Deriving Heisenberg's operator algebra from a classical model of stochastic mechanics

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A feature of quantum mechanics that distinguishes it from classical Newtonian mechanics is that momentum and position are described by non-commuting operators on a Hilbert space. It has long been known that certain diffusion theories provide stochastic models for Schrödinger's equation, together with a natural way to understand this non-commutative structure. This theory is revisited here. It suggests that the origin of quantum theory might be found in an algebraic extension of the geometry of space-time to complex numbers. It is a possible stepping stone to an emergent explanation for quantum mechanics and to a unification of classical and quantum physics.

I. INTRODUCTION[1]

Heisenberg introduced the concept of a time-dependent and non-commuting algebra for dynamic variables like position and momentum in his landmark 1925 paper [1]. Stochastic mechanics (SM) provides a stochastic interpretation for quantum mechanics [2–4]. It also provides a toolkit of rigorous mathematical

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theorems in stochastic processes for proving theorems about quantum mechanics. This is particularly relevant to the path-integral approach [4–8]. Another interesting reason for studying it is that it can help in the search for an emergent explanation for quantum mechanics, as envisioned by Einstein [9] and many others [10–14]. It may also provide insight into quantum noise and vacuum fluctuations. Although stochastic mechanics has focused much more on the wave function formalism of Schrödinger than on the operator formalism of Heisenberg, there is an operator approach to generalized stochastic mechanics (GSM) that is closely related to the Heisenberg operator formalism in quantum mechanics [15–17]. Whereas SM provides a diffusion model for quantum mechanics with a unique value for the diffusion constant, GSM provides many diffusion models that all have different diffusion constants [17, 18]. Jaekel and Pignon generalized this result even further by allowing the diffusion constant to depend on position and time [19]. Position-dependent diffusion constants are a well-known phenomenon in condensed matter physics ([20], § 3.7), and GSM might find applications there as well. GSM is based on the mathematical framework of SM [2–4, 21], and it depends on a nonlinear transformation of Schrödinger's equation and a modification of Nelson's force equation in stochastic mechanics [16, 18]. The reason why any diffusion constant value can be used in GSM, as explained below, is that the combination of wave-function collapse combined with the Heisenberg uncertainty principle makes it probably impossible to directly measure the diffusion constant for a quantum particle. A review of wave-function collapse is given in [22]. This is a contentious subject as there are quantum interpretations that attempt to remove this collapse from physics such as Many Worlds, Decoherence, Bayesian, and Spontaneous Collapse interpretations. None of these

have yet successfully predicted experiments that were confirmed, and that differ from standard Copenhagen quantum mechanics. Therefore we can take them as effectively equivalent to wave-function collapse as far as the measurability of the diffusion parameter is concerned here. Should this change in the future, and this is certainly a possibility, then it may be possible to measure the diffusion parameter in GSM, and this would be great.

It is fair to say that GSM has been somewhat ignored by mathematicians in favor of a unique value for the diffusion constant. One reason for this is the action principle of Nelson [3, 23] that many have argued supports the claim of uniqueness of stochastic mechanics with the diffusion constant of $\hbar/2m$. But actually, it's possible to generalize the action principle to allow for a variable diffusion constant as is shown below.

Heisenberg introduced the wave-function collapse in 1927 [24]. It was made into a rigourous mathematical postulate by Von Neumann [25]. Dirac discussed the measurement process and the wave function collapse postulate in his 1930 book [26]. The desire to have maximum mathematical rigor in stochastic mechanics made it difficult to incorporate it, so it was mostly ignored in the SM literature. The measurement problem is still a fundamental problem of quantum theory [27]. Petroni and Morato [28] remarked on the deficient lack of attention to this topic in the SM literature. The physical interpretation of SM or GSM can be taken to be similar to Bohmian mechanics, and good sources for this are [29–32]. Both theories are essentially non-local when multi-particle states are considered. Wave function collapse simply doesn't easily fit neatly into rigorous mathematical stochastic analysis. But nature and experiments require it. At the very core of quantum theory is this mysterious collapse, and it suggests

that the diffusion parameter cannot be measured in an experiment. There have been many efforts to demystify it. Many Worlds, Decoherence and Spontaneous Collapse are prime examples, but none of these efforts have achieved unanimous acceptance by physicists.

In this paper the non-commutative approach is reviewed, and it is shown how the usual Heisenberg algebra can be constructed. The analytic continuation to complex values for the diffusion constant can be interpreted as a diffusion in complex space (or space and time in a relativistic framework). This adds to a growing list of reasons that we should take the arena of space-time to be a complex 4D manifold, where the real subspace plays a prominent role at the macroscopic level, but the complex embedding plays a large role when quantum mechanics is necessary.

There are similarities between the techniques used here and the path integral approach of Feynman in his thesis [33]. Feynman derives the commutation rules of quantum mechanics by time-ordering products of position and momentum at infinitesimally different times, and summing over real-valued paths with a complex weighting factor. This is the same technique we use here, but our weighting factor in GSM is positive definite, and it qualifies therefore as a true stochastic process. We find here that the commutator between position and momentum is real-valued for a real diffusion constant. Only by analytically continuing to a complex space-time can we obtain an imaginary commutator between position and momentum, whereas Feynman obtains an imaginary commutator for paths in real space-time. So GSM seems to favor the introduction of complex space-time into physics more strongly than the Feynman path integral method does. It's also obvious that one can generalize the Feynman path integral approach by

using the same nonlinear gauge transformation of the Schrödinger equation used in deriving GSM.

Stochastic mechanics, like Bohmian mechanics, is a non-local theory because of wave-function collapse for entangled multi particle states. Moreover, since the drift term in GSM for multi particle states depends on the coordinates of all the particles, IE. on configuration space, they can be considered to be non-local even without considering collapse. There are good arguments that quantum mechanics is inherently non-local, and this doesn't only apply to hidden variable interpretations [32, 34–36]. Nelson eventually became very worried about this and recommended concentrating on field theory instead of wave function analysis of particle motion [37]. The non-locality problems of stochastic mechanics are no worse than those faced by Bohmian mechanics, and since a fair number of physicists accept Bohmian mechanics as a viable interpretation, they would probably find stochastic mechanics equally acceptable. See, for example, [30, 38]. One possible way of avoiding non-locality is if underlying the stochastic behavior is a hidden fully-deterministic theory, yet still a local one. An example of this is 't Hooft's Planck scale cellular automata theory [39]. Finding the algorithm that nature might use for this is a difficult problem though. The stochastic motion of particles in stochastic mechanics might be caused by such a chaotic deterministic theory at the Planck scale that allegedly eliminates free-will in the analysis of Bell type experiments. This can avoid the non-locality, as 't Hooft has argued. Another possibility is that we are living in a complex space-time manifold that appears real to us because the events that we are aware of are happening on or near to a real subspace [40], but the space-time is overlayed with multiple Riemann sheets of fields. Generalized stochastic mechanics on complex manifolds

provides a continuous path to the Heisenberg algebra by analytic continuation to imaginary diffusion constant. This requires embedding the theory in complex space-time. The Feynman path integral approach also derives the Heisenberg algebra, but it does so in real space-time [33]. So the Feynman path integral picture might be considered superior since it doesn't require a complex space-time to derive the Heisenberg algebra, but on the other hand it's not a true diffusion theory since the path integrals are weighted by a complex measure. For a true diffusion theory such as GSM we must analytically continue it to complex space-time to get the Heisenberg algebra. If this is how nature actually works, then GSM is giving us a clue to it that the Feynman path integral approach does not give.

In this paper, we shall mainly use the Markov transition function, the forward and backward differential equations for it, the Chapman-Kolmogorov equations, and the multiple-time densities of generalized Brownian motion. The mathematics needed is therefore familiar to most physicists, and is sufficient to do calculations. We treat only the 1 dimensional case here for simplicity, but we use the ∇_x and $\triangle_x$ operator symbols to represent the first and second derivatives in 1D. For simplicity of notation, we shall refer to a Wiener process as having an arbitrary diffusion constant ν, and denote it by $W(t,\nu) = \sqrt{2\nu}W(t)$, where $W(t)$ is the standard Wiener process with diffusion constant of $1/2$. When writing out operators that involve derivatives, we shall use the operator notation used in quantum mechanics. For example $\hat{\nabla} f(x) = \nabla f + f\hat{\nabla}$.

II. ON MEASURING OR PLACING AN EXPERIMENTAL BOUND ON THE DIFFUSION CONSTANT OR THE DRIFT TERM IN STOCHASTIC MECHANICS

A widely accepted assertion about quantum mechanics is that all measurements ultimately boil down to positions measurements. It is then argued that a consequence of this is that if we can develop a stochastic model of quantum mechanics that correctly describes probability densities as they evolve in position and time, then this is sufficient to describe nature. For example, in ([41], §16, p. 111) Nelson writes:

"It is a triviality that all measurements are reducible to position measurements, since the outcome of any experiment can be described in terms of the approximate position of macroscopic objects. Let us suppose that we observe the outcome of an experiment by measuring the exact position at a given time of all the particles involved in the experiment, including those constituting the apparatus. This is a more complete observation than is possible in practice, and if the quantum and stochastic theories cannot be distinguished in this way then they cannot be distinguished by an actual experiment. However, for such an ideal experiment stochastic and quantum mechanics give the same probability density $|\psi|^2$ for the position of the particles at the given time."

In a similar vein, Feynman and Hibbs ([42], §5, 96) wrote:

"Indeed, all measurements of quantum-mechanical systems could be made to reduce eventually to position and time measurements (e.g., the position of a needle on a meter or the time of flight of a particle). Because of this possibility a theory formulated in terms of position measurements is complete enough in

principle to describe all phenomenon."

Without this reasonable assertion, the path integral approach would not be equivalent to other interpretations of quantum mechanics, nor would Bohmian mechanics or stochastic mechanics. A formal mathematical investigation of the degrees of freedom in describing quantum position-equivalent nonlinear wave equations was performed by Doebner and Goldin [43]. GSM is possible because it exploits a particular nonlinear gauge transformation of Schrödinger's equation which is included in Doebner and Goldin's more general framework.

Quantum mechanics does not agree with experiments unless the postulate of wave-function collapse is included in some form or another. An ideal experiment is viewed in the Copenhagen interpretation as manufacturing a change in the quantum state succinctly described by a projection operator acting on the quantum Hilbert state vector. This projection imprints the result of a measurement onto the canvas of reality [25]. This is exploited routinely in quantum computers. When wave function collapse is combined with the Heisenberg uncertainty principle, it makes it seemingly impossible to measure the diffusion constant. This section is partially a response to an argument that the diffusion constant cannot be larger than $\hbar/2m$ without abandoning the "usual assumptions" ([3], §14, p. 66). It is not an observable. The reason why is quite simple, but worth presenting here. For the purpose of this argument, I assume that the collapse as described by Von Neumann occurs. If it doesn't then the following argument is not correct, and there may be a way to measure the diffusion constant. Consider a non-relativistic particle without spin. Let us consider a real and positive diffusion constant in GSM. It's important to note that we are talking about GSM here, and not the general diffusion situation. GSM is very special in that the

Schrödinger equation is the same for all values of the diffusion constant. The diffusion constant (in 1D for simplicity) is defined by the stochastic expectation

$$\nu \;=\; \frac{1}{2}\lim_{\epsilon\to 0}\left(\frac{(E(x(t+\epsilon)-x(t))^2)}{\epsilon}\right) \tag{1}$$

The expectation value here is the stochastic process expectation, and it's real-valued (for real ν), unlike a quantum expectation which would be complex valued because of the non-commuting property of position operators at different times. For each sample path in the expectation $x(t)$ is a continuous real-valued function of t (almost surely). Now let's ask how we can measure this. We shall accept with Nelson and Feynman that we must make position measurements. We must measure the same particle in quick succession, first at time t and then later at time $t+\epsilon$. We take that the particle is described by a wave function $\psi(x,t)$ undergoing unitary time evolution up until the first measurement is performed, and this can be described as a diffusion process up until that point defined by the standard form of stochastic differential equation as

$$dx = b(x,t,\nu)dt \;+\; dw(t,\nu) \tag{2}$$

where $b(\nu,x,t)$ is a sufficiently well-behaved function called the drift term, $w(t,\nu)$ is a Wiener function with diffusion constant ν, and the second term is called the diffusion term [41]. The drift term depends on the wave function, and it also depends on what value of ν we are considering. If we precisely measure the particle position at time t, and then instantly the wave function collapses, the drift term will become huge after this measurement because of the Heisenberg

uncertainty principle. The collapsed ψ will be localized at a point x, but it will have a large spread in momentum, and so the drift term will dominate the diffusion term even during the short time delay ϵ to the next measurement. This situation persists no matter how small we make ϵ, and consequently, we can never measure the diffusion constant. This was pointed out, for example, in [19]. In [3] it was argued that for large values of ν, the diffusion term would dominate and could thereby be ruled out with a measurement, but this assumes that the b term doesn't grow with increasing values of ν, and this is exactly what happens in GSM, as the probability density in GSM is independent of ν. After measuring the position at time t, the wave function collapses and evolves to time t+ε through the Schrödinger equation, which is independent of ν. Therefore, the argument against very large values of ν does not apply to GSM. One might try to get around this situation with some sort of weak measurement [44] that might indirectly determine the diffusion constant without disturbing the wave function so much, but thus far there's no known way to do this. One can't even place a bound on the magnitude of ν, as we have just shown. GSM is possible because of this inability. Nelson ([3], §, p.117) agrees that strictly speaking, the diffusion constant cannot be measured, but he suggests that a reasonable assumption is that the forward velocity b right after a position measurement should be no bigger than p/m, where p here denotes the uncertainty in the momentum of the particle after measurement as required by the Heisenberg uncertainty principle. This assumption is not satisfied in GSM, where b would also depend on the diffusion constant after a measurement because of (22), and for a large diffusion constant, it would be bigger than Nelson's suggested upper bound. So in this paper, we abandon the "usual assumptions" used to dismiss large diffusion constants as

metaphysical. Thus, we are free to analytically continue the stochastic functions of GSM into the entire complex ν plane.

