a Julia set (e.g. Sekovanov, 2013). The conducted research allowed us to obtain two-dimensional radii of convergence for values of the rate of change of world GDP in various time ranges. To build these radii of convergence was used a standard method. According to this method, some initial value for X_{n+1} was required. This value was substituted into equation (4), which was decided relative to X_n, with the resulting solution used as the initial value. The new initial value was then substituted into equation (4) which was solved relative to X_{n+1}. The obtained solution was used as the initial value for the next step, etc. As studies have shown, clear Julia sets were formed after 8 - 9 iterations.

4. Results and discussions

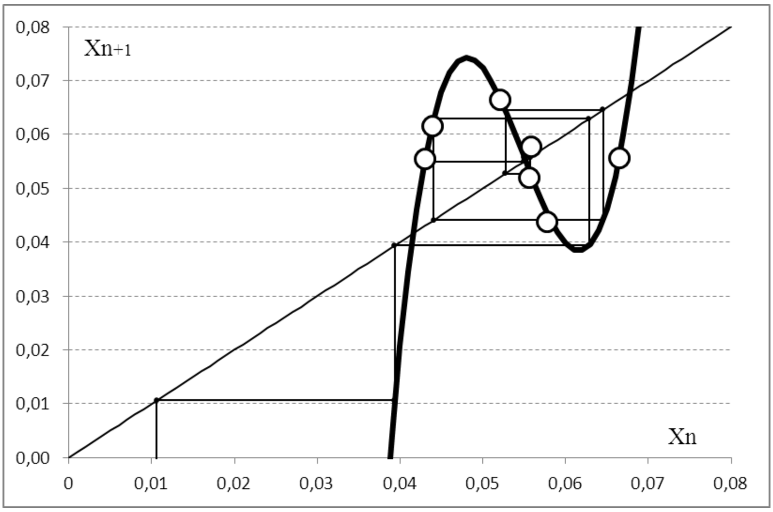
The main characteristics of non-overlapping time intervals, in which the accuracy of approximation by polynomials of degree 3 give the best result and approximating polynomials of these ranges are shown in Table 1. The Figures 1 - 9 show Verhulst diagrams and convergence areas (Julia sets) corresponding to polynomials that approximate the rate of change of world GDP at these time intervals.

TABLE 1. THE MAIN CHARACTERISTICS OF THE TIME RANGES AND THE POLYNOMIALS OF DEGREE 3 (a_3, a_2, a_1, a_0, R - SQUARED VALUE), APPROXIMATING THE BEHAVIOR OF THE RATE OF WORLD GDP CHANGE

Time range	a_3	a_2	a_1	a_0	R	Type of convergence	Key events
1961-1967	28 890	-4 753.7	256.76	-4.4927	0.9019	The lack of convergence	The Cuban missile crisis (1962)
1968-1974	-6 626.6	780.31	-26.793	0.2855	0.8385	The lack of convergence	The crisis of the Bretton woods monetary system, the transition to the Jamaican system
1975-1981	-3 886	422.2	-13.41	0.1423	0.7372	Strange attractor	Reaganomics, the transition from Keynesianism to neo-conservatism
1982-1986	3 876	-326.9	7.86	-0.0068	0.9996	Attractive fixed points, transition to periodic points	No significant key economic events
1987-1991	-9 690	910.2	-25.295	0.2183	0.7129	Attractive fixed point	No significant key economic events
1992-1997	-39 906	3 300.2	-86.799	0.7427	0.9194	The lack of convergence	The failure of the planned economy in the USSR and the countries of Eastern Europe (1989-1991)
1998-2003	-2 046	24.4	4.786	-0.0666	0.986	The lack of convergence	The Asian financial crisis (1997)
2004-2009	-3 770.4	232.03	-0.8456	-0.0564	0.854	Attractive fixed points	The bankruptcy of dot.com companies (2004), the mortgage crisis (2008)
2010-2015	10 291	-984.06	30.539	-0.2832	0.934	Attractive fixed points	The global economic crisis (from 2010)

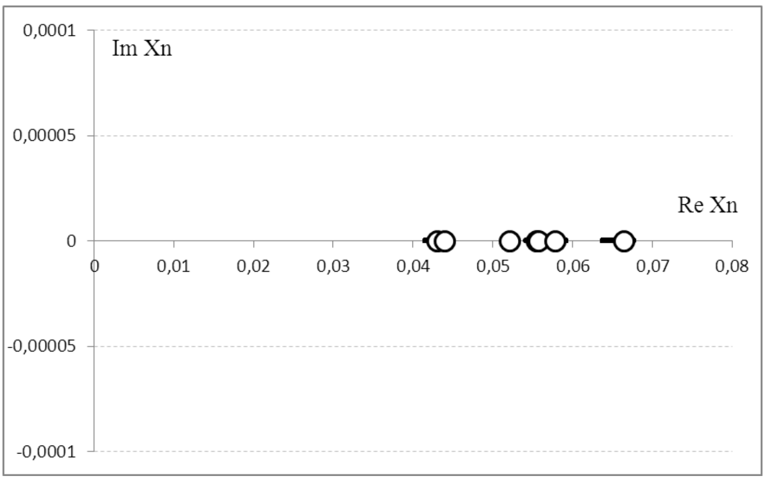
Source: Own calculations based on data from World Bank Open Data. GDP growth (annual %).

