# On the physical parity transformation and antiparticles

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**Abstract**

It was argued a long time ago by T.D. Lee and G.C. Wick that there is a basis where the usual CP (charge-parity) transformation becomes a parity transformation. Indeed, at the quantum level all fields are real representations of the group of symmetries, because CP (charge-parity) is a linear transformation. From a canonical quantization point of view, there are no complex-conjugate representations, and all particles are their own antiparticles. The Poincare group is the quotient group of the group of symmetries by the normal subgroup group of internal symmetries (which may include the charge conjugation). Thus, CP is a candidate for a physical parity transformation (included in the Poincare group). We check explicitly what this implies for the Standard Model and discuss the implications for Left-Right symmetric models and models featuring a CP order-4 symmetry.

*...if one performs a mirror reflection and converts all matter into antimatter, then physical laws remain unchanged. This combined transformation which leaves physical laws unchanged could thus be defined as the true mirror reflection process. According to this definition, mirror reflection symmetry is restored.[...]*
*Of course the question remains why it is necessary in order to have symmetry, to combine the operation of switching matter and antimatter with a mirror reflection. The answer to such a question can only be obtained through a deeper understanding of the relationship between matter and antimatter. No such understanding is in sight today.*

*C. N. Yang (1961) [1]*

## 1 Introduction

It was argued a long time ago [2], that there is a basis where CP becomes a parity transformation (where CP and P are the transformations usually identified as charge-parity and parity transformations). Majorana spinors and the Majorana basis for the Dirac matrices were used. It was explicitly said: "*Any non-Hermitian field can always be decomposed into a sum of two Hermitian fields. The use of Hermitian fields simplifies some of the general discussions, and there is no loss of generality*". In such a basis there are no fields which are complex-conjugate representations (usually associated with the charge conjugation), all particles are their own antiparticles (from a canonical quantization point of view) and the difference between CP and P dissolves.

The authors of reference [2] were well aware that Weyl spinors can be used to parametrize a Majorana spinor as much as the Majorana spinor can be used to parametrize a Weyl spinor, stating "*the use of a Hermitian field to describe a two-component spin*$-\frac{1}{2}$ *particle was made by E. Majorana*". The advantage of the Majorana spinor is that it is a representation of parity, while the Weyl spinor is not, thus Majorana spinors are relevant when studying parity violation.

Note that the Poincare group is the quotient group of the group of symmetries by the normal subgroup of internal symmetries. The charge-conjugation is not a distinguished transformation in Quantum Field Theory, and depending on the Lagrangian it may not even make sense to define it (this is the case of the Standard Model, as we will see). Thus, CP is a candidate for a physical parity transformation (included in the Poincare group) [2, See Definition 3 of parity]. If charge conjugation is defined, it is part of the group of internal symmetries and there will be parity transformations which are charge conjugating and parity transformations which are not charge conjugating. If the group of internal symmetries is linearly reductive (as is usually the case), we can without loss of generality [3] use the composite operators which are invariant under the internal symmetries to determine whether or not parity is conserved.

However, reference [2] is ambiguous about the role of fields which are complex representations (where particles are not their anti-particles) at the quantum level and it does not address the Standard Model (which was still being developed at the time). In this paper, we address these questions very explicitly. In particular, it is clear nowadays that fields which are complex representations play no role at the quantum level (except as an intermediate step in defining the real representations).

The notion of "generalized" parity has been used since then, but only in part as "generalized" CP or "generalized" P without ever relating CP and P [4]. The reason is that in order to relate CP and P we need to define the fields as real representations, e.g. use Majorana spinors instead of Weyl spinors [5]. Note that classical fields can indeed be defined as complex representations, but these classical fields cannot be quantum fields. At the quantum level, all fields are real representations of the group of symmetries, because CP is a linear transformation. In the classical Action (used in the path integral quantization), if we would consider the fields as complex representations then CP would be necessarily anti-linear [6] instead of a linear transformation as it should be [7].

This happens even if the vacuum is treated as a complex representation in canonical quantization, since the fields are self-adjoint operators and not quantum states. More generally, a quantum system is defined by the real numbers which are the expectation values of self-adjoint operators representing observables; thus a physical transformation of the system transforms the self-adjoint operators into other self-adjoint operators and the linear space of self-adjoint operators is a real representation space (otherwise the condition of self-adjointness is not preserved).

Of course, we can do linear combinations of the self-adjoint fields using complex coefficients, but the group action always preserves the condition of self-adjointness (so they are real representations, since a real structure is preserved [8]). The situation is analogous to the fact that in a complex vector space, the adjoint representation of a compact Lie group preserves a real structure and thus it is equivalent to a real representation.

