ORTHOGONALIZATION OF NONORTHOGONAL VECTOR COMPONENTS

Ludvik Pfeifer

Defense Mapping Agency
Washington, DC 20305-3000

ABSTRACT

Whenever a three-dimensional frame of reference is physically instrumented by electromechanical sensors, as for example a triad of accelerometers in an inertial measurement unit, the inevitable misalignment of the effective input axes of these sensors from the idealized orthogonal frame of reference must be taken into consideration. This paper investigates in detail the linear transformation required to obtain orthogonal components of a vector given the nonorthogonal components of that vector, as produced by a physical sensor triad.

INTRODUCTION

The operation of an inertial navigation or guidance system is ultimately predicated on the accurate, real-time sensing of the acceleration and rotation vectors with respect to inertial space. The total acceleration vector is sensed by a triad of mutually orthogonal accelerometers, and the total rotation vector by a triad of mutually orthogonal gyroscopes, which together comprise a fully instrumented inertial platform.

Great pains are taken in the manufacture of these electromechanical sensors to define precisely their sensitive (input) axes, and in the assembly of these sensors into an inertial platform to have the three accelerometer input axes mutually orthogonal, the three gyro input axes mutually orthogonal, and the two orthogonal triads parallel to a conceptual three-dimensional rectangular frame of reference. While perfection is often approached, it cannot be fully attained in practice, and hence one must deal with the problem of having a vector expressed in terms of three nonorthogonal components and needing to have it expressed in terms of three orthogonal components, e.g. along the conceptual platform axes.

This paper investigates in detail the linear transformation required to obtain orthogonal components of a vector given its nonorthogonal components, e.g. as produced by a triad of real-world sensors. Although this is a very basic problem in the field of inertial navigation and guidance, a thorough perusal of the available open literature has revealed that the treatment of this subject, in most instances, is conspicuous by its absence. Where treated at all (e.g. PARVIN, 1962), it is glossed over and confusing. The only cogent treatment to be found is that by BRITTING (1971); however, the steps of the brief derivation given therein could not be followed by this author, and hence the motivation for a detailed derivation which may be of general interest.

PARVIN (1962, pp. 132-133) gives the inverse transformation matrix, i.e., the linear transformation of orthogonal vector components into their nonorthogonal equivalents, for the case of infinitesimal misalignments in which the small-angle approximation is valid. He then argues that this inverse transformation matrix, although not orthogonal, is close enough to being orthogonal that it may be treated as such without incurring much error. The implication is that the desired direct transformation (of nonorthogonal vector components into their orthogonal equivalents) may be effected by the transpose of this matrix since, as is well known, an orthogonal matrix has the property that its inverse is equal to its transpose.

BRITTING (1971, pp. 39-41), using a two-dimensional example, clearly shows that the transpose of the inverse transformation matrix is not the desired orthogonalization matrix. He then proceeds to derive the small-angle approximation of the correct orthogonalization matrix by means of an elegant, albeit "handwaving" argument, the central assertion of which is the difficult part which could not be understood.

NOTATION

In what follows, the three-dimensional coordinate reference frames are taken to be right-handed, and the right-handed convention for rotation angles about the coordinate axes is followed. Vectors will be denoted by underlined lower-case letters, and matrices by underlined upper-case letters. Use will be made of the elementary rotation matrices (e.g. MUELLER, 1969, pp. 42-43):

$$\underline{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{1}$$

$$\underline{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2}$$

$$\underline{R}_z(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

which effect the transformation of vector components from a right-handed rectangular coordinate frame to a second right-handed rectangular coordinate frame which differs from the first by a rotation, respectively, through angle α about the common X-axis, through angle β about the common Y-axis, and through angle γ about the common Z-axis. The elementary rotation matrices, as well as any product of two or more such matrices, are orthogonal and hence the inverse transformation is given by the transpose of the direct transformation matrix:

$$\underline{R}_j(-\delta) = \underline{R}_j^{-1}(\delta) = \underline{R}_j^{t}(\delta) \qquad (j=x,y,z) \tag{4}$$

Finally, the notation for rotation angles will be θ_{ij} denoting rotation component of axis i ($i=u, v,w$) about axis j ($j=x,y,z$) -- see Figure 1. To save space in the transformation equations involving sines and cosines of the rotation angles, the notation $S_{ij}=\sin\theta_{ij}$ and $C_{ij}=\cos\theta_{ij}$ will be employed.

