

LFT Formulation for Multivariate Polynomial Problems

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Abstract

Robust control system analysis and design is based on an uncertainty description, called a linear fractional transformation (LFT), which separates the uncertain (or varying) part of the system from the nominal system. Low-order LFT models are difficult to form for nonlinear parameter-dependent systems. This paper presents a numerical computational method that can be used to construct low-order LFT models for multivariate polynomial and rational problems based on simple matrix computations. This LFT modeling method makes current robust and linear parameter varying (LPV) control analysis and design methods accessible to a broad class of difficult practical problems.

1.0 Introduction

Formulation of linear fractional transformation (LFT) models of systems involving nonlinear parameter variations in the state-space model is of interest for robust control system analysis and design, as well as for gain-scheduled control of linear parameter varying (LPV) systems. The dimension of the LFT models should be as small as possible (i.e., low order) for efficient computation during control system analysis and design.

A matrix singular value decomposition (svd) approach was presented in 1985 in references [1] and [2] for computing LFT's for problems involving linear parameter variations. Construction of low-order LFT models for nonlinear parameter-dependent problems is very difficult, because it is equivalent to a multidimensional minimal state-space realization problem for which there is no general theory. The approach that has been taken in practice for solving nonlinear parameter-dependent problems is to successively decompose the system using symbolic methods until all components are linear, and then to compute an LFT for each linear component based on the results presented in [1] and [2]. The LFT's associated with each system component are then combined using LFT properties to form the LFT model of the full system. One-dimensional (1-D) state-space model reduction (i.e., removing uncontrollable and/or unobservable modes) is usually required using this approach, because unnecessary

repetitions of the varying parameters can result. A decomposition method for LFT modeling of nonlinear parameter-dependent systems was first presented in reference [3], and later refined in reference [4]. This latter paper presented a special decomposition approach which reduces the number of unnecessary repetitions of the varying parameters, although 1-D model reduction is still employed to reduce the dimension of the resulting LFT model of the full system.

The approach presented in this paper is an extension of the computational approach of references [1] and [2] for nonlinear parameter-dependent systems, and is based on reference [5]. The computational approach is developed for multivariate matrix polynomial problems, although multivariate rational problems can be solved using this approach by reformulating the rational problem into a multivariate polynomial form. References [5] and [6] present a method for doing this. The LFT modeling approach presented in this paper requires no symbolic matrix decompositions for multivariate polynomial problems, and achieves a low-order LFT model directly - i.e., without the use of 1-D model reduction. Moreover, the computations are based on simple numerical matrix operations, including singular value decomposition (svd) and solving generalized linear matrix equations.

2.0 LFT Modeling Problem Definition

The LFT modeling problem to be addressed in this paper is defined below

Given: A linear parameter varying (LPV) model of a nonlinear parameter-dependent system, as represented by the following equation

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\delta) & \mathbf{B}(\delta) \\ \mathbf{C}(\delta) & \mathbf{D}(\delta) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{S}(\delta) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \\ = (\mathbf{S}_0 + \mathbf{S}_\Delta(\delta)) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \quad (2.1a)$$

$$\delta = [\delta_1, \delta_2, \dots, \delta_m] \in \mathbf{R}^m \quad (2.1b)$$

where $S(\delta)$ has been separated into nominal and varying components, and each element of $S_{\Delta}(\delta)$ is a multivariate polynomial function of the varying parameters, δ ;

Find: A low-order state-space uncertainty model as depicted below in Figure 1:

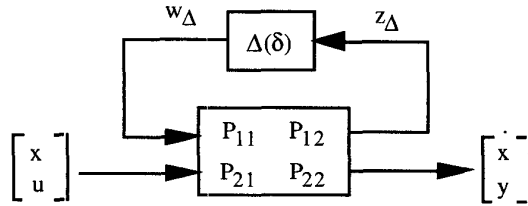


Figure 1. LFT Model of the Uncertain System

in which the P_{22} matrix represents the nominal part of the system, and the $S_{\Delta}(\delta)$ matrix is decomposed as follows:

$$\begin{aligned} S_{\Delta}(\delta) &= P_{21} \Delta (I - P_{11} \Delta)^{-1} P_{12} \\ &= P_{21} (I - \Delta P_{11})^{-1} \Delta P_{12} \end{aligned} \quad (2.2)$$

