Derivation of the Order of Scattering Model with Reflective Surface

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1. Model Description

I make several important approximations and assumptions in developing this model:

1. The basis for this model is the assumption that the contributions to the radiation field from the second and higher scatterings can be neglected compared to contributions from the zeroth and first scatterings. I include only the first-order scatter in the model.
2. I neglect thermal radiation, including only solar and scattered solar radiation.
3. I assume plane-parallel geometry.
4. I assume the scattering properties of the atmosphere are constant with altitude (or at least that the scattering properties can be represented by a vertical average).

With the above assumptions, the basic radiative transfer (RT) equation is:

$$\mu \frac{dI(\tau, \hat{\Omega})}{d\tau} = I(\tau, \hat{\Omega}) - S(\tau, \hat{\Omega}) \tag{1}$$

where μ is the cosine of the angle between the observation line of sight (LOS) and the normal to the plane, $I(\tau, \hat{\Omega})$ is the intensity headed into direction $\hat{\Omega}$, τ is the nadir extinction (absorption + scattering) optical depth and is 0 at $z = \infty$ and τ_{tot} at $z = 0$. $S(\tau, \hat{\Omega})$ represents any sources (scattering, emission, etc., although I ignore emission here). μ is assumed positive, and I included the appropriate signs in the equations as necessary below.

The approximation employed here is the Order of Scattering (OOS) Approximation. This approximation is discussed in detail in Thomas & Stamnes (2002), ch. 6-7 (although I use an alternative notation here). Essentially, I assume the intensity is a sum of beams that have been scattered a different number of times, i.e.

$$I = I^{(0)} + I^{(1)} + I^{(2)} + ... \tag{2}$$

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where $I^{(0)}$ represents beams that experience NO atmospheric scattering, $I^{(1)}$ represents beams that have ONLY been scattered by the atmosphere once, etc. In theory, this approximation can be arbitrarily accurate (see Section 7.2.1. from Thomas & Stamnes (2002)), but for this derivation, I will only include up to the first order scattering, i.e.

$$I \approx I^{(0)} + I^{(1)} \tag{3}$$

NOTE: Beams can be REFLECTED by the surface any number of times and will STILL be counted as zeroth-order. That is, a beam reflected once by the surface is NOT counted as part of $I^{(1)}$.

With this approximation, Equation 1 becomes:

$$\mu \frac{d(I^{(0)} + I^{(1)})}{d\tau} = (I^{(0)} + I^{(1)}) - \frac{\omega}{4\pi} \int_{\Omega'} p(\hat{\Omega}', \hat{\Omega}) \left(I^{(0)}(\hat{\Omega}') + I^{(1)}(\hat{\Omega}') \right) d\Omega' \tag{4}$$

where the second term represents the beams scattered from any direction $\hat{\Omega}'$ into the observational line of sight $\hat{\Omega}$. The scattering phase function $p(\hat{\Omega}', \hat{\Omega})$ is proportional to the probability for a beam of light from $\hat{\Omega}'$ to be scattered into $\hat{\Omega}$ and is normalized so that $\int_{\Omega} \int_{\Omega'} p(\hat{\Omega}', \hat{\Omega}) \frac{d\Omega'}{4\pi} \frac{d\Omega}{4\pi} = 1$.

ω is the single scattering albedo for an atmospheric plane. Note that this is NOT necessarily the same as the single scattering albedo for a Titanian haze particle. In fact, ω is given by

$$\omega = \frac{\sigma_{scat} n_{scat}}{\sigma_{scat} n_{scat} + \sigma_{abs} n_{abs}} \tag{5}$$

where σ_i is the scattering/absorption cross-section for a scattering/absorbing particle (gas or haze) and n_i is the corresponding number density. This ratio is probably not constant with altitude, although I will assume it is constant (or incorporate a vertically averaged version of it) in what follows. Presumably, ω doesn't change too much along observational LOS that are near one another, so whether it's a constant with altitude or not shouldn't matter for comparing the radiative profiles for such LOS.