SMALL-ANGLE APPROXIMATION

Because of the care taken in the manufacture of accelerometers and gyroscopes, and in the assembly of the inertial platform, it is the usual case that the misalignment angles, compared to the right angle, are in the differential range and may be treated as infinitesimals. Such a nonorthogonal reference frame, involving only "small" misalignment angles, will be called "quasiorthogonal." In what follows, the desired orthogonalization transformation will be derived in complete generality; however, in order to avoid undue complexity in the resulting equations, transition will be made to the respective small-angle approximations at an appropriate point in the derivation. The small-angle approximation, as used herein, involves the following:

1. Replacement of the cosine of a small angle by unity.
2. Replacement of the sine of a small angle by the angle itself (in radians).
3. Replacement of a product of two or more sines of the same or different small angles by zero.

****

DERIVATION OF THE ORTHOGONALIZATION TRANSFORMATION

Consider an orthogonal (e.g. platform) reference frame XYZ and a quasiorthogonal (e.g. accelerometer) reference frame UVW which share the common origin O (Figure 1). An arbitrary vector $\underline{r}$ will be denoted as $\underline{r}^o=(r_x,r_y,r_z)^t$ when expressed in terms of its components in the orthogonal reference frame, and as $\underline{r}^q=(r_u,r_v,r_w)^t$ when expressed in terms of its components in the quasiorthogonal reference frame.

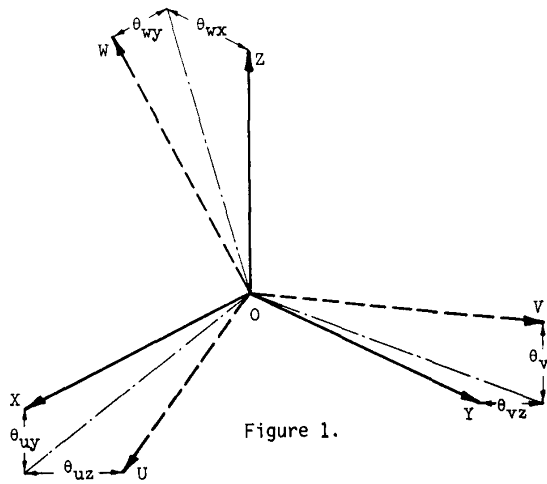

Figure 1.

The derivation will be made in three stages. In the first stage, the orthogonal reference frame XYZ is rotated to become reference frame X'Y'Z' such that the X'-axis is coincident with the U-axis of the quasiorthogonal system. In this rotated orthogonal frame, $r_{x'} = r_u$. Substituting r_u for $r_{x'}$, the frame X'Y'Z' is then rotated back into the original orthogonal frame XYZ. In the second and third stages, this two-rotation process is repeated to obtain rotated orthogonal frames X"Y"Z" in which $r_{y''} = r_v$ and X'''Y'''Z''' in which $r_{z'''} = r_w$, which are then rotated back into coincidence with the original reference frame XYZ, after substituting r_v for $r_{y''}$ and r_w for $r_{z'''}$, respectively.

<u>Stage 1</u>: Rotation of the XYZ-axes by θ_{uy} about the Y-axis and by θ_{uz} about the Z-axis into X'Y'Z'≡UY'Z'-axes; i.e., to bring the X-axis of the orthogonal system into coincidence with the U-axis of the quasiorthogonal system:

$$\begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix} = \underline{R}_z(\theta_{uz})\underline{R}_y(\theta_{uy}) \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} C_{uz} & S_{uz} & 0 \\ -S_{uz} & C_{uz} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{uy} & 0 & -S_{uy} \\ 0 & 1 & 0 \\ S_{uy} & 0 & C_{uy} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} C_{uz}C_{uy} & S_{uz} & -C_{uz}S_{uy} \\ -S_{uz}C_{uy} & C_{uz} & S_{uz}S_{uy} \\ S_{uy} & 0 & C_{uy} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} =$$

$$= \underline{R}_{yz}(\theta_{uy},\theta_{uz}) \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} (C_{uz}C_{uy}r_x + S_{uz}r_y - C_{uz}S_{uy}r_z) \\ (-S_{uz}C_{uy}r_x + C_{uz}r_y + S_{uz}S_{uy}r_z) \\ (S_{uy}rx + \qquad C_{uy}r_z) \end{bmatrix} = \begin{bmatrix} r_{x'} \\ r_{y'} \\ r_{z'} \end{bmatrix} = \begin{bmatrix} r_u \\ r_{y'} \\ r_{z'} \end{bmatrix}$$

At this point note that $\underline{r_{x'} = C_{uz}C_{uy}r_x + S_{uz}r_y - C_{uz}S_{uy}r_z = r_u}$ (5)

The X'Y'Z'≡UY'Z'-axes are now rotated back into coincidence with the XYZ-axes:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{R}^t_{yz}(\theta_{uy},\theta_{uz})\begin{bmatrix} r_u \\ r_{y'} \\ r_{z'} \end{bmatrix} = \begin{bmatrix} C_{uz}C_{uy} & -S_{uz}C_{uy} & S_{uy} \\ S_{uz} & C_{uz} & 0 \\ -C_{uz}S_{uy} & S_{uz}S_{uy} & C_{uy} \end{bmatrix}\begin{bmatrix} r_u \\ (-S_{uz}C_{uy}r_x + C_{uz}r_y + S_{uz}S_{uy}r_z) \\ (S_{uy}r_x + C_{uy}r_z) \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