This implies that a representation theory which is consistent at the quantum level for the Standard Model is more complicated than what appears in most introductory books, for instance: complex irreducible representations of the group $G \times H$ are a direct product of complex irreducible representations of $G$ and of $H$ respectively, while this does not happen for real irreducible representations. Thus, addressing the Standard Model with real representations is not trivial and (to the knowledge of the author) it was not done before. References such as [2, 3, 9, 10], where it is understood that real representations are the most suitable for general discussions are rare in the physics literature, while literature making sound claims [11] which are explicitly false (and do not even make sense at the quantum level) is abundant. Even books dedicated to group theory in physics [12] only discuss classical fields which are complex representations, because they are easier to discuss [1].

On the other hand, with complex representations we cannot define all linear operators available in Quantum Field Theory, some of these operators have important phenomenological consequences, for instance: the approximate custodial symmetry of the Higgs potential [9, 13, 14], pseudo-goldstone bosons [9], Majorana neutrinos [15], or the parity transformation [2]. Fortunately, there is a well known map from the complex to the real irreducible representations of any Lie group [2], so that using real representations requires just one correction with respect to the complex representations.

The fact that all particles are their own antiparticles (from a canonical quantization point of view), does not exclude the existence of conserved quantum numbers. Namely, global $U(1)$ symmetries corresponding to the lepton or baryon quantum numbers may or may not be conserved, depending on the Lagrangian [25] and the path-integral measure [26]. Note that in the Standard Model, the baryon number is anomalous (it is not conserved by the path-integral measure) which implies that the baryon asymmetry of the Universe is a non-perturbative problem and thus necessarily complicated [26] and beyond the scope of this paper.

In section 2, we review why the quantum fields are self-adjoint operators and thus real representations, despite that the vacuum may be an element of a complex Hilbert space. In section 3, we check explicitly what this implies for the Standard Model. In section 4, we state some consequences for extensions to the Standard Model.

[1] In the particular case of the book [12], the reason invoked to avoid real representations is that "Schur's lemma does not hold for a real representation." which is misleading, what it happens is that Schur's lemma is simpler for complex representations than for real representations. This is one of the many examples where Wikipedia is more trustworthy than the physics literature.

[2] The map is well known in the case of finite-dimensional representations [8, 16–19] and it was first established by Ellie Cartan in 1914. The analogous map for unitary infinite-dimensional representations was established in reference [20]. The relation between real and complex algebras was also discussed in the references [21–24].

## 2 Majorana spinors in canonical quantization and antiparticles

Given a real Hilbert space $V$ with inner product $<,>$ we can always construct an associated complex Clifford C* algebra [22]. Let $C(V)$ be the associated complex Clifford algebra, i.e. $C(V)$ is a unital associative complex algebra such that there is an injective linear map $a : V \to C(V)$, verifying $a^2(v) =< v, v > 1$ and $a^*(v) = a(v)$, $C(V)$ admits a unique involution $*$ and it is generated by the operators $a(v)$, for all $v \in V$. The algebra has a natural norm given by $||a(v_1)...a(v_n)|| \equiv \sqrt{||a(v_n)...a(v_1)a(v_1)...a(v_n)||} = ||v_1||...||v_n|| \cdot ||1||$. The C* algebra $C[V]$ is the completion $C(V)$ with respect to its natural norm.

Consider now a skew-symmetric operator $J$ acting on the Hilbert space $V$, with $J^2 = -1$. The vacuum functional verifies $< o\, a(v+iJv) >= 0$ for any operator $o$ and $v \in V$. The condition defining the vacuum involves complex numbers, despite that the Hilbert space $V$ is real. The operator $a(v + iJv)$ is an annihilation operator, while $a(v - iJv)$ is a creation operator. The operators $a(v) = a(v+iJv)+a(v-iJv)$ are self-adjoint and represent a particle which is its own antiparticle, for any $v \in V$. Thus we see that it is always possible to do the canonical fermionic quantization of a real Hilbert space, with the corresponding operators $a(v)$ being self-adjoint.

For bosons, we have a similar situation, except that a commutation relation holds $[a(v), a(w)] =< v, Jw > i$ instead of $\{a(v), a(w)\} =< v, w > 1$; a sympletic product [27–29] $< v, Jw > i$ replaces the inner product $< v, w > 1$; and the space $V$ is a real symplectic space instead of a real Hilbert space. The operators $a(v) = a(v + iJv) + a(v - iJv)$ are again self-adjoint and represent a particle which is its own antiparticle.

We stress once again that we can do linear combinations of the self-adjoint fields using complex coefficients, but the action of the group of symmetries always preserves the condition of self-adjointness (so they are real representations, since a real structure is preserved [8]), that is $a(v) \to a(Tv)$ for some real operator $T$ acting on the real space $V$.

We can then relate the canonical quantization with the path integral quantization. In the path integral, the real fermionic fields (Majorana spinors) are Grassmann (anti-commuting) fields while the bosons are classical (commuting) fields.