FIGURE 1 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1961 - 1967



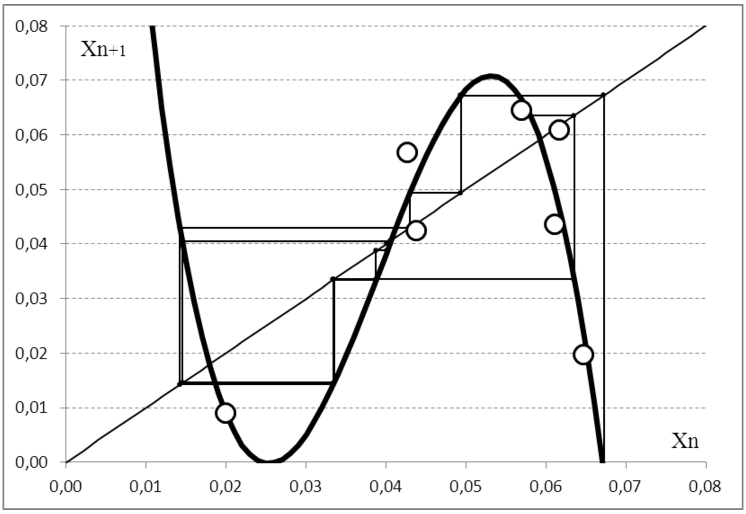
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.042957.

FIGURE 1 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1961 - 1967



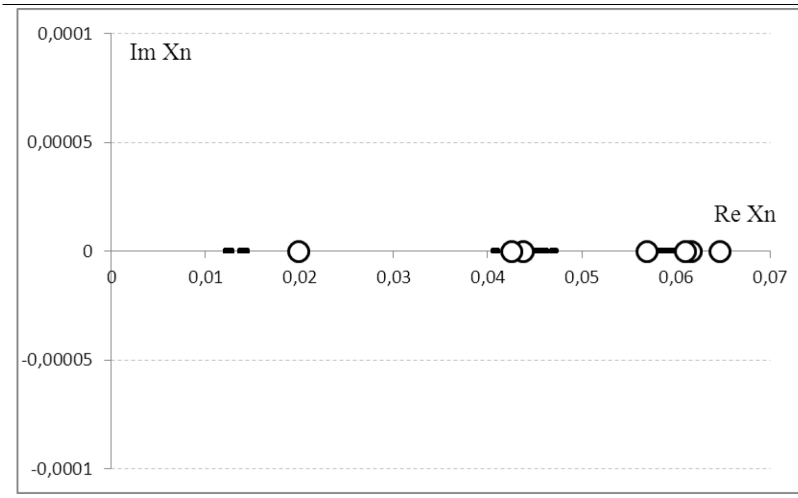
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 2 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1968 - 1974



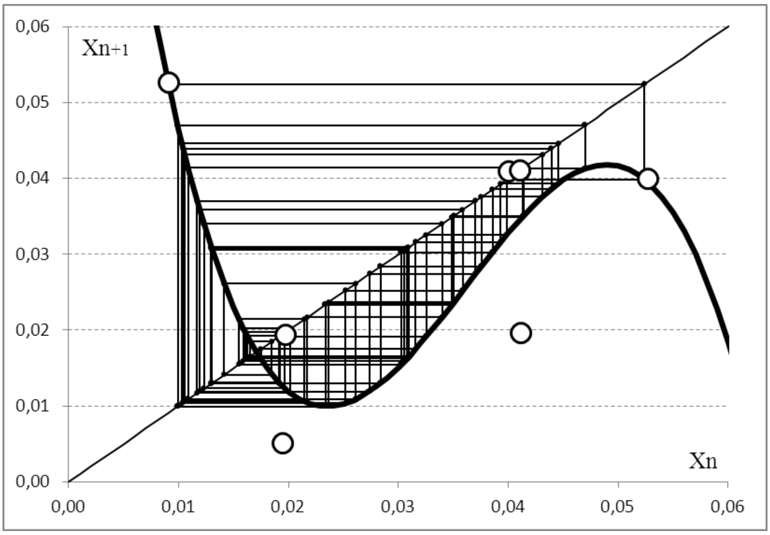
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value $X_n = 0.058$.