The $\Delta(\delta)$ matrix associated with Figure 1 is defined below.

$$\Delta(\delta) = \text{diag} [\delta_1 I_{n_1}, \delta_2 I_{n_2}, \dots, \delta_m I_{n_m}] \in \mathbb{R}^{n_{\Delta} \times n_{\Delta}} \quad (2.3)$$

$$n_{\Delta} = \sum_{i=1}^m n_i, \quad n_i = \dim(I_i) \quad (2.4)$$

The LFT modeling problem consists of solving equation (2.2) for P_{21} , P_{12} , and P_{11} such that the dimension of the Δ matrix is as small as possible.

3.0 Main Results: LFT Model Computation

A process for constructing an LFT model for the problem defined in Section 2 is outlined in the following sub-sections.

3.1 Numerical LFT Solution Approach

As can be seen in equation (2.2), solving for the matrices P_{21} , P_{12} , P_{11} and $\Delta(\delta)$ involves the inversion of the matrix $[I - \Delta(\delta)P_{11}]$. For multivariate polynomial problems, this matrix inversion can be exactly replaced by a finite series and an associated nilpotency condition. This is expressed in the following equations.

$$(I - \Delta(\delta)P_{11})^{-1} = I + (\Delta(\delta)P_{11}) + (\Delta(\delta)P_{11})^2 + \dots + (\Delta(\delta)P_{11})^r \quad (3.1)$$

$$[\Delta(\delta)P_{11}]^{r+1} = 0 \quad (3.2)$$

Substituting equation (3.1) into equation (2.2) results in the following equation for $S_{\Delta}(\delta)$.

$$\begin{aligned} S_{\Delta}(\delta) &= P_{21} \Delta(\delta) P_{12} + P_{21} [\Delta P_{11} + (\Delta P_{11})^2 \\ &\quad + \dots + (\Delta P_{11})^r] \Delta(\delta) P_{12} \end{aligned} \quad (3.3)$$

The first term on the right side of equation (3.3) represents the linear uncertain components of $S_{\Delta}(\delta)$, and the remaining terms are nonlinear. For the case of multivariate polynomial uncertainties, the nonlinear terms of $S_{\Delta}(\delta)$ consist of crossterms of the δ parameters as well as n^{th} -order δ_i terms (i.e., δ_i^n). Thus, the order (r) of the highest term in the series of equation (3.3) is determined by the degree of the highest term appearing in $S_{\Delta}(\delta)$, where crossterm degree can be defined as follows.

$$\begin{aligned} \text{degree}(\delta_1^{\xi_1} \delta_2^{\xi_2} \delta_3^{\xi_3} \dots \delta_i^{\xi_i}) \\ = (\xi_1 + \xi_2 + \dots + \xi_i) - 1; \quad i \leq m \end{aligned} \quad (3.4)$$

Then, the exponent r in equation (3.3) can be defined by the following inequality.

$$r \leq (\eta_1 + \eta_2 + \dots + \eta_m) - 1 \quad (3.5)$$

where η_i is the maximum degree of δ_i in $S_{\Delta}(\delta)$.

Since the uncertain system matrix, $S_{\Delta}(\delta)$, has as its elements multivariate polynomial functions of δ , it can be easily expanded in a similar manner as the right side of equation (3.3), i.e.:

$$S_{\Delta}(\delta) = S_{\Delta_0}(\delta) + S_{\Delta_1}(\delta) + \dots + S_{\Delta_r}(\delta) \quad (3.6)$$

Then like terms from equations (3.3) and (3.6) can be equated as follows.

$$S_{\Delta_i}(\delta) = P_{21} (\Delta(\delta) P_{11})^i \Delta(\delta) P_{12}, \quad i = 0, 1, \dots, r \quad (3.7)$$

The uncertainty modeling problem therefore requires that equations (3.7) be solved for P_{21} , P_{12} , P_{11} , and $\Delta(\delta)$ such that the nilpotency condition of equation (3.2) is satisfied.