## 3 Majorana spinors in the Standard Model

The group of gauge symmetries is the semi-direct product $SU(2)_L \times U(1)_Y \times SU(3)_C$. Consider[3] for the moment just the weak-Higgs sector: there are one weak $SU(2)_L$ Higgs doublet $\phi$ and

[3]We follow the conventions used in the reference [30] for the signs and constants.

the gauge field $W^j_\mu$ with $j,k,l=1,2,3$. The Lagrangian is:

$$\begin{aligned}
\mathcal{L}_{\mathrm{W}} &\equiv ((D^\mu\phi)^\dagger(D_\mu\phi)-V(\phi)-\frac{1}{4}W^j_{\mu\nu}W^{j\mu\nu}-\frac{1}{4}B_{\mu\nu}B^{\mu\nu},\\
D_\mu\phi &\equiv \left(\partial_\mu+igW^j_\mu\frac{\tau_j}{2}+ig'\frac{\sigma_3}{2}B^\mu\right)\phi\\
W^j_{\mu\nu} &\equiv -\frac{i}{g}\mathrm{tr}([D_\mu,D_\nu]\tau^j)=\partial_\mu W^j_\nu-\partial_\nu W^j_\mu-g\epsilon^{jkl}W^k_\mu W^l_\nu,\\
B_{\mu\nu} &\equiv -\frac{i}{g'}\mathrm{tr}([D_\mu,D_\nu]\sigma_3)=\partial_\mu B_\nu-\partial_\nu B_\mu
\end{aligned}$$

where $V(\phi)$ is the Higgs Potential, $D_\mu$ is the covariant derivative dependent on the gauge fields $W^j_\mu$ and $B_\mu$. The gauge field strength tensors are $W^j_{\mu\nu}$ and $B_\mu$, while $g$ and $g'$ are coupling constants, $\epsilon^{jkl}$ is the Levi-Civita tensor. The Pauli matrices $i\tau_j$ are the generators of the $SU(2)_L$ gauge group and $i\sigma_j$ are the generators of the $SU(2)$ custodial group. The Higgs field verifies a Majorana condition $i\sigma_2\phi=i\tau_2\phi^*$ and thus it is a real representation of the symmetries.

Consider now a Majorana spinor field which is a fermionic (Grassmannian) field $Q_L$ it has $3\times3\times2\times4=72$ real components, containing 3 flavours, it is a triplet of $SU(3)_C$ (with the imaginary unit replaced by $i\gamma_5$), it is an $SU(2)_L$ doublet and it verifies $i\gamma_5 Q_L=i\sigma_3 Q_L$, and the Majorana condition $i\sigma_2 Q_L=i\tau_2 Q_L^*$. Introduce also Majorana fermions $d_R$, $u_R$ which have 36 real components each, containing 3 flavours, they are anti-triplets of $SU(3)_C$ (with the imaginary unit replaced by $i\gamma_5$) and invariant under $SU(2)_L$. We set the hyper-charges of the gauge symmetry $U(1)_Y$ as $Q_L(1/6_Y)$, $i\gamma_0 d_R(1/3_Y)$, $i\gamma^0 u_R(-2/3_Y)$, i.e. for $\phi\to e^{i\frac{\sigma_3}{2}\vartheta}\phi$ then $Q_L\to e^{i\gamma_5\frac{\vartheta}{6}}Q_L$ and $u_R(x)\to e^{-i\gamma_5\frac{2\vartheta}{3}}u_R(x)$. Hence, these are quarks.

The most general $SU(2)_L$ gauge invariant products of $\phi$ and $Q_L$ are linear combinations of $\overline{Q_L}\phi$, $\overline{Q_L}i\sigma_2\phi$[4]. The most general gauge-invariant form for the Yukawa couplings with the quarks is then

$$\begin{aligned}
-\frac{1}{2}\mathcal{L}_{\mathrm{Y}} &= Q_L^\dagger\gamma^0\ M_d\phi\ d_R+Q_L^\dagger\gamma^0\ i\sigma_2 M_u\phi\ u_R\\
M_w &\equiv M_{w\,r}-M_{w\,i}i\gamma_5
\end{aligned}$$

with $M_{wr}$ and $M_{wi}$ real $3\times3$ matrices, where $w=u,d$.

The matrices $M_d\equiv U_L\mathrm{diag}(m_d,m_s,m_b)U_R^{d\dagger}$ and $M_u\equiv U_L V^\dagger\mathrm{diag}(m_u,m_c,m_t)U_R^{u\dagger}$ are the quark mass matrices. The conventional Cabibbo–Kobayashi–Maskawa (CKM) matrix (except with $-i\gamma_5$ replacing the imaginary unit) is given by $V$.