Note that the Heisenberg uncertainty principle alone, without collapse of the wave function, is not sufficient to prevent us from measuring the diffusion constant. A visualization of this comes from the bouncing drop experiments of Yves Coudet [45–47]. In these experiments we see a wave influencing a particle's motion as envisioned by de Broglie. The wave equation is not exactly the same as quantum mechanics, but it shows some aspects of quantum behavior. It's quite possible that these wave functions can in some configuration satisfy an approximate form of the Heisenberg uncertainty principle as far as the wave function alone is concerned, but nevertheless we can easily measure the position and momentum of the bouncing drops without affecting them so there's no collapse. We might be called super-observers because of this. If there were ways to measure a real electron without disturbing its wave function in this way, then the diffusion constant in GSM could be measured. It's conceivable that someday this might be possible, but this would be revolutionary. The analog of a quantum observer in the bouncing drop framework would have to be an apparatus built from only bouncing drops that could somehow perform a measurement on other bouncing drops. This might lead to a wave-function collapse or change and to something like a Heisenberg uncertainty principle. The subject of noninvasive measurability in quantum mechanics has been researched intensively, and there is very strong evidence that it is impossible to achieve [48–50]. It might still be possible to infer a value for the diffusion constant in some special situations. In X several speculative candidate systems are discussed.

So one can construct a stochastic model for Schrödinger's equation with any

positive diffusion constant whatsoever. Negative values of the diffusion constant could correspond to a time reversal or to a diffusion in complex space-time. If you use the value, $\nu = \hbar/2m$, you get standard stochastic mechanics. If you take the limit $\nu \to 0$ you get Bohmian mechanics, a result first shown by Shucker [51]. If you analytically continue ν to imaginary values using GSM, you get the Heisenberg algebra when $\nu = \pm i\hbar/2m$. This requires complex space-time which we discuss below. From a strictly empirical point of view all of these theories are equivalent, at least given our current understanding of measurement limitations. This non-uniqueness of stochastic mechanics reflects a kind of nonlinear gauge invariance of Schrödinger's equation, as proved in [18], which is a subset of [43]. Modern physics is happy with gauge invariance, and so striving to fix an unmeasurable gauge with an arbitrary mathematical bias might not be the wisest strategy. If any value of the diffusion constant is preferred, it is a purely imaginary value of $\pm i\hbar/2m$, for with this value, the Heisenberg algebra is perfectly mirrored in GSM. Unlike the Feynman path integral approach though, such an imaginary value requires embedding the diffusion in a complex space-time manifold.

If we can't measure the diffusion constant if it is real-valued and positive, we certainly can't measure it if it is complex-valued because we have no way of measuring the imaginary part of the position of a particle even if we are embedded in a complex space-time. We can only observe the real part of a particle's position. Our macroscopic tools for measurement are all located on the real subspace of complex space-time, or at least that's the picture that is emerging. It's like we are in a higher dimensional flatland and not aware of it [52].

Similar arguments can be applied to the forward velocity b and the backward

velocity b_*. They cannot apparently be measured directly, either. It is remarkable how effectively quantum mechanics manages to cover its tracks in this way. Perhaps someday a way to get around these limitations will be found, and a measurement of these hidden parameters might be possible. But considerable time has passed since the original publications on GSM [16, 18], and so far no experiments have been performed or even proposed to actually measure the diffusion constant.

III. A SHORT REVIEW OF STOCHASTIC MECHANICS WITH ARBITRARY DIFFUSION CONSTANT

Most of this can be found in [41], especially in §13. We have two stochastic differential equations: a forward and a backward equation in 1D for simplicity, although the higher dimensions are straightforward.

$$dx(t,\nu) = b(x(t),t,\nu)dt + dw(t,\nu) \tag{3}$$

$$dx_*(t,\nu) = b_*(x_*(t),t,\nu)dt + dw_*(t,\nu) \tag{4}$$

$b(x,t,\nu)$ can be interpreted as the stochastic expectation value of the velocity just after the particle has passed through the point x, and $b_*(x,t,\nu)$ as the velocity expectation just before the particle passes through x. If the Markov trajectories were differentiable, then these would be equal. But they are not differentiable, and so the forward and backward velocities are different, and this leads to great richness and complexity in this theory. The existence of this pair as both Markov processes is assured by a theorem by Doob (see [41], §13, p. 85).

We mention it because the backward velocity b_* plays are large role in creating a time symmetric diffusion which is necessary for a stochastic model of quantum mechanics, and otherwise we would have to make it a separate assumption. As Nelson points out, the usual approach to diffusion in Brownian motion, that is not time symmetric, is not suitable for time reversible stochastic mechanics. In Appendix A a list of formulas that are useful in stochastic mechanics are presented. The parameter ν is the time independent diffusion parameter defined by

$$\nu = \frac{1}{2} E\left((dw(t,\nu))^2\right)/dt \tag{5}$$

and $w(t,\nu)$ is a Wiener process. Although b and b_* depend on ν, we shall not always include it in the argument list of these and other functions to simplify the notation.

Central to our discussion is the Markov transition density function. We follow the convention used in [15] such that the earlier time is to the right of the semicolon which is opposite to the convention in [3], so we have

$$P(x,t;y,s) = \lim_{dx\to 0} \frac{1}{dx} P(x(t) \in dx \mid x(s) = y), \quad t > s \tag{6}$$

The notation $P(A|B)$ means the probability of A conditioned by B. This transition function satisfies the Chapman-Kolmogorov equation (unless otherwise specified, the domain of integration is taken to be $-\infty$ to $+\infty$)

$$P(x,t;y,r) = \int P(x,t;z,s)P(z,s;y,r)dz, \ for\, times\, t > s > r \tag{7}$$

Continuity of paths leads to the limiting behavior

$$\lim_{t \downarrow s} P(x,t;y,s) \;=\; \delta(x-y) \tag{8}$$

where δ denotes the Dirac delta function-distribution. The transition function is normalized.

$$\int P(x,t;y,s)dx = 1 \tag{9}$$

The time reversed process has a different transition density function P_*, but it is simply related to P provided that the density function is never zero by the formula from ([3], eqn. 6.2, p 35)

$$P_*(y,t;x,s) = P(y,t;x,s)\frac{\rho(x,s)}{\rho(y,t)},\ t > s \tag{10}$$

which also satisfies a Chapman-Kolmogorov equation. We have for the probability density ρ:

$$\rho(x,t) = \int P(x,t;y,s)\rho(y,s)dy,\ t > s \tag{11}$$

$$\rho(y,s) = \int P_*(x,t;y,s)\rho(x,t)dx,\ t > s \tag{12}$$

In general this does not describe a stationary process, and therefore the time reversed process is different from the forward process, even though the Schrödinger equation is invariant under time reversal. The Markov transition function plays a role similar to the time evolution operator in quantum mechanics. We can calculate the probability densities for multiple times very easily by forming products of Markov transition functions ([3], eqn. 6.3, p 35).

It is customary to write the probability density as

$$\rho(x,t) = exp(2R(x,t)) \tag{13}$$

We define S_N up to a function independent of x by

$$2\nu\frac{\partial}{\partial x}S_N(x,t,\nu) \equiv \frac{b(x,t,\nu)+b_*(x,t,\nu)}{2} \tag{14}$$

and similarly ([41], §15)

$$2\nu\frac{\partial}{\partial x}R(x,t,\nu) \equiv \frac{b(x,t,\nu)-b_*(x,t,\nu)}{2} \tag{15}$$

and it follows that (in 3D this assumes that $curl(b) = 0$)

$$b(x,t) = 2\nu\nabla\left(R + S_N\right) \tag{16}$$

$$b_*(x,t) = 2\nu\nabla\left(-R + S_N\right) \tag{17}$$

The wave function will be written as

$$\psi(x,t) = exp(R(x,t) + iS_Q(x,t)) \tag{18}$$

In stochastic mechanics, $S_N = S_Q$ and $\nu = \hbar/2m = 1/2$ in our units. The wave function will be independent of ν in GSM. This is because the following two equations are equivalent, as shown for real-valued constant z in [18]

$$\left(-\frac{\hbar^2}{2m}\triangle + V\right)exp(R+iS_Q) = i\hbar\frac{\partial}{\partial t}exp(R+iS_Q) \tag{19}$$

$$\left(-\frac{(z\hbar)^2}{2m}\triangle+\left(V+\frac{\hbar^2}{2m}\left(z^2-1\right)\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right)\right)exp(R+iS_Q/z)$$
$$=i\left(z\hbar\right)\frac{\partial}{\partial t}exp(R+iS_Q/z) \quad (20)$$

and where $\rho = exp(2R)$. This equivalence is also true for all complex values of z by simple analytic continuation from the real values. The extra nonlinear term in the potential in (20) is just Bohm's quantum potential up to a multiplicative constant. The second equation looks like a modified Schrödinger equation with the replacements $\hbar \to z\hbar$ and $V \to \left(V+\frac{\hbar^2}{2m}\left(z^2-1\right)\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right)$, so we can immediately, model this with a stochastic process of the form (2) with

$$\nu = z\frac{\hbar}{2m} \quad (21)$$

$$b(x,t,\nu) = 2\nu\nabla\left(R(x,t)+S_Q(x,t)/z\right) = \nabla\left(2\nu R(x,t)+\frac{\hbar}{m}S_Q(x,t)\right) \quad (22)$$

This then defines the generalized stochastic process for any value of the diffusion constant. The generalized "equation of motion" is given in (31) along with (32).