According to the OOS approximation, the second term in Equation 4 doesn't contribute to $\frac{dI^{(0)}}{d\tau}$, and so, at zeroth order, the equation is:

$$\mu \frac{dI^{(0)}}{d\tau} - I^{(0)} = 0 \tag{6}$$

the solution to which is

$$I^{(0)}_{+/-} = I^{(0)}_{0,+/-} e^{-\tau/\mu} \tag{7}$$

Breaking the solution up into upward ($I^{(0)}_{+}$) and downward ($I^{(0)}_{-}$) components allows me to easily determine the boundary conditions. At $z = \infty$, $\tau = 0$, and the only downward contribution comes from Sun, which is assumed to be collimated along a direction $\hat{\Omega}_\odot$. This consideration gives:

$$I_-^{(0)} = \left(F_\odot \delta(\hat{\Omega}_\odot - \hat{\Omega})\right) e^{-\tau/\mu} \tag{8}$$

where $\delta(\hat{\Omega}_\odot - \hat{\Omega})$ is the Dirac delta function. $F_\odot$ is the solar flux at normal incidence.

The boundary condition for the upward intensity $I_+^{(0)}$ depends on reflection from the surface. By $z = 0$, $\tau = \tau_{tot}$, and $I_-^{(0)}$ has been reduced to $\left(F_\odot \delta(\hat{\Omega}_\odot - \hat{\Omega})\right) e^{-\tau_{tot}/\mu}$. This beam is reflected back up to give the boundary condition.

$$I_+^{(0)}(\tau = \tau_{tot}) = \int_{\Omega'=up} \rho(\hat{\Omega}', \hat{\Omega})\mu' \left(F_\odot \delta(\hat{\Omega}_\odot - \hat{\Omega}')\right) e^{-\tau_{tot}/\mu'} d\Omega' = \mu_\odot F_\odot e^{-\tau_{tot}/\mu_\odot} \rho(\hat{\Omega}_\odot, \hat{\Omega}) \tag{9}$$

$\rho(\hat{\Omega}', \hat{\Omega})$ is the bidirectional reflectance function. It is defined as the ratio of the intensity to the incident energy reflected by the surface from direction $\hat{\Omega}'$ into $\hat{\Omega}$ (see Thomas & Stamnes (2002), p. 134). This definition gives ρ units of inverse steradians. I honestly don't understand why it is defined in a different way from the phase function, but there you go.

Now I can write $I_+^{(0)}$:

$$I_+^{(0)} = \left(\mu_\odot F_\odot \rho(\hat{\Omega}_\odot, \hat{\Omega}) e^{-\tau_{tot}/\mu_\odot}\right) e^{-\frac{\tau_{tot}-\tau}{\mu}} \tag{10}$$

A sanity check: at $z = \infty$, $\tau = 0$, the intensity that has passed into and out of the atmosphere unscattered by the atmosphere is:

$$I_+^{(0)}(\tau = 0) = \left(\mu_\odot F_\odot \rho(\hat{\Omega}_\odot, \hat{\Omega})\right) e^{-(\frac{1}{\mu_\odot} + \frac{1}{\mu})\tau_{tot}} \tag{11}$$

This is (1) the sunlight reduced by one passage through the atmosphere (at an airmass $1/\mu_\odot$), (2) reflected from the ground, and (3) again reduced by passage through the atmosphere (at airmass $1/\mu$).

Now I have to solve for $I_{+/-}^{(1)}$. At this order, only $I_{+/-}^{(0)}$ enters into the integral in Equation 4:

$$-\mu \frac{dI_-^{(1)}}{d\tau} = I_-^{(1)} - \frac{\omega}{4\pi} \int_{\Omega'} p(\hat{\Omega}', \hat{\Omega}) I^{(0)}(\hat{\Omega}') d\Omega' \tag{12}$$

$$\mu \frac{dI_+^{(1)}}{d\tau} = I_+^{(1)} - \frac{\omega}{4\pi} \int_{\Omega'} p(\hat{\Omega}', \hat{\Omega}) I^{(0)}(\hat{\Omega}') d\Omega' \tag{13}$$