Performing the indicated matrix-by-vector multiplication:

$$\begin{aligned} r_x &= C_{uz}C_{uy}r_u + S^2_{uz}C^2_{uy}r_x - S_{uz}C_{uz}C_{uy}r_y - S^2_{uz}S_{uy}C_{uy}r_z + S^2_{uy}r_x + S_{uy}C_{uy}r_z = \\ &= C_{uz}C_{uy}r_u + (S^2_{uz}C^2_{uy} + S^2_{uy})r_x - S_{uz}C_{uz}C_{uy}r_y + (S_{uy}C_{uy} - S^2_{uz}S_{uy}C_{uy})r_z = \\ &= C_{uz}C_{uy}r_u + (1 - C^2_{uz}C^2_{uy})r_x - S_{uz}C_{uz}C_{uy}r_y + C^2_{uz}S_{uy}C_{uy}r_z \end{aligned}$$

$$r_y = S_{uz}r_u - S_{uz}C_{uz}C_{uy}r_x + C^2_{uz}r_y + S_{uz}C_{uz}S_{uy}r_z$$

$$\begin{aligned} r_z &= -C_{uz}S_{uy}r_u - S^2_{uz}S_{uy}C_{uy}r_x + S_{uz}C_{uz}S_{uy}r_y + S^2_{uz}S^2_{uy}r_z + S_{uy}C_{uy}r_x + C^2_{uy}r_z = \\ &= -C_{uz}S_{uy}r_u + (S_{uy}C_{uy} - S^2_{uz}S_{uy}C_{uy})r_x + S_{uz}C_{uz}S_{uy}r_y + (S^2_{uz}S^2_{uy} + C^2_{uy})r_z = \\ &= -C_{uz}S_{uy}r_u + C^2_{uz}S_{uy}C_{uy}r_x + S_{uz}C_{uz}S_{uy}r_y + (1 - C^2_{uz}S^2_{uy})r_z \end{aligned}$$

This can be written as

$$\begin{bmatrix} C_{uz}C_{uy} \\ S_{uz} \\ -C_{uz}S_{uy} \end{bmatrix} r_u + \begin{bmatrix} 1 - C^2_{uz}C^2_{uy} & -S_{uz}C_{uz}C_{uy} & C^2_{uz}S_{uy}C_{uy} \\ -S_{uz}C_{uz}C_{uy} & C^2_{uz} & S_{uz}C_{uz}S_{uy} \\ C^2_{uz}S_{uy}C_{uy} & S_{uz}C_{uz}S_{uy} & 1 - C^2_{uz}S^2_{uy} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

wherefrom

$$\begin{bmatrix} C_{uz}C_{uy} \\ S_{uz} \\ -C_{uz}S_{uy} \end{bmatrix} r_u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} 1 - C^2_{uz}C^2_{uy} & -S_{uz}C_{uz}C_{uy} & C^2_{uz}S_{uy}C_{uy} \\ -S_{uz}C_{uz}C_{uy} & C^2_{uz} & S_{uz}C_{uz}S_{uy} \\ C^2_{uz}S_{uy}C_{uy} & S_{uz}C_{uz}S_{uy} & 1 - C^2_{uz}S^2_{uy} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} =$$

$$= \begin{bmatrix} C^2_{uz}C^2_{uy} & S_{uz}C_{uz}C_{uy} & -C^2_{uz}S_{uy}C_{uy} \\ S_{uz}C_{uz}C_{uy} & S^2_{uz} & -S_{uz}C_{uz}S_{uy} \\ -C^2_{uz}S_{uy}C_{uy} & -S_{uz}C_{uz}S_{uy} & C^2_{uz}S^2_{uy} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{B}_u\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

which can be written in the form

$$\underline{a}_u r_u = \underline{B}_u \underline{r}^o \simeq \begin{bmatrix} 1 & \theta_{uz} & -\theta_{uy} \\ \theta_{uz} & 0 & 0 \\ -\theta_{uy} & 0 & 0 \end{bmatrix} \underline{r}^o \tag{6}$$

where the matrix on the right is the small-angle approximation of the matrix $\underline{B}_u$.

