

MAXIMUM-ENTROPY SPECTRUM FROM A NON-EXTENDABLE AUTOCORRELATION FUNCTION

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INTRODUCTION

In the first "published" reference on maximum entropy spectra, an enormously influential and seminal symposium reprint, Burg [1] announced his new method based upon exactly known, error free autocorrelation samples. Burg showed that if the first n samples were indeed the beginning of a legitimate autocorrelation function (ACF) that the next sample ($n+1$) was restricted to lie in a very small range. If that ($n+1$) sample were chosen to be in the center of the allowed range, then the sample number ($n+2$) would have the greatest freedom to be chosen, again in a small range. In the same paper, Burg also showed that the extrapolation to the center point of the permissible range corresponds to a maximum entropy situation in which the available data were fully utilized, while no unwarranted assumptions were made about unavailable data. In fact unmeasured data were to be as random as possible subject to the constraint that the power spectral density produce ACF values in agreement with the known, exact ACF.

Some time later it was recognized that the realization of exactly known ACF values rarely if ever occurs in practice and that the ACF is usually estimated from a few samples of the time series or from some other experimental arrangement and is, therefore, subject to measurement error. Thus the concept of exact matching of given ACF values was weakened to approximate matching: up to the error variance. Since then there have been several extensions of the Maximum Entropy Method (MEM) to include error prone ACF estimates. Ables [2] suggested an extension but gave no practical method or results. The earliest practical extension was due to Newman [3] who showed that with a slightly generalized definition, one could obtain maximum entropy spectra given noisy ACF estimates.

The Newman method appeared to work well - at least for small problems. I am very much indebted to Bill Newman for sending me a copy of his FORTRAN program which I compiled to run on the AFGL CYBER - 750 computer. The problems that Newman had written-up as test cases worked very nicely. However, when I attempted to use the method on larger problems involving noisy ACF measurements, obtained from an interferometer (an interferogram), I had difficulties.

For example, if we ignore errors in the ACF and simply solve the Yule-Walker equations using the simple program of Ulrych and Bishop (1975), at order 20 the Yule-Walker equations failed, i.e. produced a reflection coefficient greater than one and a resultant negative error power.

Now applying Newman's program, I was able to extend the valid range to 30 or so but with difficulty. The program allows an initial adjustment of the zero-order correlation (ρ_0) upward until a solution is obtained and then tries to lower this value of ρ_0 as much as possible. In order to obtain convergence for Newman's iterative procedure, increasingly large ρ_0 values were required and even so convergence was painfully slow. I then decided that it is really improper to vary the ρ_0 lag, because the entropy will increase without bound as ρ_0 is increased, and thus there will be no maximum entropy solution.

Other methods have appeared in the literature, notably Schot and McLellan [4]. This paper was written in such a general fashion, to allow for application to multichannel and/or multidimensional data, that I have been unable to utilize it properly for a test. Also the program was written in "C" programming language (not spoken by the CDC Cyber): an additional roadblock; even if the authors had made the program available it would have been a great labor to convert it to FORTRAN.

It was then decided to search for some viable and simple alternative to the above mentioned methods.

THE NEW METHOD

At the very heart of the maximum entropy method of BURG is the Fundamental Autocorrelation Theorem. This theorem, announced in the 1967 paper and proved by BURG in his Ph.D thesis [5], can be stated "the first n numbers $\rho_0, \rho_1, \dots, \rho_n$ constitute the beginning of an Autocorrelation Function (ACF) if, and only if, the Toeplitz autocorrelation matrix is nonnegative definite. If we now write down the modern Yule-Walker equations as:

$$\begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \dots & \rho_n \\ \rho_1 & \rho_0 & & & \\ \rho_2 & & \rho_0 & & \\ \vdots & & & \ddots & \\ \rho_n & \dots & \dots & \dots & \rho_0 \end{pmatrix} \begin{pmatrix} 1 \\ g_{n1} \\ \vdots \\ g_{nn} \end{pmatrix} = \begin{pmatrix} \rho_{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (1)$$

the square matrix is the Toeplitz autocorrelation matrix where the prediction error filter is written 1, $g_{n1}, g_{n2}, \dots, g_{nn}$ and the error power at stage n is ρ_{n+1} . The numbers $g_{11}, g_{22}, \dots, g_{nn}$ are the so-called reflection coefficients (due to a geophysical analogy involving seismic reflection at interfaces). The fundamental ACF theorem is completely equivalent to the statement that the first $(n+1)$ numbers, $\rho_0, \rho_1, \dots, \rho_n$, are the beginning of an ACF if, and only if, the reflection coefficients, $(g_{jj}, j=1, n)$ are all of magnitude less than or equal to unity; that is, $|g_{jj}| < 1$. This extremely powerful and simple condition was the basis for the successful and optimal non-linear methods of Fougere [6,7] which solve the line splitting and line shifting problem associated with the Burg technique. We will now proceed to derive the new technique using a method which parallels the non-linear method exactly. For if we have been given some numbers $\rho_0, \rho_1, \dots, \rho_n$ and we use the Yule-Walker equations (see for example the extremely simple program in Ulrych and Bishop [8]) and find any $g_{jj} > 1$, we know that there are errors in some or all of the ρ 's. The given numbers simply cannot be the beginning of an ACF! But if we look at our Yule-Walker equations again, we can see that using the bottom row, a very simple recursion for the ACF can be derived.