First $I_-^{(1)}$. Changing τ to τ', inserting $I^{(0)}$, and multiplying through by the integrating factor $e^{\tau'/\mu}$:

$$\frac{d}{d\tau'}\left(\mu e^{\tau'/\mu} I_-^{(1)}\right) = \frac{\omega}{4\pi}\left(\int_{\Omega'=up} p(\hat{\Omega}',\hat{\Omega}) I_-^{(0)}(\hat{\Omega}')d\Omega' + \int_{\Omega'=down} p(\hat{\Omega}',\hat{\Omega}) I_+^{(0)}(\hat{\Omega}')d\Omega'\right) e^{\tau'/\mu} \tag{14}$$

Plugging in $I^{(0)}$:

$$\begin{aligned}\frac{d}{d\tau'}\left(\mu e^{\tau'/\mu} I_-^{(1)}\right) = \frac{\omega}{4\pi}\Bigg[&p(\hat{\Omega}_\odot,\hat{\Omega}) F_\odot e^{-(\frac{1}{\mu_\odot}-\frac{1}{\mu})\tau'}\\ &+\int_{\Omega'=down}\left(p(\hat{\Omega}',\hat{\Omega})\mu_\odot F_\odot \rho(\hat{\Omega}_\odot,\hat{\Omega}') e^{-(\frac{1}{\mu'}+\frac{1}{\mu_\odot})\tau_{tot}}\right) e^{(\frac{1}{\mu'}+\frac{1}{\mu})\tau'} d\Omega'\Bigg]\end{aligned} \tag{15}$$

Next, I integrate over τ' from 0 to τ. The boundary condition for $I_-^{(1)}$ comes from $\tau' = 0$ (the top of the atmosphere), where there is no downward, scattered radiation. Thus, $I_{-,0}^{(1)} = 0$ and:

$$\begin{aligned}I_-^{(1)} = \frac{\omega}{4\pi}\Bigg[&p(\hat{\Omega}_\odot,\hat{\Omega}) F_\odot \left(\frac{\mu_\odot}{\mu-\mu_\odot}\right)\left(e^{-\tau/\mu} - e^{-\tau/\mu_\odot}\right)\\ &+\int_{\Omega'_-}\left(p(\hat{\Omega}',\hat{\Omega})\mu_\odot F_\odot \rho(\hat{\Omega}_\odot,\hat{\Omega}') e^{-(\frac{1}{\mu'}+\frac{1}{\mu_\odot})\tau_{tot}}\right)\left(\frac{\mu'}{\mu+\mu'}\right)\left(e^{\tau/\mu'} - e^{-\tau/\mu}\right) d\Omega'\Bigg]\end{aligned} \tag{16}$$

which I can rearrange to make a little more intuitive:

$$\begin{aligned}I_-^{(1)} = \frac{\omega}{4\pi}\Bigg[&p(\hat{\Omega}_\odot,\hat{\Omega}) F_\odot \left(\frac{\mu_\odot}{\mu-\mu_\odot}\right)\left(e^{-\tau/\mu} - e^{-\tau/\mu_\odot}\right)\\ &+\int_{\Omega'_-}\left(p(\hat{\Omega}',\hat{\Omega})\mu_\odot F_\odot \rho(\hat{\Omega}_\odot,\hat{\Omega}') e^{-\tau_{tot}/\mu_\odot}\right)\left(\frac{\mu'}{\mu+\mu'}\right)\left(e^{-\frac{(\tau_{tot}-\tau)}{\mu'}} - e^{-\left(\frac{\tau_{tot}}{\mu'}+\frac{\tau}{\mu}\right)}\right) d\Omega'\Bigg]\end{aligned} \tag{17}$$

Another sanity check: The first term in Equation 17 looks like it could become negative, say, if $e^{-\tau/\mu} < e^{-\tau/\mu_\odot}$, but this condition can only be satisfied if $\mu_\odot > \mu$ because then $\tau/\mu > \tau/\mu_\odot$. However, if $\mu_\odot > \mu$, then $\mu - \mu_\odot < 0$ and $\left(\frac{\mu_\odot}{\mu-\mu_\odot}\right)$ flips sign, preventing the first term in Equation 17 from going negative.