IV. AN ACTION PRINCIPAL COMPATIBLE WITH ARBITRARY DIFFUSION CONSTANT

Here we utilize the theorem and technique of Yasue [53], but with a modified Lagrangian. Given an action of the form

$$I = E\left[\int_a^b L(x(t),Dx(t),D_*x(t))dt\right] \quad (23)$$

where the operators D and D_* are defined in (A10) and (A11). He obtained the extended Euler-Lagrange equation

$$D\left(\frac{\partial L}{\partial D_* x(t)}\right) + D_*\left(\frac{\partial L}{\partial D x(t)}\right) - \frac{\partial L}{\partial x(t)} = 0 \tag{24}$$

(see also Zambrini, equation 25 in [54]). The operators D and D_* are the forward and backward time derivatives of [41]. Yasue showed that the following time reversible Lagrangian

$$L_N = \frac{m}{2}\left(\frac{(Dx(t))^2 + (D_* x(t))^2}{2}\right) - V(x) \tag{25}$$

leads to the Euler-Lagrange equation if $Dx(t)$ and $D_* x(t)$ are varied independently

$$m\frac{DD_* + D_* D}{2}x(t) = -\frac{\partial V(x(t))}{\partial x(t)} \tag{26}$$

and this is the same equation of motion that Nelson found which gives the Schrödinger equation provided the diffusion constant is given by

$$\nu_N = \frac{\hbar}{2m} \tag{27}$$

and where we have chosen units so that $\hbar = 1$. The following acceleration is called the mean acceleration

$$a_N = \frac{DD_* + D_* D}{2}x(t) \tag{28}$$

Now consider modifying the Lagrangian to read

$$L_\nu = A\left(L_N + V(x)\right) + B\left(Dx(t)\right)\left(D_*x(t)\right) - V(x) \tag{29}$$

where A and B are constants. Note that this modified Lagrangian is still invariant under time reversal. The Euler-Lagrange equations for this Lagrangian are

$$A\left(m\frac{DD_* + D_*D}{2}x(t)\right) + B\left(m\frac{DD + D_*D_*}{2}x(t)\right) = -\frac{\partial V(x(t))}{\partial x(t)} \tag{30}$$

It was shown in [18] that the following "equation of motion" also leads to Schrödinger's equation

$$\left(m\frac{DD_* + D_*D}{2}x(t)\right) + m\frac{\beta}{8}\left(D - D_*\right)^2 x(t) = -\frac{\partial V(x(t))}{\partial x(t)} \tag{31}$$

provided that

$$\nu = \frac{\hbar}{2m}\frac{1}{\sqrt{1-\beta/2}} \tag{32}$$

or equivalently

$$\beta = 2\left(1 - \left(\frac{\hbar}{2m\nu}\right)^2\right) \tag{33}$$

and therefore, we get Schrödinger's equation provided that

$$A = 1 - \frac{\beta}{4},\ and\ B = \frac{\beta}{4} \tag{34}$$

and therefore we can have an action principal for any value of the diffusion constant. Nelson has argued in [3] that although we cannot physically measure the diffusion constant according to existing quantum theory, maybe someday

a means will be found to get around this, which would be a violation of our current understanding of what is possible. The action chosen in [3] for simplicity is divergent, but when the infinite part is subtracted off, one gets a remainder that gives a unique diffusion constant that is Nelson's original value of $\hbar/2m$. It is argued that the infinite part of the action depends only on the diffusion constant, and therefore does not enter into the variation. One might question the mathematical rigor of this argument because of the equality: $\infty + a = \infty + b$ for all finite values of a and b. Since the diffusion constant cannot be measured, this argument is metaphysical, but it could nevertheless be the way that nature works. Jaekel has also given a rendition of this result [23]. For the purpose of tying together the Heisenberg operator approach to stochastic mechanics, we consider the variable diffusion constant general theory of GSM in this paper.

V. DERIVING THE HEISENBERG OPERATOR FORMALISM FROM GENERALIZED STOCHASTIC MECHANICS

The simplest way to illustrate or derive the Heisenberg operator formalism in the framework of stochastic mechanics is to use diffusion in a complex space or space-time. Here we start with real ν and use results from [15, 16], but we alert the reader that the definition of ν in these two papers differ by a factor of 2. Here in (5) we use the definition in [15] which coincides with ([41], §13). The basic idea of the derivation of non-commuting operators in stochastic mechanics is that the order of operations at equal times is simply the infinitesimal time-ordering of certain expectation values in the limit where the difference between the times tends to zero. Consider the random variable

$x(t)$ in SM setting. The time derivative does not exist for these functions as is well known. However, if we consider the two-point function $E(x(t)x(s))$,then this function is typically differentiable with respect to either time if the forward velocity $b(x,t)$ is sufficiently well-behaved which we assume. Now to get an idea of where the noncommutativity comes from in this framework, consider the following limiting procedure with $t > s$

$$Commutator = \lim_{t \uparrow s} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right) E(x(t)x(s)) \tag{35}$$

The expectation $E()$ here is just the classical expectation for a stochastic process $x(t)$. It is not a quantum expectation. The sample paths for $x(t)$ are continuous but non-differentiable trajectories. They are fractal curves if you like. It is imperative that the expectation value be taken first, before the differentiation, because of this. The only exception is the Bohmian limit where $\nu = 0$ and the paths become differentiable. This microscopic time ordering is essentially the same technique employed by Feynman in his thesis [33], but for a true diffusion process here. Our treatment here is based entirely on stochastic mechanics, though as modified in GSM. If the sample paths were smooth, this commutator would vanish. It is a difference between the two time orderings of a velocity times a position. So It's like a commutator. It can be evaluated by means of the forward and backward derivatives in the present context

$$Commutator = E\left(\left(b_*(x,s) - b(x,s) \right) x(s) \right) \tag{36}$$

Now we use (A6) to obtain by using integration by parts

$$Commutator = -\int 2\nu \frac{grad\,\rho(x,t)}{\rho(x,t)} x\rho(x,t)dx = 2\nu \tag{37}$$

This simple result suggests that we can use the microscopic time ordering of expectations to define a non-commuting operator algebra. If we consider Nelson's value of $\nu = \hbar/2m$, and that the momentum is m times the velocity, it looks like the commutator of p and x is simply $\hbar$ up to a sign at least, since we could have chosen the opposite ordering. Thus the time-ordering of expectations of this type induce a non commutative structure as was described in [15, 16], but the commutators are real-valued if the diffusion constant is real. Note that the definition of ν in [16] differs by a factor of 2 compared with [15] and here. In order to get the usual Heisenberg commutation relations, we must analytically continue to imaginary values of ν and set

$$\nu = -i\frac{\hbar}{2m} \tag{38}$$

The interpretation of this analytical continuation is non-trivial, and the reader should not be surprised if he has trouble visualizing what this means. It's basically an area of active research. A few papers dedicated to this problem are discussed briefly below. We saw that Schrödinger's equation was invariant as a function of ν in GSM for positive ν. But then, assuming that both sides of the equation are analytic in ν, the equality extends automatically to all complex values of ν. We assume this is the case. In this way we can achieve three desirable goals:

1. We describe quantum mechanics with a bona fide diffusion process with

positive definite probability densities.

2. We can describe these processes utilizing non-commuting operators as in quantum mechanics, but for real values of the diffusion constant, the commutator of $\hat{x}$ and $\hat{\hat{x}}$ is real. We can replicate the Heisenberg algebra with an imaginary commutator.

3. We can describe Bohmian mechanics, stochastic mechanics, and generalized stochastic mechanics in a single framework.

Given that the Schrödinger equation is always the same, independent of ν in GSM, then according the Nelson and Feynman's position mandate, it doesn't matter what value of ν we use. But, if we desire to replicate the Heisenberg algebra in this way, it requires that the particle diffuses in a complex space, and if we consider that space and time mix under Lorentz transformations, then this would ultimately require motion in a complex space-time. In order to develop the Heisenberg algebra further, we follow the approach in [15, 16]. In the Heisenberg picture the coordinates and momenta operators are time-dependent, but the wave function is fixed, as opposed to the Schrödinger picture where the wave function is time-dependent and the coordinates and momenta operators are time independent. Towards this end we introduce a base time s which will be fixed, and we derive time-dependent operators from this point using the Markov transition functions. This base time is arbitrary. We introduce it so that we can immediately utilize the various equations of the stochastic process which all depend on time. It is assumed that all other choices of base time are equivalent and can be mapped from one to another using the equations of Markov processes. After the commutation rules for operators are established, we can transform them by

similarity transformations that leave them unchanged, and in this way recover the usual quantum mechanical Hilbert space formulation. So first let's introduce a Hilbert space $\mathcal{H}_s$ with the following inner product:

$$(f,g) = E\left(f^*(x(s),s)g(x(s),s)\right) = \int \rho(x,s)f^*(x,s)g(x,s)dx \tag{39}$$

The functions f and g that we consider are assumed to be differentiable to all orders in both independent variables. Now let us define operators on these functions. The operator for x we denote by simple multiplication, and since the base time s is fixed, we drop it from the argument of $\hat{x}$ notational simplicity

$$\hat{x}f(x,s) \equiv xf(x,s) \tag{40}$$

Next we define an operator for $\dot{x}$ that acts on elements of the Hilbert space $\mathcal{H}_s$ and is defined as the following limit of a derivative of a conditional expectation

$$\hat{\dot{x}}f(x,s) \equiv \lim_{t\downarrow u}\lim_{u\downarrow s}\frac{\partial}{\partial u}E\left(x(u)f(x(t),t)|x(s)=x\right),\ t>u>s,\ f\in\mathcal{H}_s \tag{41}$$

where the notation $\lim_{t\downarrow u}$ means take the limit as t approaches u from above. Now it's convenient to write this expectation in terms of the Markov transition function densities because we can then use the forward and backward equations to simplify it. Using (A20) we can write the probability density for three times, where $t_1 < t_2 < t_3$, as

$$\rho(x_3,t_3;x_2,t_2;x_1,t_1) = P(x_3,t_3;x_2,t_2)P(x_2,t_2;x_1,t_1)\rho(x_1,t_1) \tag{42}$$

and this approach can be extended to any number of different times. To get the conditional expectation we simply drop the term $\rho(x_1, t_1)$. So we rewrite (41) as

$$\hat{\hat{x}} f(x,s) = \lim_{t \downarrow u} \lim_{u \downarrow s} \frac{\partial}{\partial u} \int f(x_t,t) x_u P(x_t,t;x_u,u) P(x_u,u;x,s) dx_t dx_u \quad (43)$$

We assume uniform convergence so that we can bring the $\frac{\partial}{\partial u}$ inside the integral.

$$\hat{\hat{x}} f(x,s) = \lim_{t \downarrow u} \lim_{u \downarrow s} \int f(x_t,t) \int x_u \frac{\partial}{\partial u} \left(P(x_t,t;x_u,u) P(x_u,u;x,s) \right) dx_u dx_t \quad (44)$$

$$\hat{\hat{x}} f(x,s) = \lim_{t \downarrow u} \lim_{u \downarrow s} \int f(x_t,t) \times$$
$$\times \int x_u \left(\frac{\partial P(x_t,t;x_u,u)}{\partial u} P(x_u,u;x,s) + P(x_t,t;x_u,u) \frac{P(x_u,u;x,s)}{\partial u} \right) dx_u dx_t \quad (45)$$

Use the backward equation A18 (with ∇ and $\triangle$ symbols for derivatives in 1D):

$$\frac{\partial P(x_t,t;x_u,u)}{\partial u} = -\left(b(x_u,u) \cdot \nabla_{x_u} + \nu \triangle_{x_u} \right) P(x_t,t;x_u,u), \; t > u \quad (46)$$

and use the forward equation A17 to obtain:

$$\frac{P(x_u,u;x_s,s)}{\partial u} = -\left(\nabla_{x_u} \cdot b(x_u,u) + b(x_u,u) \cdot \nabla_{x_u} - \nu \triangle_{x_u} \right) P(x_u,u;x_s,s), \; u > s \quad (47)$$

and integrating this by parts, results in :

$$\int x_u \left(\frac{\partial P(x_t,t;x_u,u)}{\partial u} P(x_u,u;x,s) + P(x_t,t;x_u,u) \frac{P(x_u,u;x,s)}{\partial u} \right) dx_u$$
$$= \int P(x_t,t;x_u,u) \left(-x_u \left(\nabla_{x_u} \cdot b(x_u,u) + b(x_u,u) \cdot \nabla_{x_u} - \nu \triangle_{x_u} \right) \right) P(x_u,u;x,s) dx_u +$$
$$\int P(x_t,t;x_u,u) \left(\left(\nabla_{x_u} \cdot b(x_u,u) + b(x_u,u) \cdot \nabla_{x_u} - \nu \triangle_{x_u} \right) x_u \right) P(x_u,u;x,s) dx_u \quad (48)$$