In order to evaluate equations (3.7) and (3.2) in more detail, an expanded definition of P_{11} , P_{12} , and P_{21} containing partitioned submatrices associated with the $\delta_i I_{n_i}$ blocks of the Δ matrix given in equation (2.3) are considered.

$$P_{11} = [P_{11\delta_i\delta_j}] ; \quad i, j = 1, 2, \dots, m \quad (3.8)$$

$$P_{12} = [P_{11\delta_i}] ; \quad i = 1, 2, \dots, m \quad (3.9)$$

$$P_{21} = [P_{11\delta_j}]^T ; \quad j = 1, 2, \dots, m \quad (3.10)$$

where: $P_{11\delta_i\delta_j} \in R^{n_i \times n_j}$,
 $P_{12\delta_i} \in R^{n_i \times n_{\text{cols}}}$, $P_{21\delta_i} \in R^{n_{\text{rows}} \times n_i}$ (3.11)

P_{11} is a partitioned square matrix, P_{12} is a block-column matrix, and P_{21} is a block-row matrix. Substituting equations (3.8) - (3.10) and (2.3) into equations (3.7) and (3.2) leads to a set of extremely complicated matrix equations to solve. In order to satisfy the nilpotency condition of equation (3.2), the matrix P_{11} must itself be nilpotent (to satisfy the case when $\Delta = I$). Allowing P_{11} to have a pre-defined nilpotent structure provides a means of somewhat simplifying these equations while assisting in satisfying the nilpotency condition of equation (3.2). It is shown in Reference [10] that block triangular matrices with nilpotent main-diagonal blocks are nilpotent. The block-triangular structure is sufficient but not necessary for nilpotency, and other special structures can also be found. (In fact, nilpotent matrices can be fully populated with nonzero elements.) For implementation, allowing the special structure to be more general than upper-block-triangular is desirable for some problems. However, for the purposes of this section the structure of P_{11} will be assumed to be upper block-triangular so that the solution can be clearly derived. Thus, let P_{11} be defined to have an upper block-triangular structure, as follows.

$$P_{11} = [P_{11\delta_i\delta_j}]; \quad i, j = 1, 2, \dots, m; \quad (3.12a)$$

$$P_{11\delta_i\delta_j} = 0 \quad \text{when } j < i \quad (3.12b)$$

where each main-diagonal block is nilpotent of index η_i :

$$(P_{11\delta_i\delta_i})^{\eta_i} = 0, \quad \eta_i \leq n_i, \quad i = 1, 2, \dots, m \quad (3.13)$$

Then substitution of equations (3.9), (3.10) and (3.12) into equations (3.7) yields the following set of equations.

Linear δ_i Terms:

$$P_{21\delta_i} P_{12\delta_i} = S_{\Delta 0 \delta_i}, \quad i = 1, 2, \dots, m \quad (3.14)$$

ξ^{th} -Degree δ_i Terms:

$$P_{21\delta_i} (P_{11\delta_i\delta_i})^{\xi-1} P_{12\delta_i} = S_{\Delta \xi-1(\delta_i)} \xi, \\ i = 1, 2, \dots, m; \quad \xi = 1, 2, \dots, \eta_i \quad (3.15)$$

Crossterms:

$$P_{21\delta_{i_1}} (P_{11\delta_{i_1}\delta_{i_1}})^{\xi_{i_1}-1} P_{11\delta_{i_1}\delta_{i_2}} (P_{11\delta_{i_2}\delta_{i_2}})^{\xi_{i_2}-1} \\ \dots P_{11\delta_{i_{n_T-1}}\delta_{i_{n_T}}} (P_{11\delta_{i_{n_T}}\delta_{i_{n_T}}})^{\xi_{i_{n_T}}-1} P_{12\delta_{i_{n_T}}} \\ = S_{\Delta \xi-1(\delta_{i_1})}^{\xi_{i_1}} (\delta_{i_2})^{\xi_{i_2}} \dots (\delta_{i_{n_T}})^{\xi_{i_{n_T}}} \quad (3.16)$$