Now I can solve for $I_+^{(1)}$ the same way I solved for $I_-^{(1)}$, except my integrating factor is $e^{-\tau'/\mu}$:

$$\begin{aligned}\frac{d}{d\tau'}\left(\mu e^{-\tau'/\mu} I_+^{(1)}\right) &= -\frac{\omega}{4\pi}\left(\int_{\Omega'=up} p(\hat{\Omega}',\hat{\Omega}) I_-^{(0)}(\hat{\Omega}')d\Omega' + \int_{\Omega'=down} p(\hat{\Omega}',\hat{\Omega}) I_+^{(0)}(\hat{\Omega}')d\Omega'\right) e^{-\tau'/\mu}\\ &= -\frac{\omega}{4\pi}\Bigg[p(\hat{\Omega}_\odot,\hat{\Omega}) F_\odot e^{-(\frac{1}{\mu_\odot}+\frac{1}{\mu})\tau'}\\ &\quad+\int_{\Omega'_-}\left(p(\hat{\Omega}',\hat{\Omega})\mu_\odot F_\odot \rho(\hat{\Omega}_\odot,\hat{\Omega}') e^{-(\frac{1}{\mu'}+\frac{1}{\mu_\odot})\tau_{tot}}\right) e^{-\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)\tau'} d\Omega'\Bigg]\end{aligned} \tag{18}$$

I integrate over τ' from τ to τ_{tot} (which is $z = 0$):

$$
\begin{aligned}
&\left(e^{-\tau_{tot}/\mu} I_{+}^{(1)}(\tau_{tot}) - e^{-\tau/\mu} I_{+}^{(1)}(\tau)\right) = \\
&-\frac{\omega}{4\pi}\Bigg[p(\hat{\Omega}_{\odot},\hat{\Omega})F_{\odot}\left(\frac{\mu_{\odot}}{\mu+\mu_{\odot}}\right)\left(e^{-\left(\frac{1}{\mu_{\odot}}+\frac{1}{\mu}\right)\tau} - e^{-\left(\frac{1}{\mu_{\odot}}+\frac{1}{\mu}\right)\tau_{tot}}\right) + \\
&\int_{\Omega'_{-}}\left(p(\hat{\Omega}',\hat{\Omega})\mu_{\odot}F_{\odot}\rho(\hat{\Omega}_{\odot},\hat{\Omega}')e^{-(\frac{1}{\mu'}+\frac{1}{\mu_{\odot}})\tau_{tot}}\right)\left(\frac{\mu'}{\mu'-\mu}\right)\left(e^{-\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)\tau_{tot}} - e^{-\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)\tau}\right)d\Omega'\Bigg]
\end{aligned}
\tag{19}
$$

Then doing a lot of algebra gives:

$$
\begin{aligned}
I_{+}^{(1)}(\tau) =& I_{+}^{(1)}(\tau_{tot})e^{-\frac{\tau_{tot}-\tau}{\mu}} \\
&+\frac{\omega}{4\pi}\Bigg[p(\hat{\Omega}_{\odot},\hat{\Omega})F_{\odot}\left(\frac{\mu_{\odot}}{\mu+\mu_{\odot}}\right)\left(e^{-\tau/\mu_{\odot}} - e^{-\tau_{tot}/\mu_{\odot}}e^{-\frac{\tau_{tot}-\tau}{\mu}}\right) \\
&+\int_{\Omega'_{-}}\left(p(\hat{\Omega}',\hat{\Omega})\mu_{\odot}F_{\odot}\rho(\hat{\Omega}_{\odot},\hat{\Omega}')e^{-\tau_{tot}/\mu_{\odot}}\right)\left(\frac{\mu'}{\mu-\mu'}\right)\left(e^{-\frac{\tau_{tot}-\tau}{\mu}} - e^{-\frac{\tau_{tot}-\tau}{\mu'}}\right)d\Omega'\Bigg]
\end{aligned}
\tag{20}
$$

A sanity check: at $\tau = \tau_{tot}$, the second term on the RHS goes to zero, and the third term (the integral) goes to zero as well, leaving only the first term, the boundary term.