This can be expressed more compactly in terms of a commutator

$$= \int P(x_t, t; x_u, u) \left[\left(\nabla_{x_u} \cdot b(x_u, u) + b(x_u, u) \cdot \hat{\nabla}_{x_u} - \nu \hat{\triangle}_{x_u} \right), x_u \right] P(x_u, u; x, s) dx_u \tag{49}$$

but $\left[\left(\nabla_{x_u} \cdot b(x_u, u) \right), x_u \right] = 0$ and so

$$= \int P(x_t, t; x_u, u) \left[\left(b(x_u, u) \cdot \hat{\nabla}_{x_u} - \nu \hat{\triangle}_{x_u} \right), x_u \right] P(x_u, u; x, s) dx_u \tag{50}$$

and we have $\left[\left(b(x_u, u) \cdot \hat{\nabla}_{x_u} - \nu \hat{\triangle}_{x_u} \right), x_u \right] = \left(b(x_u, u) - 2\nu \hat{\nabla}_{x_u} \right)$ (51)

$$= \int P(x_t, t; x_u, u) \left(b(x_u, u) - 2\nu \nabla_{x_u} \right) P(x_u, u; x, s) dx_u \tag{52}$$

Now we can take the limits

$$\lim_{t \downarrow u} P(x_t, t; x_u, u) = \delta(x_t - x_u) \tag{53}$$

$$\lim_{u \downarrow s} P(x_u, u; x_s, s) = \delta(x_u - x_s) \tag{54}$$

$$\hat{\dot{x}} f(x, s) = \lim_{t \downarrow u} \lim_{u \downarrow s} \int f(x_t, t) \delta(x_t - x_u) \left(b(x_u, u) - 2\nu \nabla_{x_u} \right) \delta(x_u - x_s) dx_t dx_u, \; t > u > s \tag{55}$$

now we integrate over x_t

$$\hat{\dot{x}} f(x, s) = \lim_{t \downarrow u} \lim_{u \downarrow s} \int f(x_u, t) \left(b(x_u, u) - 2\nu \nabla_{x_u} \right) \delta(x_u - x_s) dx_u, \; t > u > s \tag{56}$$

now we integrate by parts once again

$$\hat{\dot{x}} f(x, s) = \lim_{t \downarrow u} \lim_{u \downarrow s} \int \delta(x_u - x_s) \left(b(x_u, u) + 2\nu \nabla_{x_u} \right) f(x_u, t) dx_u, \; t > u > s \tag{57}$$

$$\hat{\dot{x}} f(x, s) = \left(b(x, s) + 2\nu \nabla_x \right) f(x, s) \tag{58}$$

$$\hat{\dot{x}} = \left(b(x,s) + 2\nu\hat{\nabla}_x \right) \tag{59}$$

It follows then that a commutator relation exists

$$\left[\hat{\dot{x}}(s), \hat{x}(s) \right] = 2\upsilon \tag{60}$$

and this is very similar to the Heisenberg commutation rule, except that its value is real and not imaginary. Note that $\hat{\dot{x}}(s)$ is non-Hermitian in general, unless ν is imaginary. Since $\hat{x}$ is just a simple multiplication operator, we will drop the hat over this symbol for simplicity. To proceed further, we note that we can write

$$\hat{\dot{x}} = \left[(\nabla_x \cdot b(x,s)) + \left(b(x,s) + \nu\hat{\nabla}_x \right) \cdot \hat{\nabla}_x, x \right] \tag{61}$$

For a smooth function $\hat{j}(x,s)$ of x and s let us define an operator corresponding to a time derivative by a commutation rule derived from this as

$$\hat{\dot{j}}(x,s) = \left[(\nabla_x \cdot b(x,s)) + \left(b(x,s) + \nu\hat{\nabla}_x \right) \cdot \hat{\nabla}_x, \hat{j}(x,s) \right] + \frac{\partial \hat{j}(x,s)}{\partial s} \tag{62}$$

To justify this equation, replace the term x_u in (43) with the function $j(x_u, u)$. Then assuming that this function is smooth, and that the integral is uniformly convergent, the result follows.

Next, we define an acceleration operator

$$\hat{\ddot{x}} f(x,s) \equiv \lim_{t \downarrow u} \lim_{u \downarrow s} \frac{\partial^2}{\partial u^2} E\left(x(u) f(x(t),t) | x(s) = x \right), \; t > u > s, \; f \in \mathcal{H}_s \tag{63}$$

This simplifies to

$$\hat{\dot{x}} = \frac{\partial b(x,s)}{\partial s} + \nu\left(\triangle_x b(x,s)\right) + \frac{1}{2}\left(\nabla_x\left(b^2(x,s)\right)\right) \tag{64}$$

Proof

$$\hat{\dot{x}} f(x,s) \equiv \lim_{t\downarrow u}\lim_{u\downarrow s}\frac{\partial^2}{\partial u^2}\int f(x_t,t)x_u P(x_t,t;x_u,u)P(x_u,u;x,s)dx_t dx_u,\ t>u>s \tag{65}$$

again we assume uniform convergence so that we can bring the $\frac{\partial^2}{\partial u^2}$ inside the integral and use the forward and backward equations to obtain

$$\begin{gathered}\hat{\dot{x}} = \lim_{t\downarrow u}\lim_{u\downarrow s}\int P(x_t,t;x_u,u)\times \\ \left[\left(\nabla_{x_u}\cdot b(x_u,u) + b(x_u,u)\cdot\hat{\nabla}_{x_u} - \nu\hat{\triangle}_{x_u}\right),\right. \\ \left.\left[\left(\nabla_{x_u}\cdot b(x_u,u) + b(x_u,u)\cdot\hat{\nabla}_{x_u} - \nu\hat{\triangle}_{x_u}\right),x_u\right]\right] \\ \times P(x_u,u;x,s)dx_u + \frac{\partial b}{\partial s}\end{gathered} \tag{66}$$

$$\begin{gathered}\hat{\dot{x}} = \lim_{t\downarrow u}\lim_{u\downarrow s}\int P(x_t,t;x_u,u) \\ \left[\left(\nabla_{x_u}\cdot b(x_u,u) + b(x_u,u)\cdot\hat{\nabla}_{x_u} - \nu\hat{\triangle}_{x_u}\right),\left(b(x_u,u) - 2\nu\hat{\nabla}_{x_u}\right)\right]\times \\ \times P(x_u,u;x,s)dx_u + \frac{\partial b}{\partial s}\end{gathered} \tag{67}$$

But we have

$$\left[\left(\hat{\nabla}_{x_u}\left(b(x_u,u) - 2\nu\hat{\nabla}\right) + \nu\hat{\triangle}_{x_u}\right),\left(b(x_u,u) - 2\nu\hat{\nabla}_{x_u}\right)\right]$$

$$= \left[\left(\hat{\nabla}_{x_u}\left(b(x_u,u) - 2\nu\hat{\nabla}\right)\right), \left(b(x_u,u) - 2\nu\hat{\nabla}_{x_u}\right)\right] + \left[\nu\hat{\triangle}_{x_u}, \left(b(x_u,u) - 2\nu\hat{\nabla}_{x_u}\right)\right]$$
$$= \left[\left(\hat{\nabla}_{x_u}\left(b(x_u,u) - 2\nu\hat{\nabla}\right)\right), \left(b(x_u,u) - 2\nu\hat{\nabla}_{x_u}\right)\right] + \left[\nu\hat{\triangle}_{x_u}, b(x_u,u)\right] \quad (68)$$

and now we use the commutator identity $[AB, B] = [A, B]\, B$ to obtain

$$\begin{aligned}
&\left[\left(\hat{\nabla}\left(b - 2\nu\hat{\nabla}\right) + \nu\hat{\triangle}\right), \left(b - 2\nu\hat{\nabla}\right)\right] \\
&= \left[\hat{\nabla}\left(b - 2\nu\hat{\nabla}\right), b - 2\nu\hat{\nabla}\right] + \left[\nu\hat{\triangle}, \left(b - \cancel{2\nu\hat{\nabla}}\right)\right] \\
&= \left[\hat{\nabla}, \left(b - \cancel{2\nu\hat{\nabla}}\right)\right]\left(b - 2\nu\hat{\nabla}\right) + \left[\nu\hat{\triangle}, b\right] \\
&= \left[\hat{\nabla}, b\right]\left(b - 2\nu\hat{\nabla}\right) + \left[\nu\hat{\triangle}, b\right] \\
&= (\nabla \cdot b)\left(b - 2\nu\hat{\nabla}\right) + \nu\left[\hat{\triangle}, b\right] \\
&= (\nabla \cdot b)\left(b - 2\nu\hat{\nabla}\right) + \nu\left(\left[\hat{\nabla}, b\right]\hat{\nabla} + \hat{\nabla}\left[\hat{\nabla}, b\right]\right) \\
&= (\nabla \cdot b)\left(b - 2\nu\hat{\nabla}\right) + \nu\left((\nabla \cdot b)\hat{\nabla} + \hat{\nabla}(\nabla \cdot b)\right) \\
&= \frac{1}{2}\nabla b^2 + \nu\Delta b
\end{aligned} \quad (69)$$

Then adding in the term $\partial b/\partial s$ from 66, the result 64 is obtained. This is the same acceleration operator that was found in both [16] and [15] which use slightly different manipulations to arrive at the same result, up the the factor of 2 difference in the definition of ν mentioned above. This can be conveniently rewritten in the following form [15] as

$$\hat{\dot{x}} = \nabla_x\left(exp(-R - S_N)\left(2\nu\frac{\partial}{\partial t} + 2\nu^2\triangle_x\right)exp(R + S_N)\right), b = 2\nu\nabla(R + S_N) \quad (70)$$

Lets compare (64) with the swapped version by interchanging b and b_* which

is

$$\hat{\dot{\bar{x}}}_* = \frac{\partial b_*(x,s)}{\partial s} + \nu \triangle_x b_*(x,s) + \frac{1}{2}\nabla_x \left(b_*^2(x,s)\right) \tag{71}$$

Now consider the difference

$$\frac{\hat{\dot{\bar{x}}} - \hat{\dot{\bar{x}}}_*}{2} = \frac{\partial u(x,s)}{\partial s} + \nu \triangle_x u(x,s) + \nabla_x \left(u(x,s)v(x,s)\right) \tag{72}$$

Comparing this with the kinematic equation (13.5) in [41] which is

$$\frac{\partial u(x,s)}{\partial s} + \nu \triangle_x v(x,s) + \nabla_x \left(u(x,s)v(x,s)\right) = 0 \tag{73}$$

We see that

$$\frac{\hat{\dot{\bar{x}}} - \hat{\dot{\bar{x}}}_*}{2} = \nu \triangle_x u(x,s) - \nu \triangle_x v(x,s) \tag{74}$$

Kinematic here means that this result does not depend on the particular equation of motion chosen. Now we can write

$$\hat{\dot{\bar{x}}} = \frac{\left(\hat{\dot{\bar{x}}} + \hat{\dot{\bar{x}}}_*\right) + \left(\hat{\dot{\bar{x}}} - \hat{\dot{\bar{x}}}_*\right)}{2} \tag{75}$$

$$\hat{\ddot{x}} = \left(\frac{\partial v(x,s)}{\partial s} + \cancel{\nu \triangle_x v(x,s)} + \frac{1}{4}\nabla_x \left(b^2(x,s) + b_*^2(x,s)\right)\right) + \left(\nu \triangle_x u(x,s) - \cancel{\nu \triangle_x v(x,s)}\right) \tag{76}$$

$$\hat{\ddot{x}} = \frac{\partial v(x,s)}{\partial s} + \frac{1}{4}\nabla_x \left(b^2(x,s) + b_*^2(x,s)\right) + \nu \triangle_x u(x,s) \tag{77}$$

Let's compare this with Nelson's mean acceleration in equation ([2] eqn.