FIGURE 2 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1968 - 1974



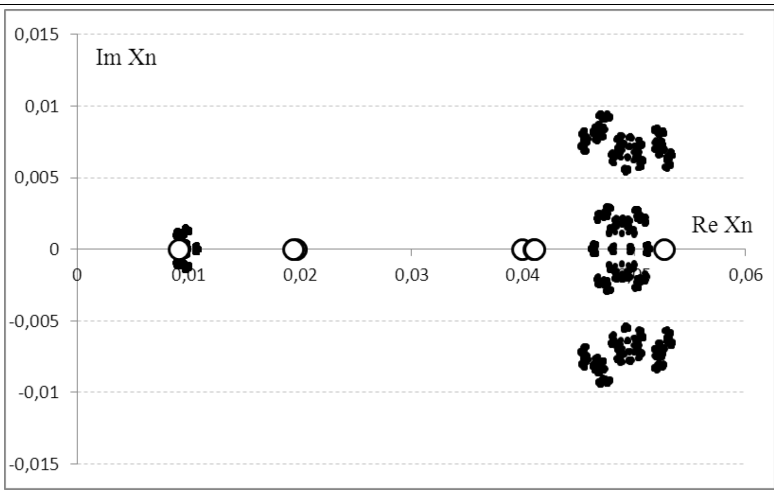
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 3 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1975 - 1981



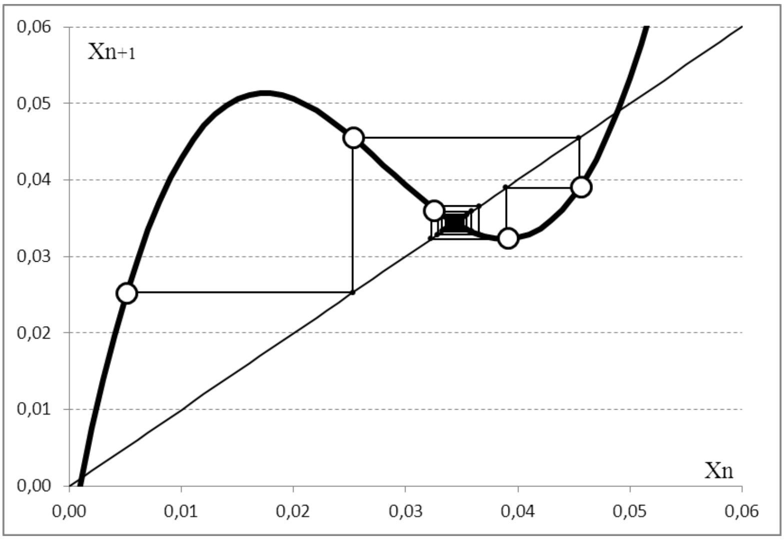
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.00909.

FIGURE 3 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1975 - 1981



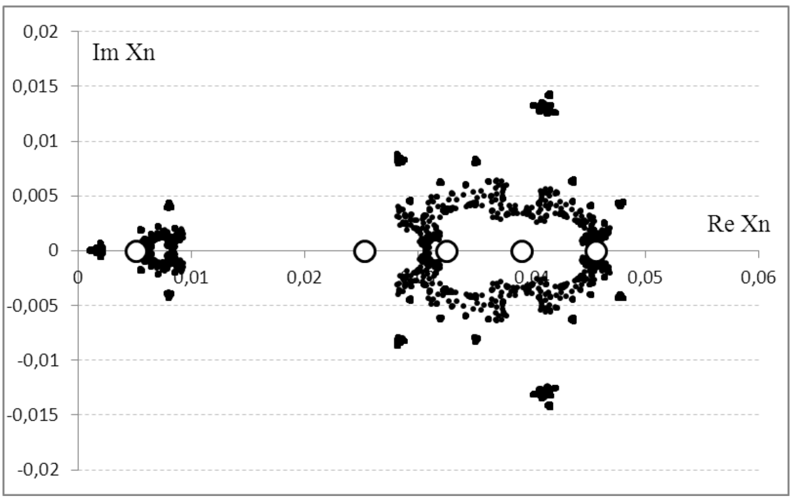
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 4 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1982 - 1986



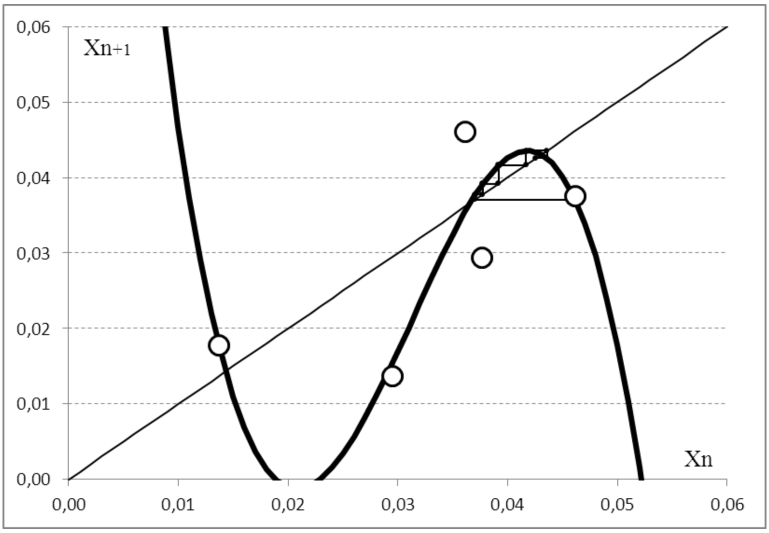
Source: Own elaboration.
Note: ○ -the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.0051.