Stage 2: Rotation of the XYZ-axes by θ_{vz} about the Z-axis and by θ_{vx} about the X-axis into X"Y"Z"≡X"VZ"-axes; i.e., to bring the Y-axis of the orthogonal system into coincidence with the V-axis of the quasiorthogonal system:

$$\begin{bmatrix} r_{x''} \\ r_{y''} \\ r_{z''} \end{bmatrix} = \underline{R}_x(\theta_{vx})\underline{R}_z(\theta_{vz})\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{vx} & S_{vx} \\ 0 & -S_{vx} & C_{vx} \end{bmatrix}\begin{bmatrix} C_{vz} & S_{vz} & 0 \\ -S_{vz} & C_{vz} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} C_{vz} & S_{vz} & 0 \\ -C_{vx}S_{vz} & C_{vx}C_{vz} & S_{vz} \\ S_{vx}S_{vz} & -S_{vx}C_{vz} & C_{vz} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} =$$

$$= \underline{R}_{xz}(\theta_{vx},\theta_{vz})\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} (C_{vz}r_x + S_{vz}r_y) \\ (-C_{vx}S_{vz}r_x + C_{vx}C_{vz}r_y + S_{vx}r_z) \\ (S_{vx}S_{vz}r_x - S_{vx}C_{vz}r_y + C_{vx}r_z) \end{bmatrix} = \begin{bmatrix} r_{x''} \\ r_{y''} \\ r_{z''} \end{bmatrix} = \begin{bmatrix} r_{x''} \\ r_v \\ r_{z''} \end{bmatrix}$$

At this point note that
$$\underline{r_{y''} = -C_{vx}S_{vz}r_x + C_{vx}C_{vz}r_y + S_{vx}r_z = r_v} \qquad (7)$$

The X"Y"Z"≡X"VZ"-axes are now rotated back into coincidence with the XYZ-axes:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{R}^t_{xz}(\theta_{vx},\theta_{vz})\begin{bmatrix} r_{x''} \\ r_v \\ r_{z''} \end{bmatrix} = \begin{bmatrix} C_{vz} & -C_{vx}S_{vz} & S_{vx}S_{vz} \\ S_{vz} & C_{vx}C_{vz} & -S_{vx}C_{vz} \\ 0 & S_{vx} & C_{vx} \end{bmatrix}\begin{bmatrix} (C_{vz}r_x + S_{vz}r_y) \\ r_v \\ (S_{vx}S_{vz}r_x - S_{vx}C_{vz}r_y + C_{vx}r_z) \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

Performing the indicated matrix-by-vector multiplication:

$$\begin{aligned} r_x &= C^2_{vz}r_x + S_{vz}C_{vz}r_y - C_{vx}S_{vz}r_v + S^2_{vx}S^2_{vz}r_z - S^2_{vx}S_{vz}C_{vz}r_y + S_{vx}C_{vx}S_{vz}r_z = \\ &= -C_{vx}S_{vz}r_v + (C^2_{vz} + S^2_{vx}S^2_{vz})r_x + (S_{vz}C_{vz} - S^2_{vx}S_{vz}C_{vz})r_y + S_{vx}C_{vx}S_{vz}r_z = \\ &= -C_{vx}S_{vz}r_v + (1 - C^2_{vx}S^2_{vz})r_x + C^2_{vx}S_{vz}C_{vz}r_y + S_{vx}C_{vx}S_{vz}r_z \end{aligned}$$

$$\begin{aligned} r_y &= S_{vz}C_{vz}r_x + S^2_{vz}r_y + C_{vx}C_{vz}r_v - S^2_{vx}S_{vz}C_{vz}r_x + S^2_{vx}C^2_{vz}r_y - S_{vx}C_{vx}C_{vz}r_z = \\ &= C_{vx}C_{vz}r_v + (S_{vz}C_{vz} - S^2_{vx}S_{vz}C_{vz})r_x + (S^2_{vz} + S^2_{vx}C^2_{vz})r_y - S_{vx}C_{vx}C_{vz}r_z = \\ &= C_{vx}C_{vz}r_v + C^2_{vx}S_{vz}C_{vz}r_x + (1 - C^2_{vx}C^2_{vz})r_y - S_{vx}C_{vx}C_{vz}r_z \end{aligned}$$

$$r_z = S_{vx}r_v + S_{vx}C_{vx}S_{vz}r_x - S_{vx}C_{vx}C_{vz}r_y + C^2_{vx}r_z$$

This can be written as
$$\begin{bmatrix} -C_{vx}S_{vz} \\ C_{vx}C_{vz} \\ S_{vx} \end{bmatrix} r_v + \begin{bmatrix} 1 - C^2_{vx}S^2_{vz} & C^2_{vx}S_{vz}C_{vz} & S_{vx}C_{vx}S_{vz} \\ C^2_{vx}S_{vz}C_{vz} & 1 - C^2_{vx}C^2_{vz} & -S_{vx}C_{vx}C_{vz} \\ S_{vx}C_{vx}S_{vz} & -S_{vx}C_{vx}C_{vz} & C^2_{vx} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

