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The Capabilities and Art of State-Dependent Riccati Equation-Based Design

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Abstract

State-dependent Riccati equation (SDRE) techniques are general design methods which provide a systematic and effective means of designing nonlinear controllers, observers, and filters. This paper provides an overview of the capabilities of SDRE control and goes into detail concerning the art of carrying out effective SDRE designs for both systems that conform and do not conform to the basic structure and conditions required by the method. The paper is centered around the SDRE nonlinear regulator. The following situations which prevent a straightforward application of the SDRE method to the control problem at hand are addressed: the existence of state-independent terms, the existence of state-dependent terms which do not go to zero as the state vector goes to zero, the existence of nonlinearity in the controls, and the existence of uncontrollable and unstable but bounded state dynamics.

1. Introduction

In recent years, state-dependent Riccati equation (SDRE) techniques are increasingly being used in a wide variety of nonlinear control and nonlinear filtering applications. These include advanced guidance law development [1, 2, 3], autopilot design [4, 5], integrated guidance and control design [6, 7], satellite and spacecraft control and estimation [8, 9, 10], process control [11, 12], magnetic levitation [13], control of systems with parasitic effects [14], control of artificial human pancreas [15], robotics [16], simultaneous state and parameter estimation [17], ducted fan control [18], and various benchmark problems [19, 20].

The SDRE techniques are defined by their linear-like structures which have state-dependent coefficient matrices. The SDRE nonlinear regulator has the same structure as the infinite-horizon linear quadratic regulator (LQR). SDRE H_2 control and SDRE H_∞ control have the same structures as linear H_2 and linear H_∞ control, respectively. The SDRE filter has the same structure as the steady-state linear Kalman filter. These SDRE methods require the solution of a state-dependent algebraic Riccati equation, or two, in the case of partial information SDRE H_2 control and SDRE H_∞ control.

The differential SDRE method [7] has the same structure as finite-horizon LQR and the differential SDRE filter [21, 22] has the same structure as the linear Kalman filter. These latter techniques require the integration of a state-dependent differential Riccati equation. While this paper is centered around the SDRE nonlinear regulator, the techniques presented for producing effective SDRE designs and for handling systems that do not conform the basic structure and conditions required for the direct application of the method apply to all of the various SDRE techniques listed above.

In the next section, the SDRE nonlinear regulator with integral servomechanism action is discussed. In Section 3, the capabilities of the SDRE method are reviewed. In Section 4, the art of designing SDRE controllers for both systems which do and do not conform to the structure and conditions needed for the direct application of the technique is presented. The paper is then closed with a Summary section.

2. The SDRE Nonlinear Regulator

Consider the autonomous, infinite-horizon, nonlinear regulator problem for minimizing the performance index

$$J = \frac{1}{2}\int_0^\infty x^T Q(x)x + u^T R(x)u \; dt \qquad (1)$$

with respect to the state x and control u subject to the nonlinear differential constraints:

$$\dot{x} = f(x) + B(x)u \qquad (2)$$

where $Q(x) \geq 0$ and $R(x) > 0$ for all x and where

Condition 1. $f(x)$ is a continuously differentiable function of x, *i.e.*, $f(x) \in C^1$ (3)

Condition 2. $f(0) = 0$. (4)

The SDRE approach for obtaining a suboptimal, locally asymptotically stabilizing solution of problem (1)-(2) is:

i) Use direct parameterization to bring the nonlinear dynamics to the state-dependent coefficient (SDC) form

$$\dot{x} = A(x)x + B(x)u \qquad (5)$$

where

$$f(x) = A(x)x \qquad (6)$$

In the multivariable case, it is well-known [23] that if $f(x) \in C^1$, there is an infinite number of ways to factor $f(x)$ into $A(x)x$ and that $A(x)$ can be parameterized as $A(x,\alpha)$, where α is a vector of free design parameters. In order to obtain a valid solution of the SDRE, the pair $\{A(x,\alpha), B(x)\}$ has to be pointwise stabilizable in the linear sense for all x in the domain of interest.

ii) Solve the state-dependent Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (7)$$

to obtain $P(x) \geq 0$.

iii) Construct the nonlinear feedback controller equation:

$$u = -R(x)^{-1}B(x)^T P(x)x. \qquad (8)$$

In order to perform command following, the SDRE controller can be implemented as an integral servomechanism as demonstrated in [5]. This is accomplished as follows. First, the state x is decomposed as

$$x = \begin{bmatrix} x_T \\ x_N \end{bmatrix} \qquad (9)$$

where it is desired for the vector components of x_T to track a reference command r_c. The state vector x is then augmented with x_I, the integral states of x_T:

$$\tilde{x} = \begin{bmatrix} x_I \\ x_T \\ x_N \end{bmatrix} \qquad (10)$$

The augmented system is given by

$$\dot{\tilde{x}} = \tilde{A}(\tilde{x},\alpha)\tilde{x} + \tilde{B}(\tilde{x})u \qquad (11)$$

where

$$\tilde{A}(\tilde{x},\alpha) = \left[\begin{array}{c|c} 0 & I \;\vdots\; 0 \\ \hline 0 & A(x,\alpha) \end{array}\right] \quad \tilde{B}(\tilde{x}) = \begin{bmatrix} 0 \\ B(x) \end{bmatrix} \qquad (12)$$

and the SDRE integral servo controller is given by

$$u = -\tilde{R}(\tilde{x})^{-1}\tilde{B}(\tilde{x})^T\tilde{P}(\tilde{x}) \begin{bmatrix} x_I - \int r_c dt \\ x_T - r_c \\ x_N \end{bmatrix} \qquad (13)$$