FIGURE 4 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1982 - 1986



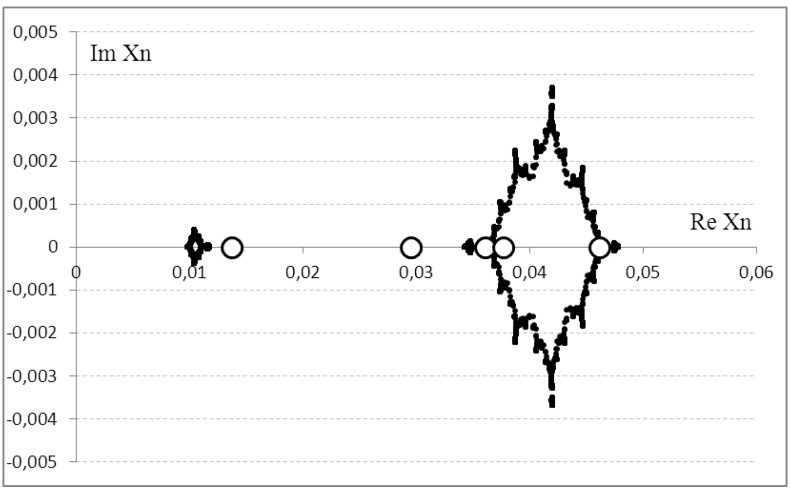
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 5 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1987 - 1991



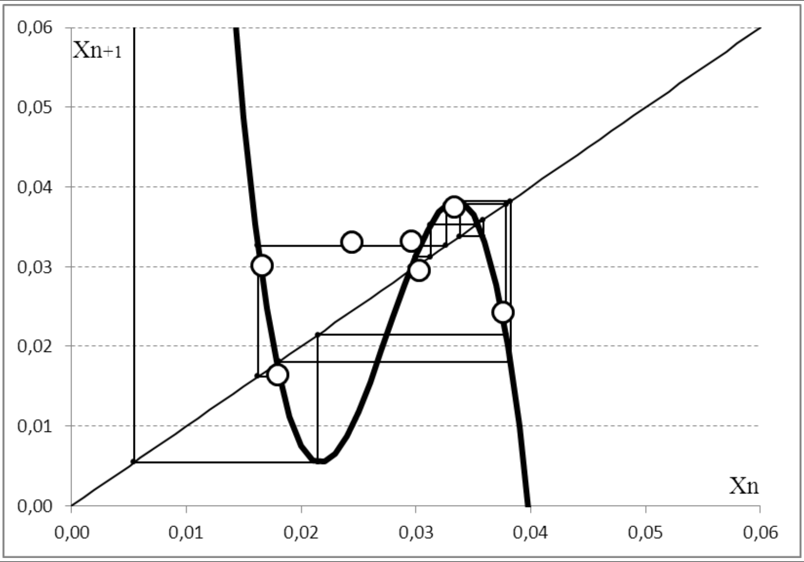
Source: Own elaboration.
Note: ○ -the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.04614.

FIGURE 5 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1987 - 1991



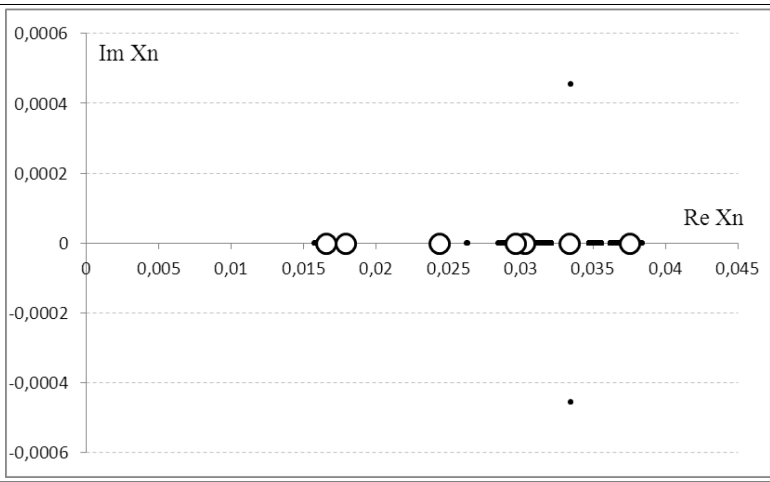
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 6 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1992 - 1997