wherefrom
$$\begin{bmatrix} -C_{vx}S_{vz} \\ C_{vx}C_{vz} \\ S_{vx} \end{bmatrix} r_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} 1 - C^2_{vx}S^2_{vz} & C^2_{vx}S_{vz}C_{vz} & S_{vx}C_{vx}S_{vz} \\ C^2_{vx}S_{vz}C_{vz} & 1 - C^2_{vx}C^2_{vz} & -S_{vx}C_{vx}C_{vz} \\ S_{vx}C_{vx}S_{vz} & -S_{vx}C_{vx}C_{vz} & C^2_{vx} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} =$$

$$= \begin{bmatrix} C^2_{vx}S^2_{vz} & -C^2_{vx}S_{vz}C_{vz} & -S_{vx}C_{vx}S_{vz} \\ -C^2_{vx}S_{vz}C_{vz} & C^2_{vx}C^2_{vz} & S_{vx}C_{vx}C_{vz} \\ -S_{vx}C_{vx}S_{vz} & S_{vx}C_{vx}C_{vz} & S^2_{vx} \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{B}_v\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

which can be written in the form
$$\underline{\underline{a}_v r_v = \underline{B}_v\underline{r}^o \simeq \begin{bmatrix} 0 & -\theta_{vz} & 0 \\ -\theta_{vz} & 1 & \theta_{vx} \\ 0 & \theta_{vx} & 0 \end{bmatrix}\underline{r}^o} \qquad (8)$$

where the matrix on the right is the small-angle approximation of the matrix $\underline{B}_v$.

Stage 3: Rotation of the XYZ-axes by θ_{wx} about the X-axis and by θ_{wy} about the Y-axis into X'''Y'''Z'''≡ ≡X'''Y'''W-axes; i.e., to bring the Z-axis of the orthogonal system into coincidence with the W-axis of the quasiorthogonal system:

$$
\begin{bmatrix} r_{x'''} \\ r_{y'''} \\ r_{z'''} \end{bmatrix} = \underline{R}_y(\theta_{wy})\underline{R}_x(\theta_{wx}) \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} C_{wy} & 0 & -S_{wy} \\ 0 & 1 & 0 \\ S_{wy} & 0 & C_{wy} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{wx} & S_{wx} \\ 0 & -S_{wx} & C_{wx} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} C_{wy} & S_{wy}S_{wx} & -S_{wy}C_{wx} \\ 0 & C_{wx} & S_{wx} \\ S_{wy} & -C_{wy}S_{wx} & C_{wy}C_{wx} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} =
$$

$$
= \underline{R}_{xy}(\theta_{wx},\theta_{wy}) \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} (C_{wy}r_x + S_{wy}S_{wx}r_y - S_{wy}C_{wx}r_z) \\ (C_{wx}r_y + S_{wx}r_z) \\ (S_{wy}r_x - C_{wy}S_{wx}r_y + C_{wy}C_{wx}r_z) \end{bmatrix} = \begin{bmatrix} r_{x'''} \\ r_{y'''} \\ r_{z'''} \end{bmatrix} = \begin{bmatrix} r_{x'''} \\ r_{y'''} \\ r_w \end{bmatrix}
$$

At this point note that

$$
\underline{r_{z'''} = S_{wy}r_x - C_{wy}S_{wx}r_y + C_{wy}C_{wx}r_z = r_w} \tag{9}
$$

The X'''Y'''Z'''≡X'''Y'''W-axes are now rotated back into coincidence with the XYZ-axes:

$$
\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{R}^t_{xy}(\theta_{wx},\theta_{wy}) \begin{bmatrix} r_{x'''} \\ r_{y'''} \\ r_w \end{bmatrix} = \begin{bmatrix} C_{wy} & 0 & S_{wy} \\ S_{wy}S_{wx} & C_{wx} & -C_{wy}S_{wx} \\ -S_{wy}C_{wx} & S_{wx} & C_{wy}C_{wx} \end{bmatrix} \begin{bmatrix} (C_{wy}r_x + S_{wy}S_{wx}r_y - S_{wy}C_{wx}r_z) \\ (C_{wx}r_y + S_{wx}r_z) \\ r_w \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}
$$