13.6) in

$$a_N = \frac{\partial v(x,s)}{\partial s} - u \cdot \nabla_x u + v \cdot \nabla_x v - \nu \triangle_x u \tag{78}$$

$$a_N = \frac{\partial v(x,s)}{\partial s} + \frac{1}{2}\nabla_x \left(v(x,s)^2 - u(x,s)^2 \right) - \nu \triangle_x u(x,s) \tag{79}$$

Let's now find the difference

$$\hat{\ddot{x}} - a_N = \left(\frac{1}{4}\nabla_x \left(b^2 + b_*^2 \right) + \nu \triangle_x u(x,s) \right) - \left(\frac{1}{2}\nabla_x \left(v(x,s)^2 - u(x,s)^2 \right) - \nu \triangle_x u(x,s) \right) \tag{80}$$

$$\hat{\ddot{x}} - a_N = 2\nu \triangle_x u(x,s) + \frac{1}{4}\nabla_x \left(b(x,s)^2 + b_*(x,s)^2 - 2b(x,s)b_*(x,s) \right) \tag{81}$$

$$\hat{\ddot{x}} - a_N = 2\nu \triangle_x u(x,s) + \nabla_x \left(u(x,s)^2 \right) \tag{82}$$

But $u = 2\nu \nabla_x R$, and therefore

$$\hat{\ddot{x}} - a_N = 2\nu \triangle_x \left(2\nu \nabla_x R \right) + \nabla_x \left(\left(2\nu \nabla_x R \right)^2 \right) \tag{83}$$

$$\hat{\ddot{x}} - a_N = (2\nu)^2 \nabla_x \left(\triangle_x R + \left(\nabla_x R \right)^2 \right) \tag{84}$$

$$\hat{\ddot{x}} - a_N = (2\nu)^2 \nabla_x \left(\frac{\triangle_x \sqrt{\rho}}{\sqrt{\rho}} \right) \tag{85}$$

It follows from the GSM equation of motion (31) that

$$m a_N = -\nabla_x V(x,t) - m\frac{\beta}{8} \left(D - D_* \right)^2 x(t) \tag{86}$$

$$m\frac{\beta}{8} \left(D - D_* \right)^2 x(t) = \nabla_x \left(m\beta\nu^2 \frac{\triangle \sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{87}$$

$$ma_N = -\nabla_x \left(V(x,t) + m\beta\nu^2 \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{88}$$

$$ma_N = -\nabla_x \left(V(x,t) + m \left(2 \left(\nu^2 - \left(\frac{\hbar}{2m} \right)^2 \right) \right) \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{89}$$

Therefore

$$\hat{\ddot{x}} = a_N + (2\nu)^2 \nabla_x \left(\frac{\triangle_x\sqrt{\rho}}{\sqrt{\rho}} \right) \tag{90}$$

$$m\hat{\ddot{x}} = -\nabla_x \left(V(x,t) + m \left(2 \left(\nu^2 - \left(\frac{\hbar}{2m} \right)^2 \right) \right) \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) + m\,(2\nu)^2 \nabla_x \left(\frac{\triangle_x\sqrt{\rho}}{\sqrt{\rho}} \right) \tag{91}$$

$$m\hat{\ddot{x}} = -\nabla_x \left(V(x,t) + m \left(2 \left(-\nu^2 - \left(\frac{\hbar}{2m} \right)^2 \right) \right) \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{92}$$

$$m\hat{\ddot{x}} = -\nabla_x V(x,t) + \nabla_x \left(2m\nu^2 + \frac{\hbar^2}{2m} \right) \frac{\triangle_x\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \tag{93}$$

This is the same formula that was found in ([15], eqn. 35). Here we have presented the tedious derivation which was omitted there. We see that the simplest possibility occurs when $\nu = \pm i\hbar/2m$, for then we get the Heisenberg formula of quantum mechanics

$$m\hat{\ddot{x}} = -\nabla_x V(x,t),\ if\ \nu = \pm i\hbar/2m \tag{94}$$

In general we have

$$m\hat{\ddot{x}} = -\nabla_x U(x,t) \tag{95}$$

for some function $U(x,t)$, and in particular, for the GSM equations of motion, we further have

$$U(x,t) = V(x,t) - \left(2m\nu^2 + \frac{\hbar^2}{2m}\right)\frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \tag{96}$$

The zero diffusion limit of the stochastic process gives Bohmian mechanics [51]. This is a significant unification of two of the principal alternative interpretations of quantum mechanics, namely the Bohmian and the stochastic interpretations. In the Bohmian limit the velocity and position operators commute.

$$\lim_{\nu\to 0} m\hat{\ddot{x}} = -\frac{\partial\left(V - \frac{\hbar^2}{2m}\frac{\triangle\sqrt{\rho}}{\rho}\right)}{\partial x} \tag{97}$$

Here V is the classical potential, and the extra term is called the quantum potential of Bohm [55]. The theoretical description of the Markov process is concisely summarized in the following two equations which were called "Markov wave equations" in [16].

$$\left[m\frac{(2\nu)^2}{2}\triangle_x + W\right]e^{R\pm S_N} = \mp m2\nu\frac{\partial}{\partial t}e^{R\pm S_N} \tag{98}$$

These are true for any given function $W(x,t)$, and not just for the GSM equations. These equations were also used in [3, 21], and were almost certainly found

earlier than [16], but I don't know who found them first. Notice that $\hat{\dot{x}}$ in (59) can be put into canonical form by a similarity transformation which does not affect the commutation rules

$$\hat{\dot{x}}_{canonical} = e^{-R-S_N}\hat{\dot{x}}e^{R+S_N} = 2\nu\nabla_x \tag{99}$$

The operator $\hat{x}$ doesn't change under this transformation.

$$\hat{x}_{canonical} = \hat{x} \tag{100}$$

So we can have the exact Heisenberg momentum operator at an imaginary value for the diffusion constant

$$\hat{P}_{Heisenberg} = m\hat{\dot{x}}_{canonical} = -i\hbar\nabla_x,\ if\ \nu = -i\hbar/2m \tag{101}$$

$$\left[\hat{x}, \hat{P}_{Heisenberg}\right] = i\hbar \tag{102}$$

We can now define a Hamiltonian operator by

$$\hat{H} = \left[m\frac{(2\nu)^2}{2}\triangle + U\right] = \left[m\frac{(2\nu)^2}{2}\left(\hat{\dot{x}}_{canonical}\right)^2 + U\right] \tag{103}$$

$$\hat{\dot{x}}_{canonical} = \frac{1}{2m\nu}\left[H, \hat{x}\right] = 2\nu\nabla_x \tag{104}$$

$$\hat{\ddot{x}}_{canonical} = \frac{1}{2m\nu}\left[H, \hat{\dot{x}}\right] = -\frac{1}{m}\nabla_x U \tag{105}$$

Comparing this with (95) we find

$$\hat{\ddot{x}} = \hat{\ddot{x}}_{canonical} \tag{106}$$

We can construct higher derivatives of order n using the recursive formula

$$\hat{x}_n = \frac{1}{2m\nu}\left[H, \hat{x}_{n-1}\right] + \frac{\partial \hat{x}_{n-1}}{\partial t},\ \hat{x}_0 = \hat{x} \tag{107}$$

and in this way we can build up a Taylor's expansion for propagating in time, very similar to the time evolution operator in quantum mechanics. At the special imaginary value of $\nu = -i\hbar/2m$ this time evolution becomes exactly the usual time evolution operator. For other values, so long as the equation of motion is that of GSM, the probability density at different times propagated in this way is always the same as Schrödinger's equation gives, provided we calculate the density only on the real subspace of complex space-time using Born's rule there. If the probability density in position is all that can be measured, as discussed above, then even if the diffusion constant is imaginary but different from the Heisenberg value, the measurements will still be the same in GSM because the probability densities for positions will be the same. It's conceivable though that someday we might find a way to measure the diffusion constant, and this would open a door to a sub-quantum world that we can only imagine now. In X we discuss some speculations about several ideas about measuring the diffusion constant, thereby going beyond conventional quantum mechanics.

VI. COMPARISON WITH THE FEYNMAN PATH INTEGRAL METHOD

It is quite a bit simpler to derive the Heisenberg operator algebra from Feynman path integrals than from stochastic mechanics [33]. The advantage of a stochastic process is that it can be understood as a proper diffusion process, unlike the Feynman path integral which uses effectively a complex measure to sum the different paths, and then requires one to square the absolute value of the resulting probability amplitude to obtain a probability. One might say that GSM is quantum mechanics made complicated, but from the point of view of emergent quantum mechanics, it's desirable to have a true diffusion process that can replicate quantum mechanics. Also, Born's rule is rigorously derivable from GSM, at least for real-valued diffusion constants. Then one can consider all sorts of possible diffusion models that might lead to quantum behavior, and the playground is enlarged.

To derive the commutation rules from Feynman path integrals, we can write simply

$$\langle \hat{x}(t)\hat{x}(s)\rangle = \int \mathcal{D}[x]x(t)x(s)e^{iS/\hbar} \tag{108}$$

and then we have

$$\lim_{s\uparrow t}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)\langle \hat{x}(t)\hat{x}(s)\rangle = \lim_{s\uparrow t}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)\int \mathcal{D}[x]x(t)x(s)e^{iS/\hbar} \tag{109}$$

$$\lim_{s\uparrow t}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)\langle\hat{x}(t)\hat{x}(s)\rangle=\frac{1}{m}\langle[p(t),x(t)]\rangle=-i\hbar \tag{110}$$

and thus we have a time ordered commutation rule expressed as a limiting procedure applied to a Feynman path integral, and from this one can reconstruct the full Heisenberg algebra. Also, this commutator is imaginary, and so it does not require any analytic continuation to a complex space. Because the Feynman path integral does not represent a true diffusion theory, however, it does not give us a picture of reality, although it feels like it does when we draw Feynman diagrams on the blackboard. This is deceptive. GSM gives us a large variety of truly stochastic realistic pictures of quantum processes, but it's more complicated and it seems to favor a complex space-time embedding, although this is not absolutely necessary. The Feynman path integral approach has proven very useful, and it is seductive to imagine that quantum particles are somehow moving along Feynman paths in our imagination, but it is not a road-map to a stochastic interpretation of quantum mechanics. In a sense it's a deterrent because it's mathematically much simpler than stochastic mechanics or GSM, and it's tempting to not bother with a rigorous diffusion model.

The limiting case of Bohmian mechanics gives a deterministic picture of reality within the framework of GSM. It is interesting to speculate that the underlying diffusion parameter may depend on the local environment. So the diffusion parameter in a vacuum may be different than in a condensed matter setting. We speculate in X below on some experiments that might possibly allow us to determine the diffusion constant in some special cases.

VII. THE CASE FOR COMPLEX SPACE-TIME

We have seen that generalized stochastic mechanics takes a particularly simple form for a diffusion constant which is purely imaginary with a value $\nu = \pm i\hbar/2m$. In this case the commutation relations also become exactly the Heisenberg rules. Such a diffusion constant would require the particle's motion to be in a complex space and because of space-time mixing in special relativity, in a complex space-time. If there are any values of the diffusion constant which are uniquely associated with quantum mechanics it is these two imaginary ones. But if the diffusion constant is imaginary, then the position of the particle will take on complex values. There is a history of suggestions that the space-time manifold might be embedded in a complex one. Perhaps the most interesting suggestion along these lines was made by Einstein. He believed that quantum theory might be derivable from classical field theory [9, 56, 57]. This is what he meant by unified field theory, and not the modern interpretation of the expression. He explored the complexification of space-time in this effort [58–60]. He used a Hermitian metric so that the metric length between points would be real-valued. Many others have pursued this line of research [61–75] An entirely different application of complex space-time comes from a fascinating method of solving difficult problems in general relativity involving analytical continuation of electromagnetic charges to complex-valued coordinates [40, 75–77]. In quantum mechanics there have been a number of applications as well [40, 71, 78–87]

David Bohm introduced the concept of the Implicate Order to physics [88]. A mathematically tangible example was proposed based on higher order algebras by Frescura and Hiley [89]. The simplest algebraic extension of space-time is to

replace the algebra of real numbers with that of complex numbers for the coordinates. The idea of the implicate order is that behind our classical perception of reality lies a hidden realm which interconnects non-locally events in our world. This is a manifestation of a web of quantum entanglement that permeates our universe. Freshly minted physicists tend to find the concepts of wholeness and implicate order to be too metaphysical to be of interest, but as one matures in the study of quantum foundations they tend to increase in perceived importance, as the other alternatives like the Many Worlds or 't Hooft determinism seem equally strange. In a complex space-time interpretation there are many Riemann sheets for the electromagnetic fields and for the particle motions. This Riemann sheet web embedded in a complex space-time manifold is what we would compare to Bohm's implicate order. This comparison resonates with recent ideas in quantum gravity interpreting quantum entanglement in terms of Einstein-Rosen bridges [40, 90].