Now to figure out the boundary condition. Heading downward into the surface (τ_{tot}), the beam that has been scattered only once is $I_{-}^{(1)}(\tau_{tot})$. I assume that reflections from the surface do NOT increase the order of scattering of a beam, so the boundary value ($I_{+}^{(1)}(\tau_{tot})$) is just the surface reflection of $I_{-}^{(1)}(\tau_{tot})$:

$$
\begin{aligned}
I_{+}^{(1)}(\tau_{tot}) &= \int_{\Omega'=up}\rho(\hat{\Omega}',\hat{\Omega})\mu' I_{-}^{(1)}(\tau_{tot})d\Omega' \\
&= \frac{\omega}{4\pi}\int_{\Omega'_{+}}\rho(\hat{\Omega}',\hat{\Omega})\mu'\Bigg[p(\hat{\Omega}_{\odot},\hat{\Omega}')F_{\odot}\left(\frac{\mu_{\odot}}{\mu'-\mu_{\odot}}\right)\left(e^{-\tau_{tot}/\mu'} - e^{-\tau_{tot}/\mu_{\odot}}\right) + \\
&\qquad \int_{\Omega''_{-}}p(\hat{\Omega}'',\hat{\Omega}')\mu_{\odot}F_{\odot}\rho(\hat{\Omega}_{\odot},\hat{\Omega}'')e^{-\tau_{tot}/\mu_{\odot}}\left(\frac{\mu''}{\mu'+\mu''}\right)\left(1 - e^{-\left(\frac{1}{\mu''}+\frac{1}{\mu'}\right)\tau_{tot}}\right)d\Omega''\Bigg]d\Omega'
\end{aligned}
\tag{21}
$$

Plugging this back into Equation 20 leads to horror, so I won't do it.

Now the final intensity measured at the top of the atmosphere at the approximation considered here $I^{+}(\tau = 0)$ is the sum of

1. Sunlight that gets to surface, is reflected, and back out, totally unscattered $= I_{+}^{(0)}(0)$
2. Sunlight scattered up by the atmosphere before reaching the surface - the second term in Equation 20
3. Sunlight that gets to surface, reflected, and is scattered up on the way out of the atmosphere - third term in Equation 20
4. Sunlight scattered to the surface, reflected, then back out, with no more scatter - first term in $I_{+}^{(1)}(\tau_{tot})$ in Equation 21
5. Sunlight that gets to surface, reflected, scattered back to surface, reflected, then back out, with no more scatter - second term in $I_{+}^{(1)}(\tau_{tot})$ in Equation 21

And so the solution is

$$I_{+}(\tau = 0) = I_{+}^{(0)}(\tau = 0) + I_{+}^{(1)}(\tau = 0) \tag{22}$$

and

$$I_{+}^{(0)}(\tau = 0) = \left(\mu_{\odot} F_{\odot} \rho(\hat{\Omega}_{\odot}, \hat{\Omega})\right) e^{-(\frac{1}{\mu_{\odot}} + \frac{1}{\mu})\tau_{tot}} \tag{23}$$

$$\begin{aligned} I_{+}^{(1)}(\tau = 0) = & \; I_{+}^{(1)}(\tau_{tot}) e^{-\tau_{tot}/\mu} \\ & + \frac{\omega}{4\pi} \Bigg[p(\hat{\Omega}_{\odot}, \hat{\Omega}) F_{\odot} \left(\frac{\mu_{\odot}}{\mu + \mu_{\odot}} \right) \left(1 - e^{-\left(\frac{1}{\mu_{\odot}} + \frac{1}{\mu}\right)\tau_{tot}} \right) \\ & + \int_{\Omega'_{-}} \left(p(\hat{\Omega}', \hat{\Omega}) \mu_{\odot} F_{\odot} \rho(\hat{\Omega}_{\odot}, \hat{\Omega}') e^{-\tau_{tot}/\mu_{\odot}} \right) \left(\frac{\mu'}{\mu - \mu'} \right) \left(e^{-\tau_{tot}/\mu} - e^{-\tau_{tot}/\mu'} \right) d\Omega' \Bigg] \end{aligned} \tag{24}$$

and $I_{+}^{(1)}(\tau_{tot})$ is given by Equation 21.