In order for the SDRE to have a solution, the pointwise detectability condition must be satisfied. This is accomplished by penalizing the integral states with the corresponding non-zero diagonal elements of $\tilde{Q}(\tilde{x})$.

3. Capabilities of the SDRE Method

The SDRE method has many capabilities that other nonlinear design methods do not have, at least collectively. These include: 1) the capability to directly specify and affect performance through the selection of the state-dependent state and control weighting matrices $Q(x)$ and $R(x)$, respectively, 2) the capability to impose hard bounds on the control or control rate, 3) the capability to satisfy state constraints, and combined state and control constraints, 4) the capability to directly handle unstable, non-minimum phase systems, 5) the capability to preserve beneficial nonlinearities, and 6) the capability to utilize the extra design degrees of freedom that are available in the non-uniqueness of the state-dependent coefficient matrix $A(x,\alpha)$ to enhance the performance of the system.

The SDRE method also allows for the systematic design of a broad class of nonlinear systems as will be seen in Section 4. In contrast, some nonlinear control techniques are restricted to systems having certain structures, such as cascaded systems; other nonlinear techniques are not systematic and require mini-designs to be carried out on one equation at a time; and yet others cannot be directly applied to unstable, non-minimum phase systems.

Increasingly, because of the capabilities and the systematic nature of the SDRE technique, control practitioners are using the SDRE method in a variety of real-world applications, in spite of the fact that the stability of the design typically has to be verified via simulation. During the conference presentation, examples of the above capabilities will be presented.

4. The Art of SDRE Design

When the system dynamics are affine in the control and $f(x) \in C^1$ with $f(0) = 0$, the system conforms to the basic structure and conditions required for the straightforward application of the SDRE method. In this case, the steps for carrying out an SDRE design are given by Eqs.(5)-(8) and additionally Eqs. (10)-(13) if the SDRE integral servomechanism is employed. The art in the design process for a conforming system consists of the selection of the state-dependent coefficient matrix $A(x, \alpha)$ with α being a vector of free design parameters, and the selection of the state-dependent state and control weighting matrices $Q(x)$ and $R(x)$. $A(x, \alpha)$ must be chosen so that the parameterized pair $\{A(x, \alpha), B(x)\}$ is pointwise stabilizable in the linear sense in order to obtain a legitimate Riccati equation solution. Pointwise stabilizability also guarantees that the SDRE controller is locally asymptotically stable [23]. In addition to satisfying the pointwise stabilizability requirement, a rule of thumb in selecting the state-dependent factorization is to place a non-zero entry in the $\{i, j\}$-element of the $A(x, \alpha)$ matrix if the i^{th} state derivative depends on the j^{th} state. For example, if $\dot{x}_3 = x_1 x_2$, two possible factorizations are $a_{31} = 0$, $a_{32} = x_1$ and $a_{31} = x_2$, $a_{32} = 0$. Neither one of these parameterizations reflect in the $A(x, \alpha)$ matrix the fact that $\dot{x}_3$ depends on both x_1 and x_2. While both of these parameterizations may work, it is expected that better responses can be obtained with the parameterization $a_{31} = \alpha_1 x_2$, $a_{32} = (1 - \alpha_1)x_1$ with α_1 being a free design parameter. Additionally, both of the previous factorizations can be tested by setting $\alpha_1 = 0$ and $\alpha_1 = 1$, respectively. A complete characterization of the possible factorizations of $f(x)$ into $A(x, \alpha)x$ is given in [24].

There are many systems that do not conform to the structure or conditions given in (2)-(4) and the SDRE technique cannot be directly applied. In these cases, there is an art in converting the given system to a system that is conforming so that an effective SDRE design can be performed. Below we present several cases where the system is non-conforming and show in each case how to convert the system to a conforming one.