There is literature from stochastic mechanics extending it to complex space and space-time. These techniques give precise mathematical meaning to an imaginary value for the diffusion constant. See for example Wang [91, 92], Rosenbrock [93–96], and Kuipers [97]. These papers all provide a framework for analytically continuing the diffusion constant to complex space and time. Therefore it seems that the combination of Heisenberg algebra with stochastic mechanics strongly favors a universe which is embedded in a complex manifold. Complex time models have also been proposed [98]. This body of research suggests a possible connection between stochastic mechanics and the geometric Langlands program as applied to physics [99]. The paper by Kuipers also considers relativistic wave equations. Complex space-time models have also been utilized in condensed

matter physics [100]. These topics are subjects of current and future research.

Wang frames the basis of his stochastic model of quantum mechanics in the language of weak measurement theory. He writes in [91]:

"In summary, we have proposed a new stochastic interpretation of quantum mechanics in which a quantum system is associated to a stochastic process in complex space. The real part of the trajectory of the stochastic process, which we call a weak trajectory, is interpreted as the trajectory of a particle in real configuration space. We showed that, on the one hand, the ensemble averaged moments of the weak trajectory are equal to the weak values of moments of the position operator of the quantum system, which are complex valued but nevertheless have well defined physical meanings according to the weak measurement theory [101, 102] and that, on the other hand, a single weak trajectory reduces to the correct classical trajectory in the classical limit."

The basis of his theory [91, 92] is exactly the same as our stochastic differential equation (2) with a diffusion constant given by $\nu = i\hbar/2m$. He shows that Schrödinger's equation can be derived with suitable choices of the backward derivative.

Wang uses a backward equation, but other than this, his stochastic differential equation is the same as GSM. The time reversal symmetry of GSM suggests that Wang's theory should have time reversal symmetry too. He finds in equation 36 of [92] an interesting result that the expectation value of a function of complex position z is equal to the usual quantum expectation value of the same function of a real position variable x. But this is only valid, he shows, if at some future time the full complex probability density $\rho(x, y, t_f)$ factorizes into a product $|\psi(x, t_f)|^2 \delta(y)$, where x and y are the real and imaginary parts of z.

Rosenbrock uses the variational formalism of dynamic programming and control theory to arrive at a stochastic model of quantum mechanics. His model is equivalent to GSM at the value of $\nu = -i\hbar/2m$. As a step towards interpreting the results of complex diffusion, and towards obtaining Born's rule for probability densities, he makes the following postulate.

"The properties of the physical particle are related to those of the complex image by the following postulate: If the expected value of some property (such as momentum $\widetilde{p}$ or energy $\widetilde{H}$) of the complex images in an ensemble has the real value α on the real axis, then α is the value of the corresponding property of the physical particles. Whereas the usual formulation of quantum mechanics deals with complex functions on a real space, the above procedure embeds this theory in a theory of complex functions on a complex space. The properties of physical particles in a real space are then regained by the postulate."

He develops an elaborate theory which boils down to GSM evaluated at an imaginary value of the diffusion constant. So he has the same stochastic process as Wang, except has has $\nu = -i\frac{\hbar}{2m}$ instead of $\nu = i\frac{\hbar}{2m}$

$$dz = b(z,t)dt + dw(t,\nu) \tag{111}$$

$$\nu = -i\frac{\hbar}{2m} \tag{112}$$

$$w(t,\nu) = \sqrt{-i\frac{\hbar}{2m}}w(t,1) = (1-i)\frac{1}{2}\sqrt{\frac{\hbar}{m}}w(t,1) \tag{113}$$

He writes the complex position z in terms of real and imaginary parts

$$z = x + iy \tag{114}$$

He changes variables and chooses to work with two real variables X and Y

$$X = x + y \tag{115}$$

$$Y = x - y \tag{116}$$

These are generated by the following stochastic differential equations

$$dX = (Re\, b(X,Y) + Im\, b(X,Y)dt \tag{117}$$

$$dY = (Re\, b(X,Y) - Im\, b(X,Y)dt + \sqrt{\hbar/m}dw(t) \tag{118}$$

where Re and Im denote real and imaginary parts. Note that the X differential does not have a diffusion term. The following equation follows from the forward equation of Kolmogorov applied to the probability density $\rho(X,Y,t)$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial X}\left((Re\, b + Im\, b)\rho\right) + \frac{\partial}{\partial Y}\left((Re\, b - Im\, b)\rho\right) - \frac{\hbar}{2m}\frac{\partial^2}{\partial Y^2}\rho = 0 \tag{119}$$

After transforming back to the original coordinates x, y Rosenbrock shows that

$$\rho(x,y)|_{y=0} = \psi^*(x,t)\psi(x,t) \tag{120}$$

So $\rho(x, y)$ serves as a probability density function for the full complex space, and $\rho(x, 0)$ serves as a probability density function on the real space, and moreover $\rho(x, 0)$ is the usual quantum position density. This seems to be the best stand-in for Born's rule in the extant literature on complex space-time and stochastic mechanics. All of this is a special case of GSM.

Kuipers' mathematical analysis is sophisticated and powerful [97]. He considers non-relativistic and relativistic particle diffusions in complex space for the non relativistic case, and complex space and time in the relativistic case. He generalizes the usual stochastic mechanics to allow stochastic processes which are semimartingales.

From Wikipedia, we have the following definition [103]

> "In probability theory, a real valued stochastic process X is called a semimartingale if it can be decomposed as the sum of a local martingale and a càdlàg adapted finite-variation process. Semimartingales are "good integrators", forming the largest class of processes with respect to which the Itô integral and the Stratonovich integral can be defined.
>
> The class of semimartingales is quite large (including, for example, all continuously differentiable processes, Brownian motion and Poisson processes). Submartingales and supermartingales together represent a subset of the semimartingales."

In particular, he allows for the possibility that the real and imaginary parts of the coordinates may be driven by independent Wiener processes. He does not derive Born's rule however. His analysis includes as a special case the stochastic

differential equation $dz(t) = b(z,t) + dw(t, \nu = -i\hbar/2m)$, which is the result of analytic continuation in GSM to the Heisenberg imaginary value for the diffusion constant, and is also the same as in Wang and Rosenbrock's papers.

He treats the relativistic case by showing the equivalence of his stochastic processes by introducing a "proper time" variable and deriving what has been termed the relativistic Schrödinger equation, or alternately the Stueckelberg-Horwitz-Piron (SHP) wave equation [104]. This yields the Klein-Gordon equation when the mass constraint is imposed on the solution. Other authors have proposed relativistic proper-time diffusion models for quantum mechanics in real space-time, For example [105–108]. The interpretation of these types of particle models is discussed in [109]. GSM has been extended to diffusions on Riemannian manifolds [110]. These results apply as well to pseudo-Riemannian manifolds like Minkowski space or the space-time in general relativity. So in particular, we could apply GSM to the relativistic diffusion models of Kuipers or those of Oron and Horwitz [105–108] and thereby obtain stochastic models of relativistic quantum mechanics which are embedded in real or complex space-time.

VIII. A POSSIBLE CONNECTION TO ADLER'S TRACE DYNAMICS

A preliminary exploration of a possible connection between stochastic mechanics and Adler's theory of emergent quantum mechanics based on trace dynamics [13] was made in [111]. It was found that the non-commutative structure of stochastic mechanics might be used to provide an explanation for trace dynamics. However, it was also found that the result tended to lead to the thermal diffusion equation rather than the Schrödinger equation unless a nonlinear po-

tential was added without any justification. However, only real-valued diffusion constants were considered in that work. If one revisits this mathematics with now an imaginary diffusion constant, it seems plausible that the Schrödinger equation can be understood as resulting from the trace-dynamics equilibrium for a suitably defined trace Lagrangian. This is still another reason to consider complex space-time to be an ontological reality.

IX. VISUALIZATIONS

The mathematics of GSM is a bit dry, and so here we will try and give some pictorial insight into the subject. Visualizations of stochastic mechanics were given by Yasue and Zambrini in [112]. We have used an open source Python library called Qmsolve [113]. This convenient software generates solutions to Schrödinger's equation. We use this to calculate the drift terms in GSM, and then using standard Wiener process simulations we generate sample trajectories for GSM. We also utilized a Python stochastic differential equation package called sdeint [114].

In atomic units, with $\hbar = 1$ and $m_e = 1$, Schrödinger's equation takes a simple form convenient for numerical analysis

$$\left(-\frac{1}{2}\triangle + V(x,t)\right)\psi(x,t) = i\frac{\partial\psi(x,t)}{\partial t} \tag{121}$$

In these units, the diffusion constant of stochastic mechanics is just 1/2, and the unit of velocity is αc so that $c \approx 137$. The diffusion model in GSM depends on $\psi(x,t)$ and on ν by the formulas $\psi = e^{R+iS}$ and $b(x,t,\nu) = 2\nu\nabla R + \nabla S$. We

generate the sample trajectories step-wise from the following stochastic equation

$$\mathbf{x}(t+\epsilon,\nu)-\mathbf{x}(t,\nu)=\mathbf{b}(x(t),t,\nu)\epsilon+\mathbf{W}(t+\epsilon,\nu)-\mathbf{W}(t,\nu) \tag{122}$$

Case 1 A simple plane wave

We consider an electron with velocity in the y direction in the x-y plane. Though we've been considering the 1D case above, it's more interesting to see images in two space dimensions. Here's a simple plane wave in the y direction. Simulations are shown in figure 1. The velocity was 10 and the initial standard deviation was 1.

$$\psi(x,y,t)=exp(i(ky-(k^2/2)t)) \tag{123}$$

$$b(x,t,\nu)=\nabla S=k\hat{y} \tag{124}$$

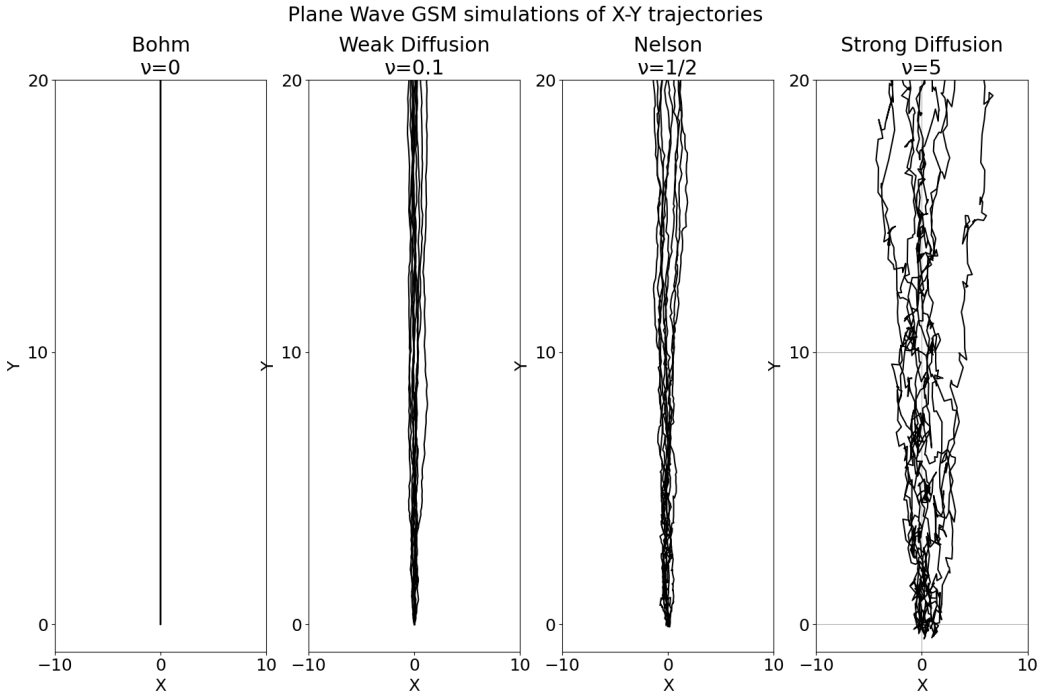

Figure 1. A plane wave as in Case 1 is plotted here. Only one starting point is shown since all others would look similar as the drift function is independent of starting x position for a plane wave. The diffusion constant is increasing from left to right, starting with a value of zero which gives the deterministic Bohmian trajectory. Next we have a weak diffusion case with $\nu < 1/2$, then we plot the standard Stochastic Mechanics case of Nelson with $\upsilon = 1/2$, and at the left we plot a strong diffusion case where $\nu > 1/2$

Case 2 A Gaussian wave function

A Gaussian wave function is an analytic solution to Schrödinger's equation of the form (in 3 dimensions)

$$\psi(\mathbf{x},t) = \left(2\pi s_t^2\right)^{-3/4} e^{i\mathbf{u}\cdot\left(\mathbf{x}-\frac{1}{2}\mathbf{u}t\right)-(\mathbf{x}-\mathbf{u}t)^2/4s_t\sigma_0} \tag{125}$$

$$s_t = \sigma_0\left(1+it/2\sigma_0^2\right) \tag{126}$$

A nice visualization and of the Bohmian orbits for this and other wave functions together with an excellent elucidation of the theory is in Holland [115]. We find

$$S_Q(x,t) = -\frac{3}{2}\tan^{-1}(\frac{1}{2}t\sigma_0^2) + \mathbf{u}\cdot(\mathbf{x}-\frac{1}{2}\mathbf{u}t) + (\mathbf{x}-\mathbf{u}t)^2 t/8\sigma_0^2\left|s_t\right|^2 \tag{127}$$

$$R(x,t) = -\frac{3}{4}\ln(2\pi\left|s_t\right|^2) - (x-ut)^2/4\left|s_t\right|^2 \tag{128}$$

We calculate the drift term from (22).

$$\mathbf{b}(x,t,\nu) = 2\nu\nabla R(x,t) + \nabla S_Q(x,t) \tag{129}$$

$$\mathbf{b}(x,t,\nu) = -2\nu\left(\mathbf{x}-\mathbf{u}t\right)/2\left|s_t\right|^2 + \mathbf{u} + (\mathbf{x}-\mathbf{u}t)t/4\sigma_0^2\left|s_t\right|^2 \tag{130}$$

Now with this drift term, we can calculate the orbits for various values of ν by integrating the stochastic differential equation $d\mathbf{x} = \mathbf{b}(x,t,\nu)dt + d\mathbf{w}(t,\nu)$.