During our conversation on 2011 Dec 28, Jason said he wanted to drop the terms related to items 3 and 5 from the list. In this case, the desired solution $I_+(\tau = 0)$ (including the expressions for $I_+^{(1)}(\tau_{tot})$) is:

$$
\begin{aligned}
I_+(\tau = 0) = (1)\ & \mu_\odot F_\odot \rho(\hat{\Omega}_\odot, \hat{\Omega}) e^{-(\frac{1}{\mu_\odot}+\frac{1}{\mu})\tau_{tot}} + \\
& (2) \frac{\omega}{4\pi} p(\hat{\Omega}_\odot, \hat{\Omega}) F_\odot \left(\frac{\mu_\odot}{\mu + \mu_\odot}\right) \left(1 - e^{-\left(\frac{1}{\mu_\odot}+\frac{1}{\mu}\right)\tau_{tot}}\right) + \\
& (4)\ \left[\frac{\omega}{4\pi} \int_{\Omega'_+} \rho(\hat{\Omega}', \hat{\Omega}) \mu' p(\hat{\Omega}_\odot, \hat{\Omega}') F_\odot \left(\frac{\mu_\odot}{\mu' - \mu_\odot}\right) \left(e^{-\tau_{tot}/\mu'} - e^{-\tau_{tot}/\mu_\odot}\right) d\Omega'\right] e^{-\tau_{tot}/\mu}
\end{aligned}
\tag{25}
$$

For convenience, the general expressions are:

$$
I_-^{(0)}(\tau) = \left(F_\odot \delta(\hat{\Omega}_\odot - \hat{\Omega})\right) e^{-\tau/\mu}
\tag{26}
$$

$$
I_+^{(0)}(\tau) = \left(\mu_\odot F_\odot \rho(\hat{\Omega}_\odot, \hat{\Omega}) e^{-\tau_{tot}/\mu_\odot}\right) e^{-\frac{\tau_{tot}-\tau}{\mu}}
\tag{27}
$$

$$
\begin{aligned}
I_-^{(1)}(\tau) = & \frac{\omega}{4\pi} [p(\hat{\Omega}_\odot, \hat{\Omega}) F_\odot \left(\frac{\mu_\odot}{\mu - \mu_\odot}\right) \left(e^{-\tau/\mu} - e^{-\tau/\mu_\odot}\right) \\
& + \int_{\Omega'_-} \left(p(\hat{\Omega}', \hat{\Omega}) \mu_\odot F_\odot \rho(\hat{\Omega}_\odot, \hat{\Omega}') e^{-(\frac{1}{\mu'}+\frac{1}{\mu_\odot})\tau_{tot}}\right) \left(\frac{\mu'}{\mu + \mu'}\right) \left(e^{\tau/\mu'} - e^{-\tau/\mu}\right) d\Omega']
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
I_+^{(1)}(\tau) = & I_+^{(1)}(\tau_{tot}) e^{-\frac{\tau_{tot}-\tau}{\mu}} \\
& + \frac{\omega}{4\pi} [p(\hat{\Omega}_\odot, \hat{\Omega}) F_\odot \left(\frac{\mu_\odot}{\mu + \mu_\odot}\right) \left(e^{-\tau_{tot}/\mu_\odot} e^{-\frac{\tau_{tot}-\tau}{\mu}} - e^{-\tau/\mu_\odot}\right) \\
& + \int_{\Omega'_-} \left(p(\hat{\Omega}', \hat{\Omega}) \mu_\odot F_\odot \rho(\hat{\Omega}_\odot, \hat{\Omega}') e^{-\left(\frac{1}{\mu'}+\frac{1}{\mu_\odot}\right)\tau_{tot}}\right) \left(\frac{\mu'}{\mu - \mu'}\right) \left(e^{-\frac{\tau_{tot}-\tau}{\mu}} - e^{-\frac{\tau_{tot}-\tau}{\mu'}}\right) e^{\tau_{tot}/\mu'} d\Omega']
\end{aligned}
\tag{29}
$$

REFERENCES

Thomas, G. E., & Stamnes, K. 2002, Radiative Transfer in the Atmosphere and Ocean

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