where: $\xi = \xi_{i_1} + \xi_{i_2} + \dots + \xi_{i_{n_T}}$
 $i_1 = 1, 2, \dots, m - (n_T - 1)$
 $i_2 = i_1 + 1, i_1 + 2, \dots, m - (n_T - 2)$
 \vdots
 $i_{n_T} = i_1 + (n_T - 1), \dots, m$

n_T = number of parameters in the crossterm $\leq m$

Note that the S_{Δ} terms on the right-hand side of equations (3.14) - (3.16) are the known constant matrix coefficients associated with the indicated parameter terms in $S_{\Delta}(\delta)$. Moreover, depending on the number of parameters and the degree of each appearing in $S_{\Delta}(\delta)$, there can be literally hundreds of S_{Δ} coefficient terms and coupled matrix equations to be solved.

3.2 Numerical LFT Model Solution

This section presents a numerical approach for solving all equations of the form defined by equations (3.14) - (3.16) such that the nilpotency condition of equation (3.2) is satisfied and the resulting P- Δ model is of low-order.

3.2.1 Simultaneous Solution of P_{21} , P_{12} , and the Main-Diagonal Blocks of P_{11} for each δ_i Parameter

The blocks of P_{21} and P_{12} , and the main-diagonal blocks of P_{11} are solved simultaneously for each uncertain parameter δ_i using the linear and ξ^{th} -degree δ_i terms defined by equations (3.14) and (3.15). The solution is accomplished such that the resulting main-diagonal blocks of P_{11} are nilpotent with the appropriate index of nilpotency, as required by equation (3.13). This solution is accomplished numerically with a matrix svd by recognizing that this part of the problem is equivalent to a 1-D state-space (minimal) realization problem and by appropriately defining the equivalent block Hankel matrices. The solution is accomplished for each δ_i parameter as shown by

the following theorem (based on Theorem 6-4, pages 268 - 272, of reference [7]).

Theorem 3.1

Consider the linear and ζ^{th} -degree δ_i terms of $S_{\Delta}(\delta)$, which can be expanded as follows

$$S_{\Delta_{L,\zeta}}(\delta) = [S_{\Delta_{0\delta_i}}] \delta_i + [S_{\Delta_{1\delta_i^2}}] \delta_i^2 + \dots + [S_{\Delta_{\eta_i-1\delta_i^{\eta_i}}}] \delta_i^{\eta_i} \quad (3.17a)$$

$$\Rightarrow S_{\Delta_{L,\zeta}} = \sum_{n=1}^{\eta_i} [S_{\Delta_{n-1\delta_i^n}}] \delta_i^n \quad (3.17b)$$

and use the constant coefficient matrices of equation (3.17) to form the block Hankel matrices defined below

$$\bar{S}_{\Delta_{0\delta_i}} = \text{Hankel} [S_{\Delta_{0\delta_i}} \quad S_{\Delta_{1\delta_i^2}} \quad \dots S_{\Delta_{\eta_i-1\delta_i^{\eta_i}}}] \quad (3.18)$$

$$\bar{S}_{\Delta_{1\delta_i}} = \text{Hankel} [S_{\Delta_{1\delta_i^2}} \quad \dots S_{\Delta_{\eta_i-1\delta_i^{\eta_i}}} \quad 0] \quad (3.19)$$

where:

$$\text{Hankel} [H_1 \quad H_2 \quad \dots H_n] = \begin{bmatrix} H_1 & H_2 & \dots & H_n \\ H_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ H_n & 0 & \dots & 0 \end{bmatrix} \quad (3.20)$$