Performing the indicated matrix-by-vector multiplication:

$$
r_x = C^2_{wy}r_x + S_{wy}C_{wy}S_{wx}r_y - S_{wy}C_{wy}C_{wx}r_z + S_{wy}r_w
$$

$$
\begin{aligned}
r_y &= S_{wy}C_{wy}S_{wx}r_x + S^2_{wy}S^2_{wx}r_y - S^2_{wy}S_{wx}C_{wx}r_z + C^2_{wx}r_y + S_{wx}C_{wx}r_z - C_{wy}S_{wx}r_w = \\
&= -C_{wy}S_{wx}r_w + S_{wy}C_{wy}S_{wx}r_x + (S^2_{wy}S^2_{wx} + C^2_{wx})r_y + (S_{wx}C_{wx} - S^2_{wy}S_{wx}C_{wx})r_z = \\
&= -C_{wy}S_{wx}r_w + S_{wy}C_{wy}S_{wx}r_x + (1 - C^2_{wy}S^2_{wx})r_y + C^2_{wy}S_{wx}C_{wx}r_z
\end{aligned}
$$

$$
\begin{aligned}
r_z &= -S_{wy}C_{wy}C_{wx}r_x - S^2_{wy}S_{wx}C_{wx}r_y + S^2_{wy}C^2_{wx}r_z + S_{wx}C_{wx}r_y + S^2_{wx}r_z + C_{wy}C_{wx}r_w = \\
&= C_{wy}C_{wx}r_w - S_{wy}C_{wy}C_{wx}r_x + (S_{wx}C_{wx} - S^2_{wy}S_{wx}C_{wx})r_y + (S^2_{wy}C^2_{wx} + S^2_{wx})r_z = \\
&= C_{wy}C_{wx}r_w - S_{wy}C_{wy}C_{wx}r_x + C^2_{wy}S_{wx}C_{wx}r_y + (1 - C^2_{wy}C^2_{wx})r_z
\end{aligned}
$$

This can be written as

$$
\begin{bmatrix} S_{wy} \\ -C_{wy}S_{wx} \\ C_{wy}C_{wx} \end{bmatrix} r_w + \begin{bmatrix} C^2_{wy} & S_{wy}C_{wy}S_{wx} & -S_{wy}C_{wy}C_{wx} \\ S_{wy}C_{wy}S_{wx} & 1 - C^2_{wy}S^2_{wx} & C^2_{wy}S_{wx}C_{wx} \\ -S_{wy}C_{wy}C_{wx} & C^2_{wy}S_{wx}C_{wx} & 1 - C^2_{wy}C^2_{wx} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}
$$

wherefrom

$$
\begin{bmatrix} S_{wy} \\ -C_{wy}S_{wx} \\ C_{wy}C_{wx} \end{bmatrix} r_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} C^2_{wy} & S_{wy}C_{wy}S_{wx} & -S_{wy}C_{wy}C_{wx} \\ S_{wy}C_{wy}S_{wx} & 1 - C^2_{wy}C^2_{wx} & C^2_{wy}S_{wx}C_{wx} \\ -S_{wy}C_{wy}C_{wx} & C^2_{wy}S_{wx}C_{wx} & 1 - C^2_{wy}C^2_{wx} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} =
$$

$$
= \begin{bmatrix} S^2_{wy} & -S_{wy}C_{wy}S_{wx} & S_{wy}C_{wy}C_{wx} \\ -S_{wy}C_{wy}S_{wx} & C^2_{wy}S^2_{wx} & -C^2_{wy}S_{wx}C_{wx} \\ S_{wy}C_{wy}C_{wx} & -C^2_{wy}S_{wx}C_{wx} & C^2_{wy}C^2_{wx} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{B}_w \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}
$$

which can be written in the form

$$
\underline{a}_w r_w = \underline{B}_w \underline{r}^o \simeq \begin{bmatrix} 0 & 0 & \theta_{wy} \\ 0 & 0 & -\theta_{wx} \\ \theta_{wy} & -\theta_{wx} & 1 \end{bmatrix} \underline{r}^o \tag{10}
$$

where the matrix on the right is the small-angle approximation of the matrix $\underline{B}_w$.

THE ORTHOGONALIZATION MATRIX

Invoking the principle of superposition, add equations (6), (8), and (10):

$$\underline{A}\,\underline{r}^q = \begin{bmatrix} \underline{a}_u & \underline{a}_v & \underline{a}_w \end{bmatrix} \begin{bmatrix} r_u \\ r_v \\ r_w \end{bmatrix} = \underline{a}_u r_u + \underline{a}_v r_v + \underline{a}_w r_w =$$

$$= (\underline{B}_u + \underline{B}_v + \underline{B}_w) \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{B}\,\underline{r}^o$$

or $\underline{A}\,\underline{r}^q = \underline{B}\,\underline{r}^o$, wherefrom $\underline{r}^o = \underline{B}^{-1}\underline{A}\,\underline{r}^q = \underline{Q}\,\underline{r}^q$,

where
$$\underline{Q} = \underline{B}^{-1}\underline{A} \qquad (11)$$

is the desired orthogonalization matrix which converts (exactly) vector components measured along the quasiorthogonal (e.g. accelerometer) UVW-axes to their equivalent components along the orthogonal (e.g. platform) XYZ-axes.