The Lagrangian for the quarks is then:

$$\mathcal{L}_{\mathrm{Q}}=iQ_L^\dagger\gamma^0\gamma^\mu D_\mu\ Q_L+iu_R^\dagger\gamma^0\gamma^\mu D_\mu\ u_R+id_R^\dagger\gamma^0\gamma^\mu D_\mu\ d_R-\frac{1}{4}G^j_{\mu\nu}G^{j\mu\nu}$$

[4]the basis of matrices commuting with the generators of $SU(2)_L$ is $\{1,i\sigma_j\}$, with $j,=1,2,3$, for a total of 4 matrices. Due to the projector in $Q_L$, we must divide the total by 2 which leaves us with 2 linearly independent products.

Where $D_\mu$ is the gauge-covariant derivative with gauge couplings corresponding to each fermion, and $G_{\mu\nu}$ is the $SU(3)_C$ gauge field strength tensor corresponding to the gauge field $G_\mu^a$ (gluons, with $a = 1, ..., 8$).

The lepton sector with three right handed neutrinos is analogous in the absence of Majorana masses. There is a global symmetry $U(1)_{nb} \times U(1)_{nl}$ related to the baryon and lepton (no Majorana masses) numbers.

The outer automorphism group of $SU(3)$ or $U(1)_Y$ is $Z_2$, while the outer automorphism group of $SU(2)_L$ is the trivial group. The parity symmetry is violated by the CKM matrix. Promoting the CKM matrix to a background field, the Lagrangian is invariant under the background symmetry given by the semi-direct product $SU(2)_L \times (SU(3)_C \times U(1)_Y) \rtimes Z_4$ and the $Z_4$ group is generated by the (generalized) parity reversal transformation $\phi(t, \vec{x}) \to i\sigma_2\phi(t, -\vec{x})$, $Q_L(t, \vec{x}) \to -\sigma_2\gamma_0 Q_L(t, -\vec{x})$, $u_R(t, \vec{x}) \to i\gamma_0 u_R(t, -\vec{x})$, $d_R(t, \vec{x}) \to i\gamma_0 d_R(t, -\vec{x})$. The gauge fields $B_\mu$ and $G_\mu^a$ transform under parity reversal according to $B_\mu(t, \vec{x}) \to -B_\mu(t, -\vec{x})$ and $G_\mu^a(t, \vec{x}) \to s^a G_\mu^a(t, -\vec{x})$ (with $s^1, s^3, s^4, s^6, s^8 = -1$ and $s^2, s^5, s^7 = 1$ in the usual basis for the Gell-Mann matrices).

Note that this is a natural parity transformation, since it acts on the Majorana spinors with $i\gamma_0$ and $\vec{x} \to -\vec{x}$. What it is uncommon is the fact that the generator of the gauge group $U(1)_Y$ is $i\gamma_5$, but this is not a transformation of the Lorentz group so it shouldn't affect our definition of parity. As for the Higgs scalar field, we have a generalized parity transformation of order-4 and the generator of the gauge group $U(1)_Y$ is also not a transformation of the Lorentz group.

Since the parity transformation is order-4 for $u_R$, then it is order-4 after Electroweak "symmetry breaking", since $u_R$ is $SU(2)_L$-gauge-invariant. See reference [10] for more details on the Electroweak "symmetry breaking"[5].

# 4 Consequences for extensions to the Standard Model

Since CP and P can be both identified as parity transformations, then parity is broken only by the CKM matrix in the Standard Model. Thus, the popular Left-Right symmetric models [4, 31] accomplish the same as any other model where CP is broken spontaneously and not explicitly, at least with respect to the parity violation. The Left-Right symmetric models may have many virtues, perhaps they can be motivated using the definition of charge conjugation (which is not defined in the Standard Model), but clearly the theoretical motivation usually stated for these models is misleading with respect to parity violation. Such theoretical motivation should also take into account that spontaneous parity violation can be described as a particular case of explicit parity violation [10].

Also, the parity transformation is order-4 both on the Higgs field and in the fermion fields. When considering the $SU(2)_L$-gauge-invariant operators (or after electroweak "symmetry breaking"), the parity transformation will still be order-4 on the fermions, despite that it acts trivially

[5] The two-Higgs-doublet model was addressed in reference [10], but the Standard Model case can be easily be inferred from there and the references therein.

on the Higgs boson. Thus the claim that models of the Higgs sector featuring an order-4 CP symmetry, broken to order-2 CP symmetry (in the Higgs sector) after electroweak "symmetry breaking" are unique with respect to its CP properties [32], since the Standard Model already has these features. That does not imply that these models do not have (other) many virtues.

This also contradicts a previous claim that depending on how the neutrinos acquire a mass, the double-cover of the Lorentz group could be $Pin(1,3)$ or $Pin(3,1)$ [33], but in fact it must be $Pin(3,1)$ since the parity is order-4 in the right-handed quarks (and thus in the right-handed neutrinos if they are introduced)

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