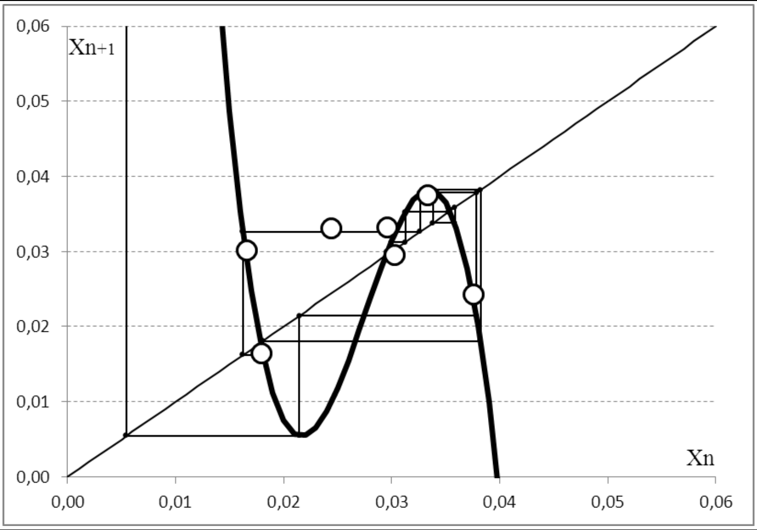
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.02958.

FIGURE 6 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1992 - 1997



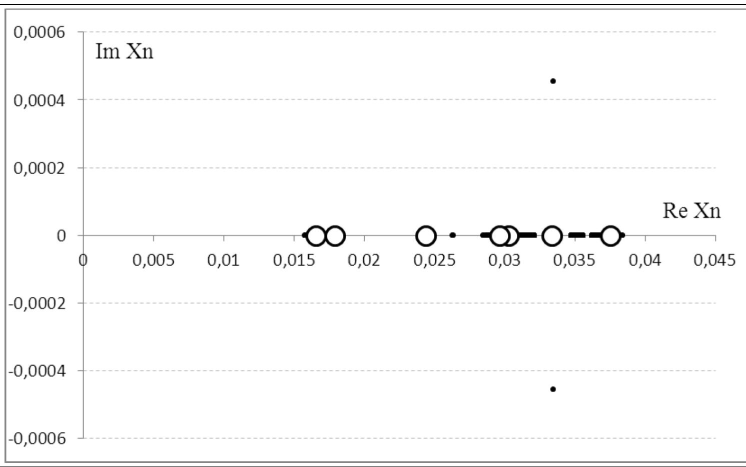
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 7 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1998 - 2003



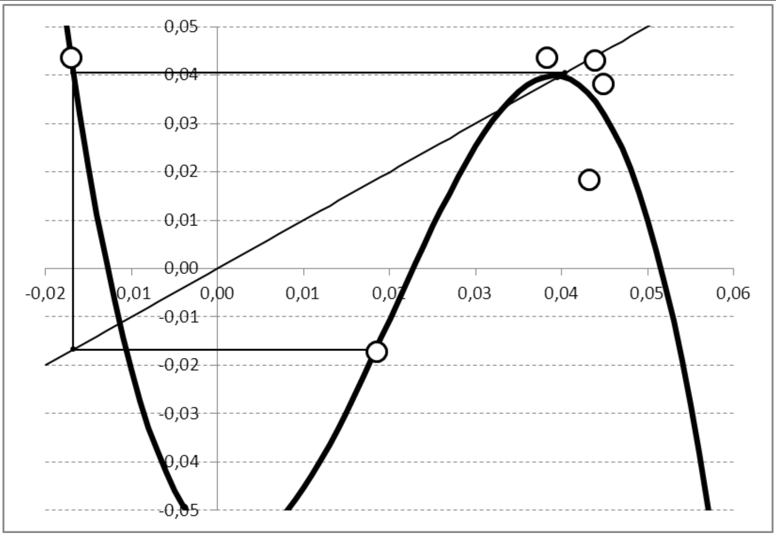
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value $X_n =$ 0.024991.

FIGURE 7 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 1998 - 2003



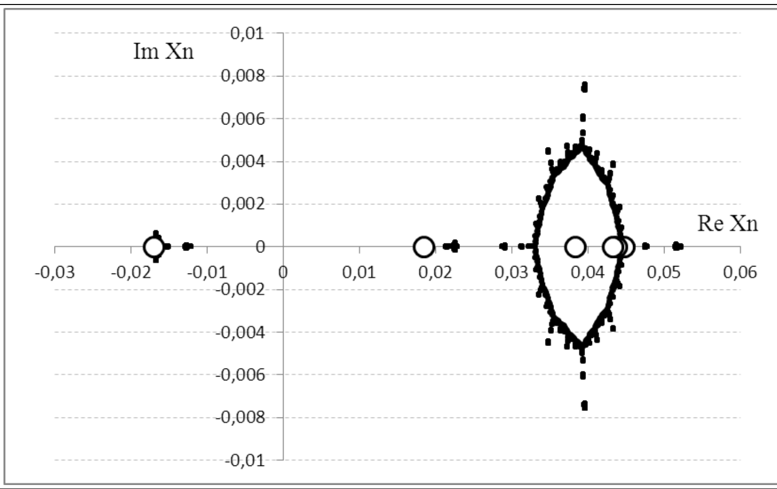
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 8 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 2004 - 2009



