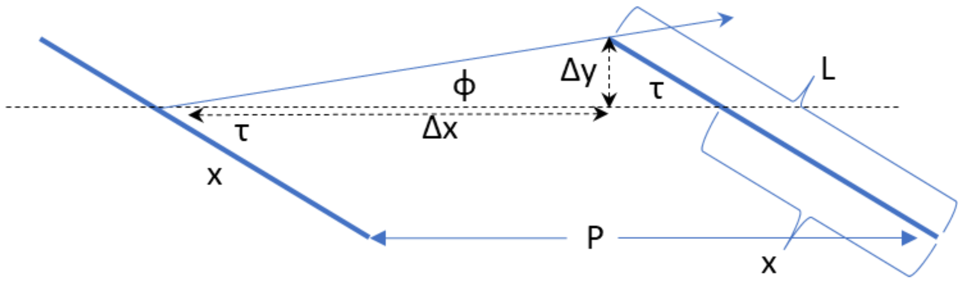
Determining the average view factor from a module to the visible sky



From a point at a distance x from the bottom edge of the slant surface L, the visible sky is the arc clockwise from the ray defined by the slanted surface to the ray defined from the point x to the top of the facing slanged surface. The view factor to the sky is given by

$$\begin{aligned} vf(x) &= \frac{1}{2}\left(\cos\left(180^\circ\right) - \cos\left(180^\circ - \phi - \tau\right)\right) \\ &= \frac{1}{2}\left(1 + \cos\left(\phi + \tau\right)\right) \end{aligned} \tag{1}$$

To compute $\cos\left(\phi+\tau\right)$, we obtain $\cos\phi$ and $\sin\phi$ from

$$\begin{aligned} \Delta y &= \left(L - x\right)\sin\tau = L\left(1 - \frac{x}{L}\right)\sin\tau = L\left(1 - f_x\right)\sin\tau \\ \Delta x &= P - \left(L - x\right)\cos\tau = L\left(\frac{P}{L} - \left(1 - f_x\right)\cos\tau\right) \\ C &= \left(\Delta x\right)^2 + \left(\Delta y\right)^2 = L^2\left[\left(\frac{P}{L} - \left(1 - f_x\right)\cos\tau\right)^2 + \left(\left(1 - f_x\right)\sin\tau\right)^2\right] \\ &= L^2\left[\left(\frac{P}{L}\right)^2 - 2\frac{P}{L}\cos\tau\left(1 - f_x\right) + \left(\left(1 - f_x\right)\right)^2\right] \end{aligned} \tag{2}$$

where $f_x = \dfrac{x}{L}$

Thus

$$\cos\phi = \frac{L\left(\frac{P}{L} - (1 - f_x)\cos\tau\right)}{C^{1/2}}$$
$$\sin\phi = \frac{L(1 - f_x)\sin\tau}{C^{1/2}} \tag{3}$$

Substituting the above and changing variables

$$\begin{aligned}
vf(x) &= \frac{1}{2} + \frac{1}{2}\cos(\phi + \tau) \\
&= \frac{1}{2} + \frac{1}{2}\cos\phi\cos\tau - \frac{1}{2}\sin\phi\sin\tau \\
&= \frac{1}{2} + \frac{1}{2}\cos\tau \frac{L\left(\frac{P}{L} - (1 - f_x)\cos\tau\right)}{\left(L^2\left[\left(\frac{P}{L}\right)^2 - 2\frac{P}{L}\cos\tau(1 - f_x) + \left((1 - f_x)\right)^2\right]\right)^{1/2}} \\
&\quad - \frac{1}{2}\sin\tau \frac{L(1 - f_x)\sin\tau}{\left(L^2\left[\left(\frac{P}{L}\right)^2 - 2\frac{P}{L}\cos\tau(1 - f_x) + \left((1 - f_x)\right)^2\right]\right)^{1/2}} \\
&= \frac{1}{2} + \frac{1}{2}\frac{\frac{P}{L}\cos\tau - (1 - f_x)}{\left[\left(\frac{P}{L}\right)^2 - 2\frac{P}{L}\cos\tau(1 - f_x) + \left((1 - f_x)\right)^2\right]^{1/2}} \\
&= \frac{1}{2} + \frac{1}{2}H(f_x)
\end{aligned} \tag{4}$$

Thus $vf(x)$ can be written in terms of a function $H(f_x)$.