Equation (3.19) can be constructed from (3.18) by shifting each block row up and filling in the bottom block row with zero blocks. Define the matrix svd of equation (3.18) as follows

$$\begin{aligned} \bar{S}_{\Delta_{0\delta_i}} &= U_{\delta_i} \Sigma_{\delta_i} V_{\delta_i}^T \\ &= (U_{\delta_i} \Sigma_{\delta_i}^{1/2})(\Sigma_{\delta_i}^{1/2} V_{\delta_i}^T) = \bar{P}_{21\delta_i} \bar{P}_{12\delta_i} \end{aligned} \quad (3.21)$$

$$\text{where: } \text{rank}(\bar{S}_{\Delta_{0\delta_i}}) = \text{rank}(\bar{P}_{21\delta_i}) = \text{rank}(\bar{P}_{12\delta_i})$$

Then, $\{P_{11\delta_i\delta_i}, P_{12\delta_i}\}$ is controllable and $\{P_{21\delta_i}, P_{11\delta_i\delta_i}\}$ is observable, and the matrices $P_{21\delta_i}$, $P_{12\delta_i}$, and $P_{11\delta_i\delta_i}$

form an irreducible realization of $S_{\Delta_{L,\zeta}}(\delta)$ as defined by equation (3.17), where:

$$P_{21\delta_i} = \begin{bmatrix} I_{n_{\text{rows}}} & 0 \end{bmatrix} \bar{P}_{21\delta_i} \quad (3.22)$$

$$P_{12\delta_i} = \bar{P}_{12\delta_i} \begin{bmatrix} I_{n_{\text{cols}}} \\ 0 \end{bmatrix} \quad (3.23)$$

$$P_{11\delta_i\delta_i} = (\bar{P}_{21\delta_i})^\dagger \bar{S}_{\Delta_{1\delta_i}} (\bar{P}_{12\delta_i})^\dagger \quad (3.24)$$

and the notation $(A)^\dagger$ designates the pseudoinverse of matrix A . Moreover, the $P_{11\delta_i\delta_i}$ matrix is nilpotent with index η_i . \mathbb{I}

Proof: See Reference [10]

3.2.2 Solution of P_{11} Off-Diagonal Blocks

The P_{11} off-diagonal blocks are each solved using the appropriate crossterms of $S_{\Delta}(\delta)$, as defined by equation (3.16). The number of off-diagonal blocks to be solved is given by the following equation.

$$n_{\text{ODB}} = \sum_{i=1}^{m-1} (m-i) \quad (3.25)$$

The equation to be solved for each off-diagonal block of P_{11} is a generalized linear matrix equation. The general equation is given below for computing the off-diagonal block $P_{11\delta_n\delta_j}$, where $n = 1, 2, \dots, m-1$ and $j = n+1, n+2, \dots, m$.

$$(\bar{P}_{21\delta_n}^{[n]} \bar{P}_{11\delta_n}^{[n]}) P_{11\delta_n\delta_j} (\bar{P}_{12\delta_j}) = \bar{S}_{\Delta_{\delta_n}}^{[n]} \quad (3.26)$$

The matrices $\bar{P}_{21\delta_n}^{[n]}$, $\bar{P}_{11\delta_n}^{[n]}$, $\bar{P}_{12\delta_j}$, and $\bar{S}_{\Delta_{\delta_n}}^{[n]}$ in equation (3.26) are comprised of known

matrices as well as matrices that have already been computed at this point in the solution process. Their explicit general definition is quite lengthy and has therefore been omitted here for brevity. However, as an illustration of equations (3.26), the off-diagonal block equations for the case of three parameters ($m = 3$) with maximum degree of 2 for each δ_i parameter ($\eta_1 = \eta_2 = \eta_3 = 2$) are given below.

Off-Diagonal Block Equations for $m = 3$ ($n_1 = n_2 = n_3 = 2$)