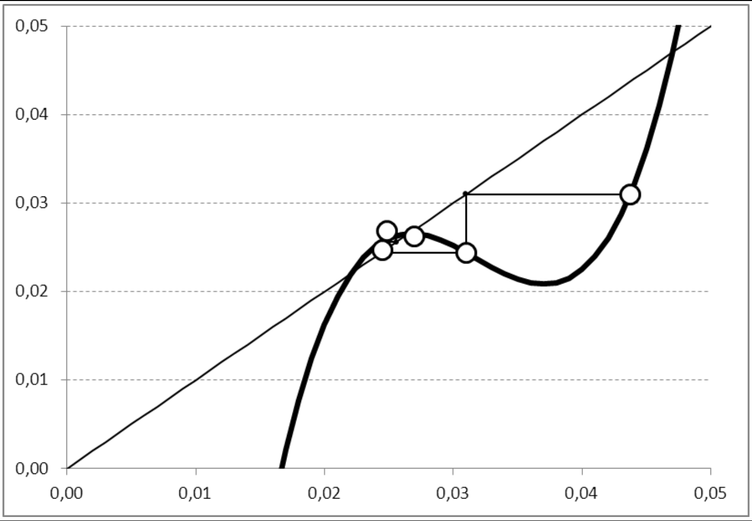
Source: Own elaboration.
Note: ○ -the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.01844.

FIGURE 8 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 2004 - 2009



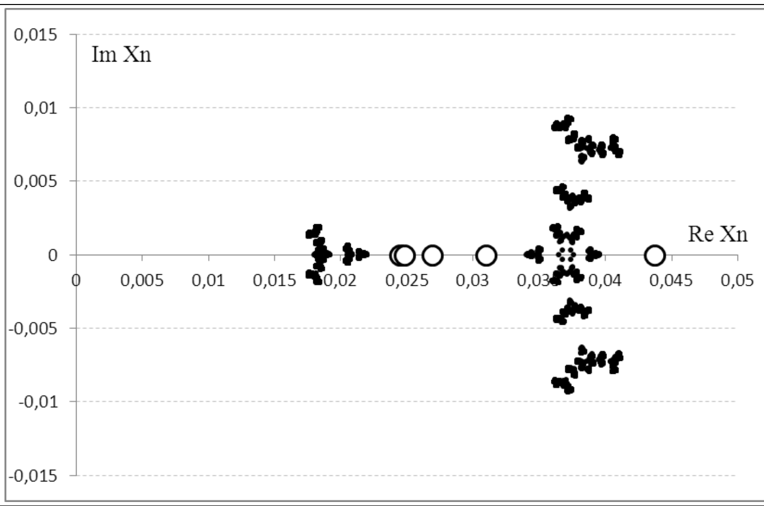
Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

FIGURE 9 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 2010 - 2015



Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.04372.

FIGURE 9 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 2010 - 2015

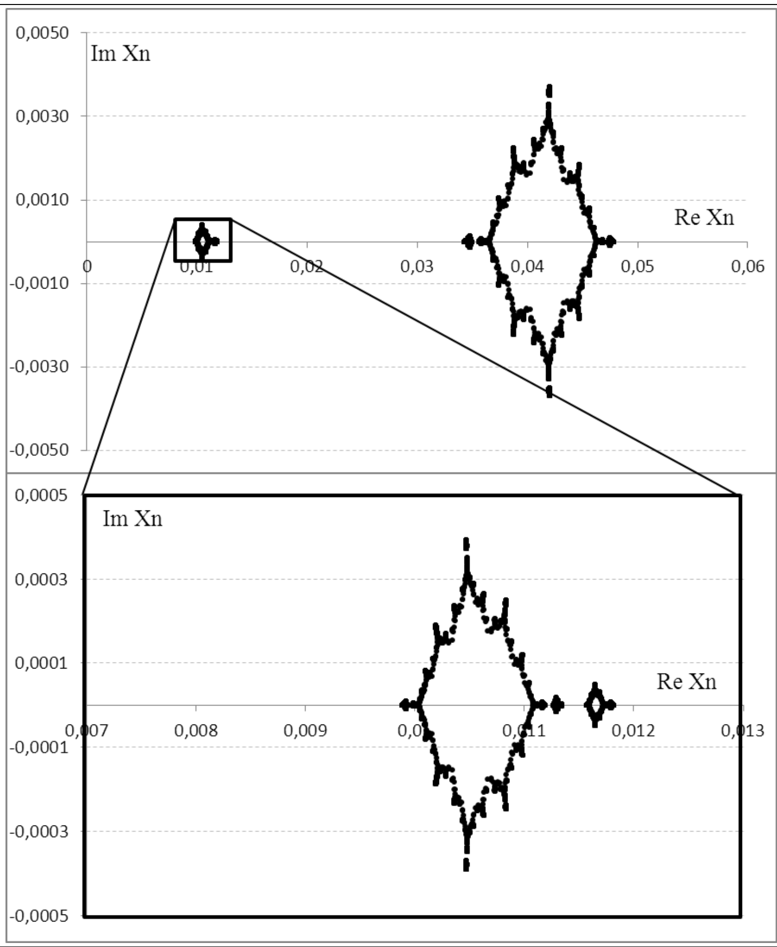


Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

The results of the study show that the radii of convergence (Julia sets) represent structures, whose forms depend on the nature of the attractor:

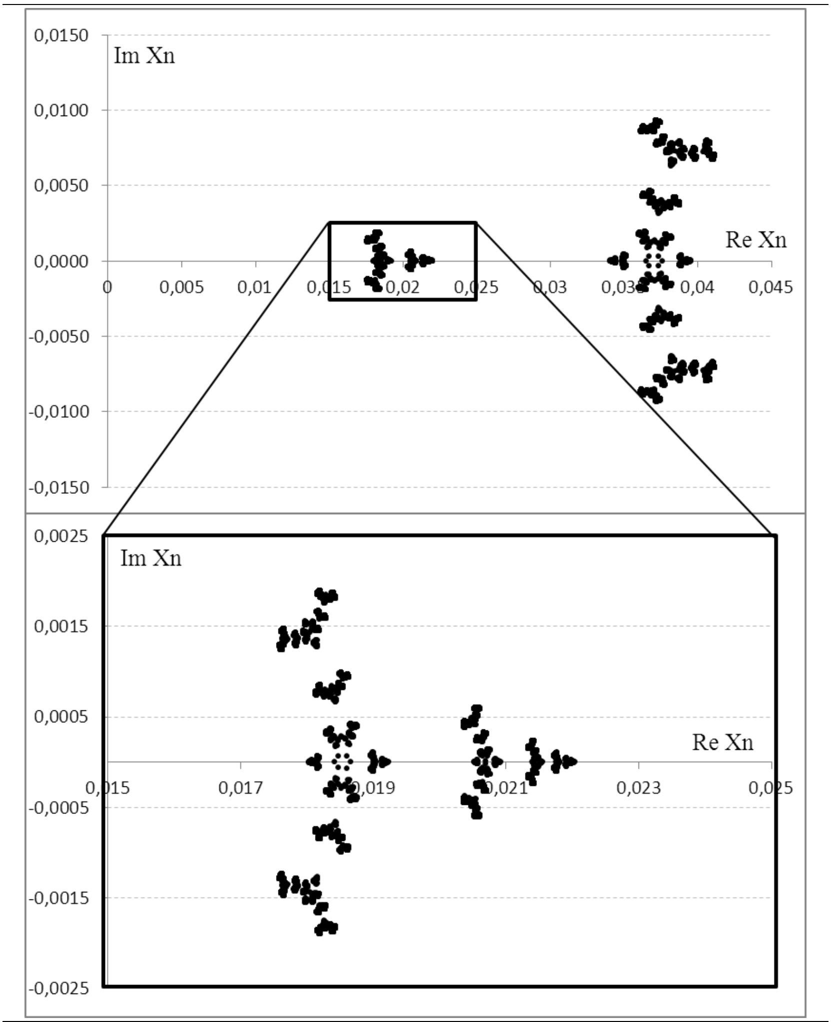
- for fixed stable points the Julia set has a strong connected structure (1987-1991; 2004 - 2009);
- in the transition to periodic points the corresponding area of Julia sets maintains the connected structure, as in the case of fixed stable points, but the border becomes "fuzzy"(1982-1986);
- Julia set, corresponding to the strange attractor, has no connected structure (1975-1981);
- Julia sets have a fractal structure (Figures 10, 11).

FIGURE 10. AN EXAMPLE OF FRACTAL STRUCTURE OF JULIA SET FOR TIME RANGE 1987 - 1991



Source: Own elaboration.

FIGURE 11. AN EXAMPLE OF FRACTAL STRUCTURE OF JULIA SET FOR TIME RANGE 2010 - 2014



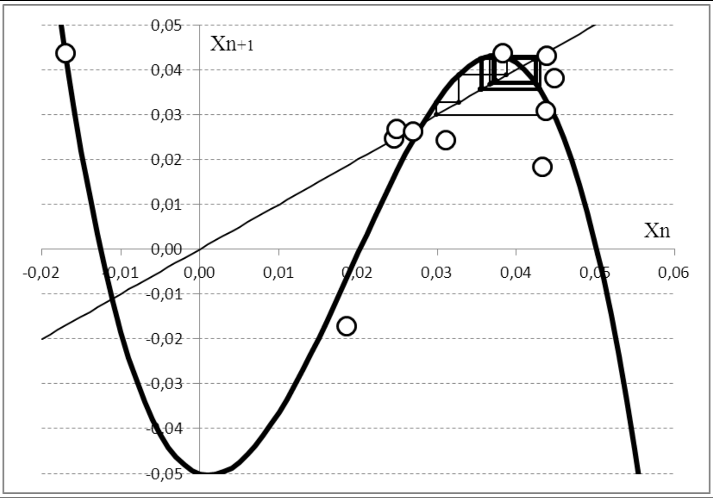
Source: Own elaboration.