$$\rho_k = - \sum_{j=1}^k \rho_{k-j} g_{kj} \quad ; k=1, 2 \dots j-1 \quad (2)$$

For example:

$\rho_1 = -\rho_0 g_{11} \quad ; \quad \rho_2 = -(\rho_0 g_{21} + \rho_1 g_{22})$
etc. Given the prediction error filter (PEF) and using the simple Levinson Recursion:

$$g_{jk} = g_{j-1,k} - g_{jj} g_{j-1,j-k} \quad ; \quad k=1, 2, \dots, j-1 \quad j=2, 3, \dots, n \quad (3)$$

we can obtain the PEF directly from the set of reflection coefficients. Thus we now see that as long as we begin with reflection coefficients, all of which lie in the range $-1 < g_{jj} < 1$ we will always get an ACF. This condition is trivial to enforce if we simply set $g_{jj} = U \sin \phi_j$ where U is a constant, very slightly less than 1, and ϕ_j is any real angle. This is the nub of the extremely simple new method. We start off by setting all $g_{jj} = U \sin \phi_j$ and then find the ACF given by these numbers. We then minimize the

distance, R , between our new acceptable ACF, $(\hat{\rho}_k)$, and the given unacceptable ACF (ρ_k) , where:

$$R^2 = \sum_{k=1}^n (\hat{\rho}_k - \rho_k)^2 \quad (4)$$

The result will always be a legitimate ACF and the extension (from n lags to ∞ lags) via the maximum entropy method will always produce a Maximum Entropy Spectrum.

It might now be argued that we have never written down an expression for entropy and maximized it. This is true but the extension of the allowable ACF will always be a "Maximum Entropy Extension". If our original given ACF were badly in error then our new ACF will fit the old ACF, but not very well. Nevertheless, the fit will always be as good as is allowed by the given ACF when the new method is allowed to converge. The starting guess for the iterative solution is obtained by extrapolating the last acceptable ACF of order j up to the full length of the original given ACF, using equation (5) also to be found in Ulrych and Bishop [8]:

$$\hat{\rho}_k = \begin{cases} \rho_k & ; k=1, 2, \dots, m \\ \sum_{j=1}^m \rho_{k-j} g_{mj} & ; k=m+1, \dots, n \end{cases} \quad (5)$$

The method, which has been programmed in FORTRAN, utilizes the IMSL subroutine ZXSSQ, which is a non-linear least squares routine. The required subroutine ZXSSQ can be obtained from IMSL; note that many computer libraries subscribe to IMSL and have copies of ZXSSQ already.

The program is available on 9 track tape on request by seriously interested scientists. Please do not send a blank tape but do include the required type density, either 800 or 1600 BPI and the required code, either EBCDIC or ASCII.

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REFERENCES

1. Burg, J.P., (1967), "Maximum Entropy Spectral Analysis", Reprinted in "Modern Spectrum Analysis", (1978), Ed D.C. Childers, IEEE Press, N.Y.
2. Ables, J.G., (1974), "Maximum Entropy Spectral Analysis" Astron. Astrophys. Suppl. Series 15, 383-393 (1972), Proc. Symp. on the Collection and Analyses of Astrophysical Data.
3. Neuman, W.I., (1977) "Extension to the Maximum Entropy Method", IEEE Trans. Inform. Theory, IT-23, 89-93.
4. Schott, J.P. and J.H. McClellan, (1984) "Maximum Entropy Power Spectrum Estimation with Uncertainty in Correlation Measurements", IEEE Trans. ASSP, 32, 410-418.
5. Burg, J.P., (1975), "Maximum Entropy Spectral Analysis" Ph.d. Thesis, Stanford University, Palo Alto, CA, 123pp
6. Fougere, P.F., (1977), "A Solution to the Problem of Spontaneous Line Splitting in Maximum Entropy Power Spectrum Analysis", J. Geophys. Res. 82, 1051,1054.
7. Fougere, P.F., (1978), "A Solution to the Problem of Spontaneous Line Splitting in Maximum Entropy Power Spectrum Analysis of Complex Signals", Proc. RADC Spectrum Estimation Workshop, Rome, N.Y.
8. Ulrych, T.J. and T.N. Bishop, (1975), "Maximum Entropy Spectral Analysis and Autoregressive Decomposition", Rev. Geophys. and Space Phys. 13, 183-200.