$$\begin{bmatrix} P_{21\delta_1} \\ P_{21\delta_1} P_{11\delta_1\delta_1} \end{bmatrix} P_{11\delta_1\delta_j} \begin{bmatrix} P_{12\delta_j} & (P_{11\delta_j\delta_j} P_{12\delta_j}) \end{bmatrix} \\ = \begin{bmatrix} S_{\Delta_1\delta_1\delta_j} & S_{\Delta_2\delta_1\delta_j^2} \\ S_{\Delta_2\delta_1^2\delta_j} & S_{\Delta_3\delta_1^2\delta_j^2} \end{bmatrix}, j = 2, 3 \\ (\bar{P}_{21\delta_2} \quad [2]\bar{P}_{11\delta_2} \quad [2])P_{11\delta_2\delta_j} \begin{bmatrix} P_{12\delta_j} & (P_{11\delta_j\delta_j} P_{12\delta_j}) \end{bmatrix} \\ = \begin{bmatrix} S_{\Delta_1\delta_2\delta_j} & S_{\Delta_2\delta_2\delta_j^2} \\ S_{\Delta_2\delta_2^2\delta_j} & S_{\Delta_3\delta_2^2\delta_j^2} \\ S_{\Delta_2\delta_1\delta_2\delta_j} & S_{\Delta_3\delta_1\delta_2\delta_j^2} \\ S_{\Delta_3\delta_1^2\delta_2\delta_j} & S_{\Delta_4\delta_1^2\delta_2\delta_j^2} \\ S_{\Delta_3\delta_1\delta_2^2\delta_j} & S_{\Delta_4\delta_1\delta_2^2\delta_j^2} \\ S_{\Delta_4\delta_1^2\delta_2^2\delta_j} & S_{\Delta_5\delta_1^2\delta_2^2\delta_j^2} \end{bmatrix}, j = 3$$

where:

$$(\bar{P}_{21\delta_2} \quad [2]\bar{P}_{11\delta_2} \quad [2]) = \begin{bmatrix} P_{21\delta_2} & & & & \\ & P_{21\delta_2} P_{11\delta_2\delta_2} & & & \\ & & P_{21\delta_1} P_{11\delta_1\delta_2} & & \\ & & & P_{21\delta_1} P_{11\delta_1\delta_1} & \\ & & & & P_{21\delta_1} P_{11\delta_1\delta_2} \\ & & & & & P_{21\delta_1} P_{11\delta_1\delta_2} \\ & & & & & & P_{21\delta_1} P_{11\delta_1\delta_2} \\ & & & & & & & P_{21\delta_1} P_{11\delta_1\delta_2} \end{bmatrix}$$

The general equation for computing each P_{11} off-diagonal block, $P_{11\delta_n\delta_j}$, given by equation (3.26) and illustrated above, can be written as a generalized linear matrix equation of the following form:

$$AXB = C \quad (3.27)$$

where A, B, and C are known constant matrices. The following Lemma is stated as an extension of Lemma 2.2 given on page 19 of Reference [8].

Lemma 1

Consider the generalized linear matrix equation given by equation (3.27), where $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{r \times p}$, and $C \in \mathbb{R}^{n \times p}$ are given matrices. Then the following statements are equivalent:

- (1) there exists a solution $X \in \mathbb{R}^{m \times r}$;
- (2) the columns of $C \in \text{Im}[A]$ and the rows of $C \in \text{Im}[B^T]$;
- (3) $\text{rank}[A \ C] = \text{rank}[A]$ and $\text{rank}[B^T \ C^T]^T = \text{rank}[B]$;
- (4) $\text{Ker}(A^T) \subset \text{Ker}(C^T)$ and $\text{Ker}(B) \subset \text{Ker}(C)$.

Furthermore, the solution, if it exists, is unique if and only if A has full column rank and B has full row rank. \square

Equation (3.27) and Lemma 1 can be used in computing a solution for each off-diagonal block of P_{11} , based on equation (3.26). This solution has the following form.

$$X^\downarrow = M \setminus N \quad (3.28)$$

$$\text{where: } M = B^T \otimes A; N = C^\downarrow \quad (3.29)$$

$$\Rightarrow X = [X_1^\downarrow \ X_2^\downarrow \ \dots \ X_r^\downarrow]; \\ X_i^\downarrow \in \mathbb{R}^{m \times 1}; i = 1, 2, \dots, r \quad (3.30)$$

Note: C^\downarrow is the column-form vector of matrix C obtained by stacking the columns of C into one column vector, \setminus represents matrix division, and \otimes is the Kronecker product

Thus, the following theorem can be stated for computing the off-diagonal blocks of P_{11} .