The standard practice (e.g. PARVIN, 1962, pp. 131-132) is to assume that matrix $\underline{B}$ is negligibly different from the identity matrix, i.e., that $\underline{B} \simeq \underline{I}$, and hence that $\underline{Q} \simeq \underline{A}$, where

$$\underline{A} = \begin{bmatrix} \underline{a}_u & \underline{a}_v & \underline{a}_w \end{bmatrix} =$$

$$= \begin{bmatrix} C_{uz}C_{uy} & -C_{vx}S_{vz} & S_{wy} \\ S_{uz} & C_{vx}C_{vz} & -C_{wy}S_{wx} \\ -C_{uz}S_{uy} & S_{vx} & C_{wy}C_{wx} \end{bmatrix} \simeq \begin{bmatrix} 1 & -\theta_{vz} & \theta_{wy} \\ \theta_{uz} & 1 & -\theta_{wx} \\ -\theta_{uy} & \theta_{vx} & 1 \end{bmatrix}$$

the last matrix being the small-angle approximation of $\underline{A}$. Matrix $\underline{A}$ will be recognized to be the transpose of the inverse transformation matrix, i.e., of the "deorthogonalization" matrix $\underline{P}$ which effects the transformation:

$$\underline{r}^q = \underline{P}\,\underline{r}^o = \underline{Q}^{-1}\underline{r}^o = \underline{A}^{-1}\underline{B}\,\underline{r}^o.$$

This transformation may be obtained explicitly by collecting and writing in matrix form equations (5), (7), and (9):

$$\underline{r}^q = \begin{bmatrix} r_u \\ r_v \\ r_w \end{bmatrix} = \begin{bmatrix} C_{uz}C_{uy} & S_{uz} & -C_{uz}S_{uy} \\ -C_{vx}S_{vz} & C_{vx}C_{vz} & S_{vx} \\ S_{wy} & -C_{wy}S_{wx} & C_{wy}C_{wx} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underline{P}\,\underline{r}^o \simeq$$

$$\simeq \begin{bmatrix} 1 & \theta_{uz} & -\theta_{uy} \\ -\theta_{vz} & 1 & \theta_{vx} \\ \theta_{wy} & -\theta_{wx} & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

wherefrom it is seen that

$$\underline{P} = \underline{Q}^{-1} = \underline{A}^{-1}\underline{B} = \underline{A}^t.$$

Matrix $\underline{B}$, in terms of its small-angle approximation, is obtained as the sum of previously derived partial matrices $\underline{B}_v$, $\underline{B}_u$, and $\underline{B}_w$:

$$\underline{B} = \underline{B}_u + \underline{B}_v + \underline{B}_w \simeq$$

$$\simeq \begin{bmatrix} 1 & \theta_{uz} & -\theta_{uy} \\ \theta_{uz} & 0 & 0 \\ -\theta_{uy} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\theta_{vz} & 0 \\ -\theta_{vz} & 1 & \theta_{vx} \\ 0 & \theta_{vx} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \theta_{wy} \\ 0 & 0 & -\theta_{wx} \\ \theta_{wy} & -\theta_{wx} & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & \theta_{uz}-\theta_{vz} & \theta_{wy}-\theta_{uy} \\ \theta_{uz}-\theta_{vz} & 1 & \theta_{vx}-\theta_{wx} \\ \theta_{wy}-\theta_{uy} & \theta_{vx}-\theta_{wx} & 1 \end{bmatrix}$$

Notice that the off-diagonal elements of $\underline{B}$ vanish (and hence $\underline{B} = \underline{I}$) if and only if

$$\theta_{vx} = \theta_{wx}, \quad \theta_{uy} = \theta_{wy}, \quad \theta_{uz} = \theta_{vz},$$

that is, if and only if the UVW-axes are also orthogonal, in which case the transformation matrix $\underline{Q} = \underline{A}$ is an orthogonal matrix with the property

$$\underline{Q}^{-1} = \underline{A}^{-1} = \underline{A}^t = \underline{Q}^t.$$

While the off-diagonal elements of $\underline{B}$ are small, they are of the same order of magnitude as the off-diagonal elements of $\underline{A}$ (since the misalignment angles θ_{ij} can be negative as well as positive) and hence the assumption that $\underline{B} = \underline{I}$ is just as inappropriate as the assumption that $\underline{A} = \underline{I}$ would be. However, since the off-diagonal elements of the small-angle version of $\underline{B}$ are small (i.e., $b_{ij} \ll 1$, $i \neq j$), we can write:

$$\underline{B}^{-1} = (\underline{I} + \delta\underline{B})^{-1} \simeq$$

$$\simeq (\underline{I} - \delta\underline{B}) = \begin{bmatrix} 1 & \theta_{vz}-\theta_{uz} & \theta_{uy}-\theta_{wy} \\ \theta_{vz}-\theta_{uz} & 1 & \theta_{wx}-\theta_{vx} \\ \theta_{uy}-\theta_{wy} & \theta_{wx}-\theta_{vx} & 1 \end{bmatrix}$$