The time interval 2010-2015 is an exception. Its Julia set has no connected structure, although it has a fixed stable point. Perhaps this indicates the formation of a strange attractor, as its Julia set looks like a Julia set for the time interval 1975-1981 (strange attractor).

A comparison of calculations made using the model and the facts of economic history indicates that the attractive fixed point is observed in periods of economic stability, whereas lack of convergence or the existence of a strange attractor corresponds to the periods of changing economic dynamics. The exception here is the period 2004-2015, when the economy has witnessed a chain of crises, and the model shows the existence of a stable fixed point. This result leads to the conclusion that these crises are manifestations of a single process and the nature of the economic dynamics since the beginning of the 2000's has changed and became to be characterized by a constant, reproducing the situation of instability. The loss of a connectivity structure of many Julia sets as noted above perhaps indicates a change in this situation in the near future.

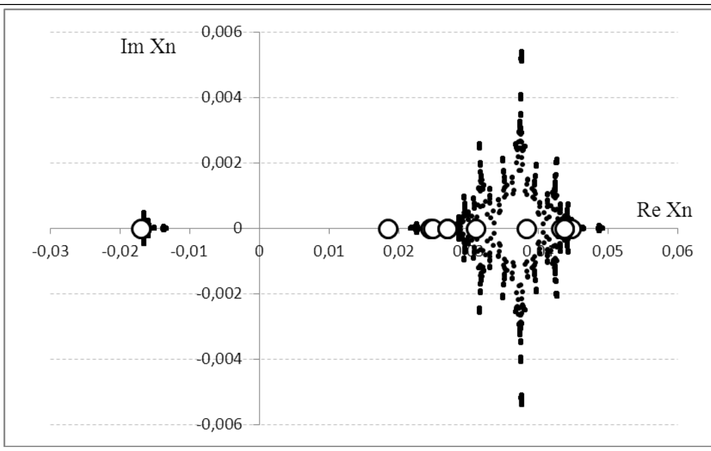
Subsequent studies confirmed this conclusion. The combination of the ranges 2004 - 2009 and 2010 - 2015 in a single range from 2004 to 2015 allows approximating the rate of change of world GDP data by a polynomial of degree 3 with an acceptable accuracy (R-squared value = 0.7132). In this case, there will be periodic points, as can be seen in Figure 12.

FIGURE 12 (A). VERHULST DIAGRAM OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 2004 - 2015



Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %). The initial value X_n = 0.04476.

FIGURE 12 (B). RADII OF CONVERGES (JULIA SET) OF APPROXIMATING POLYNOMIAL FOR TIME RANGE 2004 - 2015



Source: Own elaboration.
Note: ○ - the value of the rate of change of world GDP; data from World Bank Open Data. GDP growth (annual %).

5. Conclusion

The article considers a discrete dynamic model (DDM) describing the rate of change in global GDP. The DDM uses the only one assumption being the inertia of the processes occurring in the economy. The rate of world GDP change was represented in the Taylor series as a function of its value in the previous year till polynomials degree 3.

The rate of global GDP change, represented by a polynomial of degree 3 well describes the behavior of global GDP at various time intervals. We determined nine non-overlapping time ranges in the time interval from 1962 to 2015, in which the accuracy of approximation by polynomials of degree 3 gave the best result. This approximation, describing the rate of change of world GDP, has identified the various attractors. This provided the opportunity to move from a qualitative explanation of the observed effects to their quantitative analysis.

Radii of convergence (Julia sets) for attractors for rates of global GDP change were found. It was shown that Julia sets that meet fixed stable points and attractive fixed sets have a connected structure and Julia sets of strange attractors have no connected structure. Moreover, taking into account the connectedness of the Julia sets of fixed stable points and attractive fixed sets, the authors hope to build a corresponding Mandelbrot set.

In the framework of the assumptions adopted by the authors, the coefficients of the approximating polynomials (a_0, a_1, a_2, a_3) are associated with the entire set of macroeconomic parameters that determine the rate of economic growth. Thus, the use of DDM does not answer the question of what factors cause economic growth or recession. However, the changing in the character of growth, identified using the model, indicates that the relative impact and the set of these factors have changed. In this sense, the DDM can be called a metatheoretic model: it does not identify the causes of observed economic phenomenon, but marks the moment when these causes have changed and requires a new explanation of the economic processes.

An example of such an application of the DDM is the conclusion that the crises that have shaken the world economy since the early 2000s, are not separate periods of instability, but is one steady crisis.

The proposed graphic images of the results obtained in the application of DDM (radii of convergence), provide an opportunity to describe the changing dynamics of the world economy as not a group of numbers and graphs, but as a certain generalized image, "icons", which after appropriate revision will greatly facilitate the perception of the situation for decision-makers.

In conclusion, we note that the authors plan to undertake further development of the DDM, by solving the following formal tasks:

- How the values of the rate of world GDP change in relation to the basin of attraction?
- How the basin of attraction changes with time?
- How can we consider in the model the time dependence of the coefficients (a_0, a_1, a_2, a_3)?

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