Simulations are shown in the figure 2. Note that these are x-y trajectory plots. The group velocity was 10 and the initial standard deviation was 1.

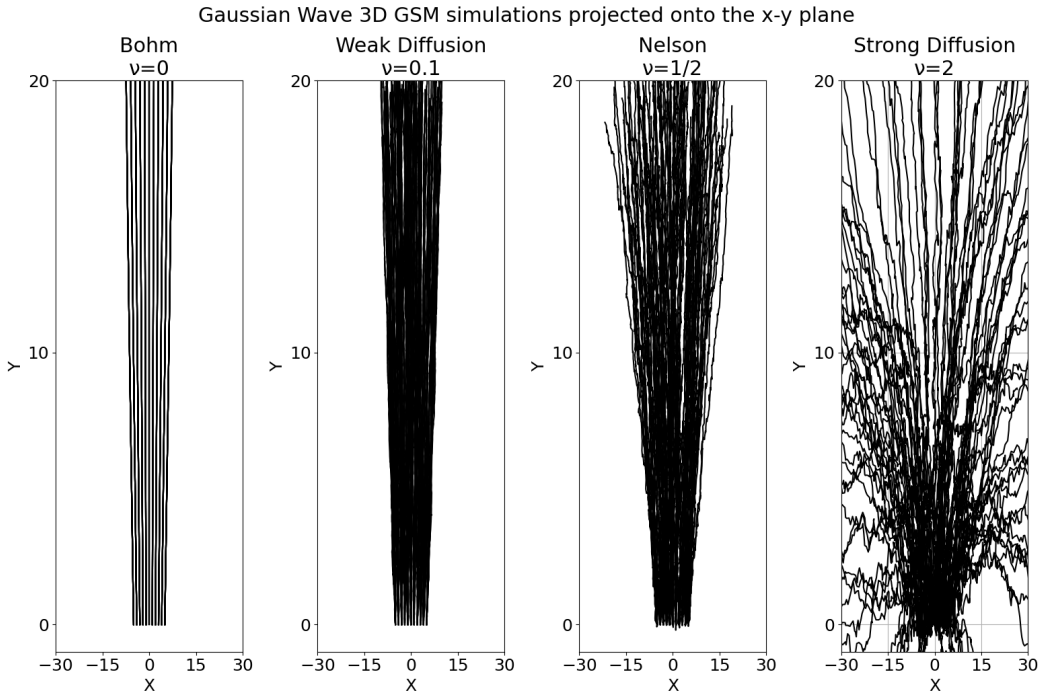

Figure 2. A Gaussian wave described in Case 2 is plotted here. We have plotted the trajectories projected onto the x-z plane. On the leftmost plot we see the Bohmian rays and there is no diffusion. The next three plots show increasing diffusion values. Although the trajectory plots look very different, when you do all the appropriate calculations to calculate the probability densities, they would all give the same result. We only have plotted results for real diffusion constants, but one can also have complex diffusion constants which still give the same probability densities.

Case 3 Two slit diffraction of a Gaussian beam

This case cannot be solved analytically, and therefore we use a numerical Schrödinger equation solver [113]. We launch a Gaussian wave towards a two-slit diffraction potential barrier, and then from the solution to this wave equation we generate various trajectories of GSM for this problem. Qmsolve can generate animations, but we can only show discrete times in the paper. The evolution of the wave function in time is illustrated in figure 3. The mass is the electron mass. The x and y limits are both $\pm 15\text{Å}$. The group velocity of the incident Gaussian wave is $80\text{Å}/\text{femtosecond}$ in the positive y direction. This is a velocity of 3.657 in atomic units. The standard deviation of the initial Gaussian is 1Å. The center to center slit separation is 2.5Å, and the width of each slit is $1/2\text{Å}$. The phase of wave function is encoded in the color hue.

Next we show ray tracing for rays starting at the two slits at about the time that the Gaussian wave is passing through them in Figure 4. We plot x vs y. This is a time-dependent simulation. You can think of the stochastic average density of the x-y trajectories as approximating the smeared probability distribution $\int_{t_1}^{t_2} \varrho(x, y, t) dt$ up to a normalization constant. In order to faithfully represent the probability density, we would need to increase the number of independent paths, and also to be precise, we would need to include rays starting from all possible initial positions at the start time, and weighted by their initial probability density. We added a color map to the Bohm orbits to make them more visible.

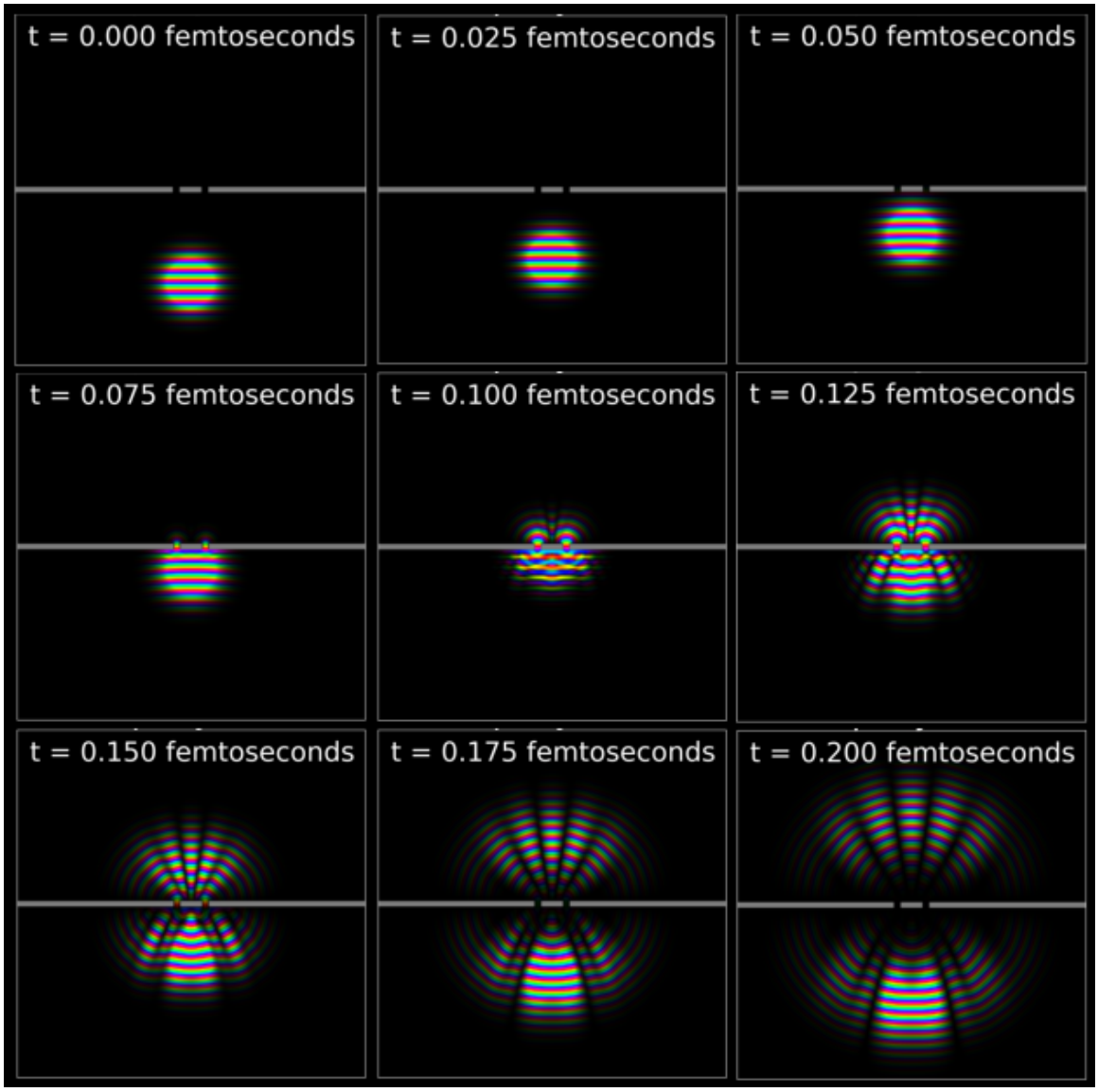

Figure 3. A Gaussian wave incident on a 2 Slit potential barrier as in Case 3

Case 4 Geometrical phase quantum waveguide of Lévy-LeBlonde

Quantum waveguide effects and associated phase delays, familiar from microwave and optical waveguides, have been a topic of interest in quantum physics [116, 117]. Here we present a GSM stochastic simulation for this process. Consider a particle with unit mass moving down a containing tube with square cross section such that ψ vanishes on the boundary of the tube. The wave function in this cases is:

$$\psi(x,y,z,t) = sin(n_x\pi x/a)sin(n_y\pi y/a)e^{ipz}e^{-iEt} \tag{131}$$

Defining a parameter $sign$ by

$$sign \equiv sgn(sin(n_x\pi x/a)sin(n_y\pi y/a)) \tag{132}$$

Then we can calculate R and S_Q by (where $m = 1$)

$$\psi(x,y,z,t) = |sin(n_x\pi x/a)sin(n_y\pi y/a)|\, e^{i\pi(1-sign)/2}e^{ipz}e^{-iEt} \tag{133}$$

$$R(x,y,z,t) = ln(|sin(n_x\pi x/a)sin(n_y\pi y/a)|) \tag{134}$$

$$R(x,y,z,t) = ln(|sin(n_x\pi x/a)|) + ln(|sin(n_y\pi y/a)|) \tag{135}$$

$$S_Q(x,y,z,t) = pz - Et + \pi\left(1 - sign\right)/2 \tag{136}$$

$$E = p^2/2m + \frac{\left(n_x^2 + n_y^2\right)\pi^2}{2ma^2} \tag{137}$$

$$\mathbf{b}(\mathbf{x},t,\nu) = 2\nu\nabla R(\mathbf{x},t) + \nabla S_Q(\mathbf{x},t) \tag{138}$$

$$\nabla R(\mathbf{x},t) = \frac{\pi}{a}\left[n_x cotan\left(n_x\pi x/a\right)\hat{x} + n_y cotan\left(n_y\pi x/a\right)\hat{y}\right] \tag{139}$$

The results of a simulation are shown in 5. The velocity used for this was 10 in AU units.