The average of $vf(x)$ on an interval $[x_0, x_1]$ is

$$\overline{VF}(x_0, x_1) = \frac{1}{x_1 - x_0}\int_{x_0}^{x_1} vf(x)\,dx \tag{5}$$

Changing variables by $f_x = \frac{x}{L}$

$$\begin{aligned}
\overline{VF}(x_0,x_1) &= \frac{1}{x_1-x_0}\int_{x_0}^{x_1} vf(x)\,dx \\
&= \frac{1}{x_1-x_0}\int_{\frac{x_0}{L}}^{\frac{x_1}{L}}\left[\frac{1}{2}+\frac{1}{2}H(f_x)\right]Ld(f_x) \\
&= \frac{1}{\frac{x_1-x_0}{L}}\int_{\frac{x_0}{L}}^{\frac{x_1}{L}}\left[\frac{1}{2}+\frac{1}{2}H(f_x)\right]d(f_x) \\
&= \frac{1}{2}+\frac{1}{2}\frac{1}{f_{x1}-f_{x0}}\int_{f_{x0}}^{f_{x1}} H(f_x)\,d(f_x)
\end{aligned} \tag{6}$$

Noting that

$$\frac{d}{df_x}\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}(1-f_x)\cos\tau+(1-f_x)^2\right]=2\left[\frac{P}{L}\cos\tau-(1-f_x)\right] \tag{7}$$

the integral in Eq. 7 is exact:

$$\begin{aligned}
\int_{f_{x0}}^{f_{x1}} H(f_x)\,d(f_x) &= \int_{f_{x0}}^{f_{x1}} \frac{\frac{P}{L}\cos\tau-(1-f_x)}{\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}(1-f_x)\cos\tau+(1-f_x)^2\right]^{1/2}}\,d(f_x) \\
&= \int_{f_{x0}}^{f_{x1}} \frac{\frac{1}{2}du}{u^{1/2}} \\
&= \left[\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}(1-f_x)\cos\tau+(1-f_x)^2\right]^{1/2}\right]_{f_{x0}}^{f_{x1}}
\end{aligned} \tag{8}$$

Thus

$$\overline{VF}(x_0,x_1)=\frac{1}{2}+\frac{1}{2}\frac{1}{f_{x1}-f_{x0}}\left[\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}(1-f_x)\cos\tau+(1-f_x)^2\right]^{1/2}\right]_{f_{x0}}^{f_{x1}} \tag{9}$$