The Presence of State-Independent Terms

If state-independent terms, which we will refer to as bias terms, are present, then Condition 2 given in (4) is violated, i.e., $f(0) \neq 0$. This prevents a direct C^1 factorization of $f(x)$ into $A(x, \alpha)x$. There are three ways to handle a bias term, denoted as $b(t)$. First, if the bias term is constant or slow-varying, then the bias term can be modeled as a stable state:

$$\dot{b}(t) = -\lambda b(t) \tag{14}$$

where λ is a small positive number. Each time through the controller, the actual value of $b(t)$ is used in calculating the control (Eq. (8)). Second, the bias term can be multiplied and divided by a state or a combination of states that it is known will not go to zero [25]. For example, in a stirred tank problem [11], if the temperature T is a state and is expressed in degrees Kelvin, then T will not go to zero, and if a bias term exists, it can be factored as

$$b(t) = \left[\frac{b(t)}{T}\right] T \tag{15}$$

In a missile control problem, any component of the velocity vector $\vec{v}$ can go to zero, but the speed of the missile will not go to zero. In this case we can multiply and divide the bias term, which may be gravity, by the magnitude squared of the velocity vector. The bias term can then be factored as

$$b(t) = \left[\frac{b(t)\vec{v}^T}{\vec{v}^T\vec{v}}\right] \vec{v} \tag{16}$$

A third alternative is to augment the system with a stable state z [5]:

$$\dot{z}(t) = -\lambda z(t) \tag{17}$$

with $\lambda > 0$. The bias term can then be factored as

$$b(t) = \left[\frac{b(t)}{z}\right] z \tag{18}$$

Each time through the controller, the initial value $z(0)$ is used in the state-dependent coefficient matrix and in calculating the control.

The Presence of State-Dependent Terms which Exclude the Origin

State-dependent terms which do not go to zero as the state goes to zero violate Condition 2 which requires that $f(0) = 0$. Like biases, these terms prevent a direct C^1 factorization of $f(x)$ into $A(x,\alpha)x$ and can be handled using the second or third way to handle biases above. However, it is desired to capture their state dependency in the proper element of the matrix $A(x,\alpha)$. For example, suppose that $\dot{x}_2 = \cos(x_1)$. It is desirable to have a non-zero entry in the (2,1)-element of the $A(x,\alpha)$ matrix that reflects the fact that $\dot{x}_2$ depends on x_1. To accomplish this, we shift the term so that it goes through the origin. This is done by adding and substracting a bias to the term. For the $\cos(x_1)$, we add and substract one:

$$\cos x_1 = [cos(x_1) - 1] + 1 \tag{19}$$

The function $\cos x_1 - 1$ goes through the origin and can then be factored as

$$\cos x_1 - 1 = \left[\frac{\cos x_1 - 1}{x_1}\right] x_1 \tag{20}$$

The bias term which was created in Eq. (19), which in this case is 1, can then be accounted for using one of the bias handling techniques above. This shifting procedure can be used for any state-dependent term which doesn't go through the origin. It is also desirable to shift state-dependent factors which exclude the origin even though they are embedded in a term which goes to zero as the state goes to zero. For example, consider $\dot{x}_2 = e^{x_1}x_3$. Obviously, this term goes to zero as x_3 goes to zero and can be factored as $a_{21} = 0$, $a_{23} = e^{x_1}$. But this factorization doesn't reflect that $\dot{x}_2$ depends on x_1 within the pointwise LQR structure since $a_{21} = 0$ and during execution of the controller, a_{23} will just be a number in the $A(x,\alpha)$ matrix. By shifting e^{x_1}, we can write

$$\dot{x}_2 = [(e^{x_1} - 1) + 1]x_3 = \left[\frac{e^{x_1} - 1}{x_1}\right] x_1 x_3 + x_3 \tag{21}$$

which allows the system to be parameterized as

$$a_{21} = \alpha_1 \left[\frac{e^{x_1} - 1}{x_1}\right] x_3, \quad a_{23} = (1 - \alpha_1)(e^{x_1} - 1) + 1 \tag{22}$$

which yields the desired non-zero entry in a_{21}.

Nonlinearity in the Controls

A system which is nonlinear in the controls can be represented as

$$\dot{x} = f(x) + g(x,u) \tag{23}$$

Such a system can be brought to the required structure given in Eq. (2) by introducing integral control [5, 26]:

$$\dot{u} = Cu + D\tilde{u} \tag{24}$$

In its simplest form, $C = 0$ and $D = I$. The augmented system then conforms to the required structure, being affine in the pseudo control $\tilde{u}$:

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} f(x) + g(x,u) \\ Cu \end{bmatrix} + \begin{bmatrix} 0 \\ D \end{bmatrix} \tilde{u} \tag{25}$$

If Condition 2 given in (4) is not satisfied in the augmented system, then the techniques given above on handling biases and shifting state-dependent terms to the origin can be employed.

Uncontrollable and Unstable but Bounded State Dynamics

A state of the system having uncontrollable and unstable but bounded dynamics results in the parameterized pair $\{A(x,\alpha), B(x)\}$ not being pointwise stabilizable, which in turn means that a legitimate Riccati equation solution is not obtainable. This situation can be handled by simply adding a stabilizing term to the dynamics of the unstable state [5, 27]; if x_1 is the unstable state, the term -λx_1 with $\lambda > 0$, is added to the dynamics.

5. Summary

An overview of the capabilities and art of SDRE-based control design was presented. It was demonstrated how numerous systems that do not meet the basic structure and conditions required for the direct application of the SDRE method can be converted to systems having the proper structure and conditions. The converted, conforming systems can then be used to produce effective SDRE designs for controlling the original plants.

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