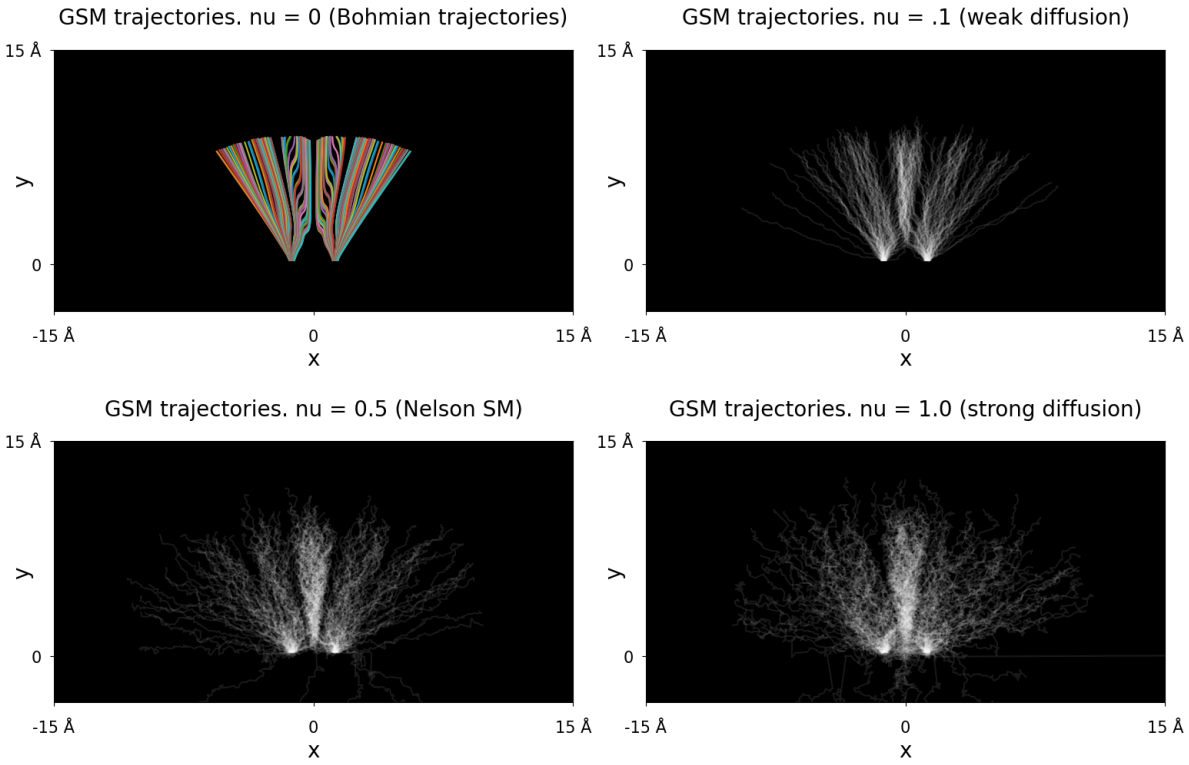

Figure 4. Ray tracing of GSM trajectories for 2 Slit diffraction as in Case 3

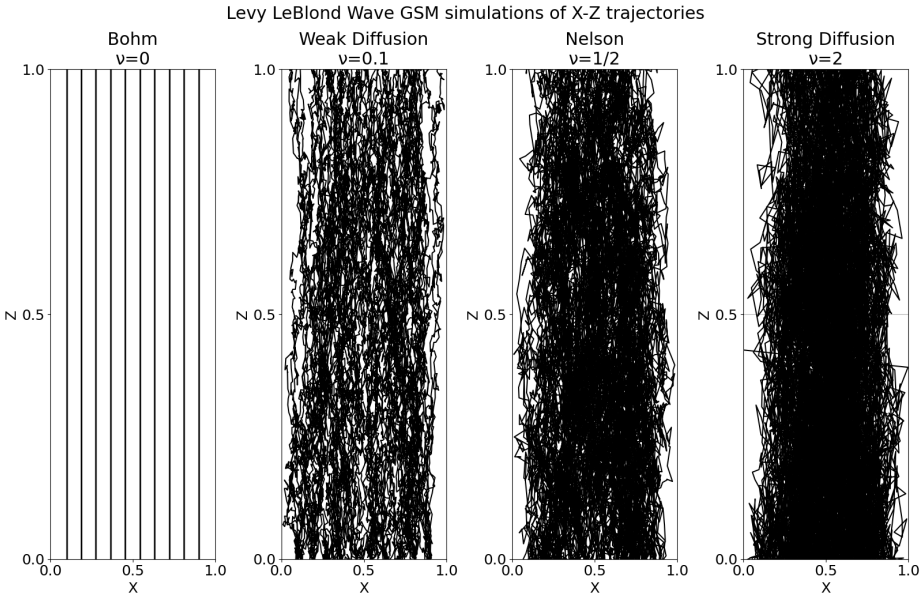

Figure 5. Lévy LeBlond GSM simulations. nx = ny = a =1, v=10, as in Case 4

X. SOME SPECULATIONS ABOUT MEASURING THE DIFFUSION CONSTANT

If the assertions about measurement limitations discussed in (II) are correct, then it is impossible to measure the diffusion constant in GSM. However, they are after all assertions, not proofs. I can think of two phenomena that might be places to look for a breakdown of this assertion.

Lattice confined fusion of deuterium One possible experimental application might be as an explanation for anomalously large fusion rates that are observed in experiments with low energy deuterium beams incident on Palladium and other

metals. After accounting for electron screening which enhances the fusion rates significantly, the extant theory still cannot explain the observed rates without some unusual additional assumptions [118]. D-D fusion is a very local phenomenon. The two fusing nuclei need to be within a Compton wavelength or so of one another in order to fuse. This is an experiment where the relative position of two quantum particles really matters critically. If we take a stochastic picture of this, then the two deuterons are both diffusing in GSM, and then the chance of two trajectories intersecting within a Compton wavelength of one another might be enhanced if the diffusion constant underlying the motion is large. If we assume that the diffusion constant in a vacuum is lower than that for deuterons moving in Palladium metal, this could explain the increased fusion rate. These ideas would require additional research, but could be topics for future publications. Although the fusion rates in these experiments are not useful for energy production yet, further research may reveal ways to increase the yields. They are quite reproducible now, and NASA has claimed that Fusion rocket engines have been prototyped, and that they have potential application for long distance space travel where chemical rockets emit too much mass be be useful [119–121].

Quantum behavior near event horizons and Hawking radiation Hawking radiation might be affected by the value of the diffusion constant in GSM. A crude explanation of Hawking radiation is that when a particle-anti-particle pair is created as a vacuum fluctuation right at the event horizon of a black hole, the two particles might be on opposite sides of the event horizon so that one particle can escape the black hole while the other is trapped. Accepting this crude picture, let us consider that the two particles are each undergoing a stochastic motion described by GSM after the creation event. As we have seen, the sample

paths of the particles are diffusing in GSM. If the diffusion constant ν is large, then the probability that the escaping particle will cross over into the interior of the black hole's event horizon will be increased, and then this particle that started off outside the event horizon has an increased chance of diffusing inside of it and thus being presumable trapped by the black hole. So we would expect that the Hawking radiation might be reduced for larger diffusion constant, or at least affected by it. The same argument could be made for Unruh radiation. In the limit of very large diffusion constant, we might expect that both Hawking radiation and Unruh radiation would tend to zero. So far neither radiation has been observed in nature. We might also expect that an isolated black hole would have a lower diffusion constant than a black hole with a significant accretion disk.

XI. CONCLUSION

Stochastic mechanics provides a thorough derivation of the non-commutative algebra of Heisenberg. The fact that an imaginary diffusion constant is preferred adds theoretical evidence that complex space-time is related to a deeper understanding of quantum mechanics. It's amazing that Einstein not only declared the need for a deeper theory but also that some of his last papers dealt with complex space-time pointing the way. Not only that, but he also invented the theory of Brownian motion, which is the basis of stochastic mechanics, and he made critical insights into quantum theory. It's also remarkable that Heisenberg's non-commuting operators should point to the same conclusion, and that his postulate of wave function collapse along with his uncertainty principle plays

a key role in it. Nelson was perhaps correct that there is a preferred value for the diffusion constant, but it might not be $\hbar/2m$, but rather it could well be $\pm i\hbar/2m$. This leads to the proposition that we are possibly living in a complex space-time, but our normal sense perception has evolved us to be aware of only the real subspace. But even if this is not the case, having GSM to provide a variety of diffusion models with which to model quantum mechanics has benefits in unifying several branches of quantum interpretations and stimulating efforts to someday-somehow perhaps measure the diffusion constant. Other models that are supported in GSM are Bohmian mechanics, diffusion parameters varying with position, complex time models, and weak and strong diffusion models, and complex diffusion models where the complex diffusion constant differs from the Heisenberg values of $\pm i\hbar/2m$.

Appendix A: Compendium of useful formulae for stochastic mechanics

(We use the notation of [41], for the most part, in this appendix)

Fokker-Planck equation or Kolmogorov forward eqn.: $$\frac{\partial \rho}{\partial t} = -div(b\rho) + \nu \triangle \rho \tag{A1}$$

Schrödinger wave function: $$\psi = e^{R+iS_Q} \tag{A2}$$

Probabilty density: $$\rho = \psi^* \psi = e^{2R} \tag{A3}$$

Fokker-Planck equation time reversed or Kolmogorov backward eqn.: $$\frac{\partial \rho}{\partial t} = -div(b_* \rho) - \nu \triangle \rho \tag{A4}$$

Backward velocity: $$b_* = b - 2\nu (grad\, \rho)/\rho \tag{A5}$$

Osmotic velocity: $$u = \frac{b - b_*}{2} = \nu \frac{grad\, \rho}{\rho} = 2\nu grad(R) \tag{A6}$$

Current velocity: $$\mathrm{v} = \frac{b + b_*}{2} = 2\nu grad(S_N) \tag{A7}$$

Equation of continuity: $\frac{\partial \rho}{\partial t} = -div(\mathsf{v}\rho)$ (A8)

Time deriviative of osmotic velocity: $\frac{\partial \mathsf{u}}{\partial \mathsf{t}} = -\nu grad\, div\, \mathsf{v} - grad\, \mathsf{v} \cdot u$ (A9)

Forward time derivative formula: $Df(x(t),t) = \left(\frac{\partial}{\partial t} + b \cdot \nabla + \nu\triangle\right) f(x(t),t)$ (A10)

Backward time derivative formula: $D_* f(x(t),t) = \left(\frac{\partial}{\partial t} + b_* \cdot \nabla - \nu\triangle\right) f(x(t),t)$ (A11)

Mean acceleration of Nelson: $\mathsf{a}(t) = \frac{DD_* + D_*D}{2} x(t)$ (A12)

Markov transition function:

$$P(x,t;y,s) = \lim_{d^3x \to 0} \frac{1}{d^3x} P(x(t) \in d^3x \mid x(s) = y), \quad t > s \tag{A13}$$

Chapman-Kolmogorov equation:

$$P(x,t;y,r) = \int P(x,t;z,s)P(z,s;y,r)d^3z, \ for\, times\, t > s > r \tag{A14}$$

Continuity of paths: $\lim_{t \downarrow s} P(x,t;y,s) = \delta^3(x-y)$ (A15)

Time evolution of the density function: $\rho(x,t) = \int P(x,t;z,s)\rho(z,s)d^3z, \ t > s$ (A16)

Forward equation for Markov transition (§3, [122]):

$$\left[\frac{\partial}{\partial t} + \nabla_x \cdot b(x,t) + b(x,t) \cdot \nabla_x - \nu\triangle_x\right] P(x,t;y,s) = 0, \ t > s \tag{A17}$$

Backward equation for Markov transition (§3, [122]):

$$\left[\frac{\partial}{\partial s} + b(y,s) \cdot \nabla_y + \nu\triangle_y\right] P(x,t;y,s) = 0, \ t > s \tag{A18}$$

Integration by parts when justified (§13, [2]):

$$\int_{-\infty}^{\infty} E(Df(x(t),t)g(x(t),t))dt = -\int_{-\infty}^{\infty} E(f(x(t),t)D_*g(x(t),t))dt \tag{A19}$$

Expectation value of a function of multiple positions at different times:

$$E(F(x(t_n),...,x(t_0))) =$$
$$\int F(x_n,...,x_0)P(x_n,t_n;x_{n-1},t_{n-1})\cdots P(x_1,t_1;x_0,t_0)\rho(x_0)\times$$
$$\times dx_0...dx_n,\ t_0 \le t_1 \le ... \le t_n \qquad \text{(A20)}$$

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