and obtain the small-angle version of the desired orthogonalization matrix as follows:

$$\underline{Q} = \underline{B}^{-1}\underline{A} \simeq$$

$$\simeq \begin{bmatrix} 1 & \theta_{vz}-\theta_{uz} & \theta_{uy}-\theta_{wy} \\ \theta_{vz}-\theta_{uz} & 1 & \theta_{wx}-\theta_{vx} \\ \theta_{uy}-\theta_{wy} & \theta_{wx}-\theta_{vx} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\theta_{vz} & \theta_{wy} \\ \theta_{uz} & 1 & -\theta_{wx} \\ -\theta_{uy} & \theta_{vx} & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

Performing the indicated matrix multiplication and enforcing the small-angle approximation conventions (i.e., suppressing second-order terms), we obtain:

$$q_{11} = 1 + (\theta_{vz}-\theta_{uz})\theta_{uz} - (\theta_{uy}-\theta_{wy})\theta_{uy} \simeq 1$$

$$q_{12} = -\theta_{vz} + \theta_{vz} - \theta_{uz} + (\theta_{uy}-\theta_{wy})\theta_{vx} \simeq -\theta_{uz}$$

$$q_{13} = \theta_{wy} - (\theta_{vz}-\theta_{uz})\theta_{wx} + \theta_{uy} - \theta_{wy} \simeq \theta_{uy}$$

$$q_{21} = \theta_{vz} - \theta_{uz} + \theta_{uz} - (\theta_{wx}-\theta_{vx})\theta_{uy} \simeq \theta_{vz}$$

$$q_{22} = -(\theta_{vz}-\theta_{uz})\theta_{vz} + 1 + (\theta_{wx}-\theta_{vx})\theta_{vx} \simeq 1$$

$$q_{23} = (\theta_{vz}-\theta_{uz})\theta_{wy} - \theta_{wx} + \theta_{wx} - \theta_{vx} \simeq -\theta_{vx}$$

$$q_{31} = \theta_{uy} - \theta_{wy} + (\theta_{wx}-\theta_{vx})\theta_{uz} - \theta_{uy} \simeq -\theta_{wy}$$

$$q_{32} = -(\theta_{uy}-\theta_{wy})\theta_{vz} + \theta_{wx} - \theta_{vx} + \theta_{vx} \simeq \theta_{wx}$$

$$q_{33} = (\theta_{uy}-\theta_{wy})\theta_{wy} - (\theta_{wx}-\theta_{vx})\theta_{wx} + 1 \simeq 1$$

Hence, the small-angle version of the desired orthogonalization matrix $\underset{\sim}{Q}$ becomes:

$$\underset{\sim}{Q} = \begin{bmatrix} 1 & -\theta_{uz} & \theta_{uy} \\ \theta_{vz} & 1 & -\theta_{vx} \\ -\theta_{wy} & \theta_{wx} & 1 \end{bmatrix} \tag{12}$$

In retrospect, the same matrix inversion argument can be applied to the small-angle version of the deorthogonalization matrix $\underline{P}$. Since the off-diagonal elements of the small-angle version of $\underline{P}$ are small (i.e., $p_{ij} \ll 1$, $i \neq j$), the small-angle version of $\underset{\sim}{Q}$ can be obtained directly as the inverse of the small-angle version of $\underline{P}$:

$$\underset{\sim}{Q} = \underline{P}^{-1} = \begin{bmatrix} 1 & \theta_{uz} & -\theta_{uy} \\ -\theta_{vz} & 1 & \theta_{vx} \\ \theta_{wy} & -\theta_{wx} & 1 \end{bmatrix}^{-1} = (\underline{I} + \delta\underline{P})^{-1} \simeq$$

$$\simeq (\underline{I} - \delta\underline{P}) = \begin{bmatrix} 1 & -\theta_{uz} & \theta_{uy} \\ \theta_{vz} & 1 & -\theta_{vx} \\ -\theta_{wy} & \theta_{wx} & 1 \end{bmatrix}$$

which serves to confirm the result.

CONCLUSION

This paper has addressed in detail the problem of transformation of vector components between non-orthogonal and orthogonal three-dimensional coordinate frames. The orthogonalization transformation was derived in general terms and then specialized to the usual case of small-angle misalignments. The small-angle version of the orthogonalization matrix verifies the result of a brief treatment by BRITTING (1971), which was the objective of this investigation.

REFERENCES

BRITTING, Kenneth R.: Inertial Navigation Systems Analysis, Wiley-Interscience, New York, 1971.

MUELLER, Ivan I.: Spherical and Practical Astronomy as Applied to Geodesy, Frederick Ungar, New York, 1969.

PARVIN, Richard H.: Inertial Navigation, Van Nostrand, New York, 1962.