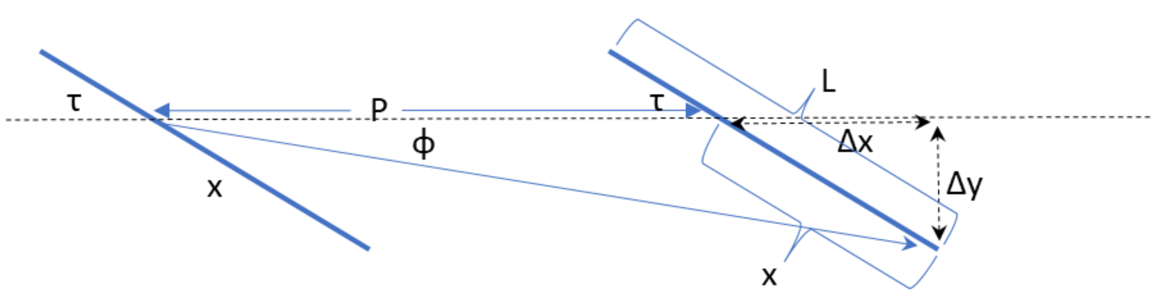
As x_0 approaches x_1,

$$
\begin{aligned}
\lim_{x_0\to x_1}\overline{VF}\left(x_0,x_1\right) &= \frac{1}{2}+\frac{1}{2}\lim_{f_{x_0}\to f_{x1}}\frac{1}{f_{x_1}-f_{x0}}\left[\begin{array}{l}\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}\cos\tau\left(1-f_{x1}\right)+\left(1-f_{x1}\right)^2\right]^{1/2}\\ -\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}\cos\tau\left(1-f_{x0}\right)+\left(1-f_{x0}\right)^2\right]^{1/2}\end{array}\right]\\
&= \frac{1}{2}+\frac{1}{2}\lim_{f_{x_0}\to f_{x1}}\frac{-\frac{1}{2}\dfrac{-2\frac{P}{L}\cos\tau\left(-1\right)+2\left(1-f_{x0}\right)\left(-1\right)}{\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}\cos\tau\left(1-f_{x0}\right)+\left(1-f_{x0}\right)^2\right]^{1/2}}}{-1}\\
&= \frac{1}{2}+\frac{1}{2}\frac{\frac{P}{L}\cos\tau-\left(1-f_{x1}\right)}{\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}\cos\tau\left(1-f_{x1}\right)+\left(1-f_{x1}\right)^2\right]^{1/2}}\\
&= vf\left(x_1\right)
\end{aligned}
\qquad (10)
$$

In particular,

$$
\begin{aligned}
\overline{VF}\left(L,L\right) &= \frac{1}{2}+\frac{1}{2}\frac{\frac{P}{L}\cos\tau-\left(1-1\right)}{\left[\left(\frac{P}{L}\right)^2-2\frac{P}{L}\cos\tau\left(1-1\right)+\left(1-1\right)^2\right]^{1/2}}\\
&= \frac{1}{2}+\frac{1}{2}\cos\tau\\
&= vf\left(L\right)
\end{aligned}
\qquad (11)
$$

Determining the average view factor from a module to the visible ground



From a point at a distance x from the bottom edge of the slant surface L, the visible ground is the arc counterclockwise from the ray defined by the slanted surface to the ray defined from the point x to the bottom of the facing slanged surface. The view factor to the ground is given by

$$\begin{aligned} vf(x) &= \frac{1}{2}\left(\cos\left(0^\circ\right) - \cos\left(\tau - \phi\right)\right) \\ &= \frac{1}{2}\left(1 - \cos\left(\tau - \phi\right)\right) \end{aligned} \tag{12}$$

To compute $\cos\left(\phi - \tau\right)$, we obtain $\cos\phi$ and $\sin\phi$ from

$$\begin{aligned} &\Delta y = x\sin\tau = L\left(\frac{x}{L}\right)\sin\tau = Lf_x\sin\tau \\ &P + \Delta x = P + x\cos\tau = L\left(\frac{P}{L} + f_x\cos\tau\right) \\ &C = \left(P + \Delta x\right)^2 + \left(\Delta y\right)^2 = L^2\left[\left(\frac{P}{L} + f_x\cos\tau\right)^2 + \left(f_x\sin\tau\right)^2\right] \end{aligned} \tag{13}$$

where $f_x = \dfrac{x}{L}$

Thus

$$\begin{aligned} \cos\phi &= \frac{L\left(\frac{P}{L} + f_x\cos\tau\right)}{C^{1/2}} \\ \sin\phi &= \frac{Lf_x\sin\tau}{C^{1/2}} \end{aligned} \tag{14}$$

Substituting the above and changing variables as above (Eq. 4) obtains

$$\begin{aligned} vf(x) &= \frac{1}{2} - \frac{1}{2}\cos\left(\tau - \phi\right) \\ &= \frac{1}{2} - \frac{1}{2}\left(\cos\phi\cos\tau + \sin\phi\sin\tau\right) \\ &= \frac{1}{2} - \frac{1}{2}\frac{\frac{P}{L}\cos\tau + f_x}{\left[\left(\frac{P}{L}\right)^2 + 2\frac{P}{L}f_x\cos\tau + \left(f_x\right)^2\right]^{1/2}} \end{aligned} \tag{15}$$

Integrating on an interval $\left[x_0, x_1\right]$, like Eq. 6 and 8, obtains

$$\overline{VF}\left(x_0, x_1\right) = \frac{1}{2} - \frac{1}{2}\frac{1}{f_{x1} - f_{x0}}\left[\left[\left(\frac{P}{L}\right)^2 + 2\frac{P}{L}f_x\cos\tau + \left(f_x\right)^2\right]^{1/2}\right]_{f_{x0}}^{f_{x1}} \tag{16}$$