Theorem 3.2

Given a general linear matrix equation of the form given by equation (3.26) for each off-diagonal block of P_{11} , i.e.:

$$(\bar{P}_{21\delta_n} \quad [n]\bar{P}_{11\delta_n} \quad [n])P_{11\delta_n\delta_j} (\bar{P}_{12\delta_j}) = \bar{S}_{\Delta\delta_n}^{[n]}$$

where: $n = 1, 2, \dots, m-1$; $j = n+1, n+2, \dots, m$

then a solution for $P_{11\delta_n\delta_j}$ of the form given by equations (3.27) - (3.30) and which satisfies rank test (3) of Lemma 1 always exists and is irreducible. \square

Proof: See Reference [10]

3.2.3 Full P-Δ Model Solution

Once the $P_{21\delta_i}$, $P_{12\delta_i}$, $P_{11\delta_i\delta_i}$, and $P_{11\delta_i\delta_j}$ matrices for each parameter have been determined, the full solution is assembled using equations (3.8) - (3.11). This is a simple matter of collecting the matrix partitions together into the full P_{21} , P_{12} , and P_{11} matrices. The Δ matrix is also known and given by equation (2.3), where the number of repetitions for each parameter, n_i , was determined in solving the $P_{21\delta_i}$, $P_{12\delta_i}$, $P_{11\delta_i\delta_i}$, and $P_{11\delta_i\delta_j}$ matrices.

4. Example: Multivariate Quadratic Problem

Several LFT modeling problems have been solved using the results of this paper. Due to space limitations, only one example is shown in this section. Consider the following compound inertia matrix problem presented in [4], and first considered in [9].

$$J = \begin{bmatrix} 0 & -2yz & 2y^2 & 4(y^2 - z^2) & -3xy & xz \\ 2yz & 0 & -2xy & -4xy & 3(x^2 - z^2) & yz \\ -2y^2 & 2xy & 0 & 4xz & -3yz & y^2 - x^2 \end{bmatrix}$$

The x , y , and z terms represent displacement parameters from some reference (zero) point for the system. Thus, the parameters x , y , and z are the uncertain parameters, δ , of the system. The model obtained using the above computational solution is shown below and in Reference [10].

$$P_{21} = [P_{21\delta_x} \ P_{21\delta_y} \ P_{21\delta_z}]$$

$$P_{12} = \begin{bmatrix} P_{12\delta_x} \\ P_{12\delta_y} \\ P_{12\delta_z} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11\delta_x\delta_x} & P_{11\delta_x\delta_y} & P_{11\delta_x\delta_z} \\ 0 & P_{11\delta_y\delta_y} & 0 \\ 0 & P_{11\delta_z\delta_y} & P_{11\delta_z\delta_z} \end{bmatrix}$$

$$\Delta = \text{diag} [\delta_x I_5, \delta_y I_7, \delta_z I_6] \quad (n_\Delta = 18)$$

The resulting LFT model had 18 parameters in Δ , with 5 repetitions for δ_x , 7 for δ_y , and 6 for δ_z . The model presented in [4] using a direct decomposition had 28 and 20 total parameters in Δ before and after 1-D model reduction, respectively. The model obtained using the specialized decomposition approach developed in [4] had 19 and 17 parameters in Δ before and after 1-D model reduction, respectively. The result presented in [9] required 27 parameters in Δ using a linear decomposition approach, and 13 parameters in Δ by recognizing that J can be factored into the product of two matrices containing only linear x , y ,

and z terms. Although this yields the lowest-order LFT model, it is specific for this particular matrix structure and can therefore not be generally applied to other problems.

5. Concluding Remarks

A numerical approach was presented in this paper to directly compute low-order LFT models for multivariate polynomial problems. The approach depends only on simple matrix computations, including singular value decomposition (svd) and solving generalized linear matrix equations. The resulting LFT model is low-order, because matrix structure is exploited during the computations in satisfying the rank conditions required for a solution. This LFT modeling method makes current robust and LPV control analysis and design methods accessible to a broad class of difficult practical problems. Future work will include developing a MATLAB implementation of this LFT modeling approach.

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