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Some Results on New Preconditioned Generalized Mixed-Type Splitting Iterative Methods

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***Abstract*—**In this paper, we present new preconditioned generalized mixed-type splitting (GMTS) methods for solving weighted linear least square problems. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give a numerical example to confirm our theoretical results.

***Keywords*—**Preconditioned, GMTS method, linear system, convergence, comparison.

I. Introduction

SOMETIMES, one has to solve a nonsingular linear system as

$$Hy = f\ , \tag{1}$$

where

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix}$$

is an invertible matrix with

$$B = \left(b_{ij}\right)_{p\times p},\ C = \left(c_{ij}\right)_{q\times q},\ L = \left(l_{ij}\right)_{q\times p},\ U = \left(u_{ij}\right)_{p\times q}.$$

Throughout the paper, we consider the following decomposition for the matrix H, $H = \hat{D} - \hat{L} - \hat{U}$, in which

$$\hat{D} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\ \hat{L} = \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix},\ \hat{U} = \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}. \tag{2}$$

In [1], authors presented a generalized mixed-type splitting (GMTS) iterative method and preconditioned generalized mixed-type splitting (PGMTS) iterative methods to solve systems of linear equations (1).They showed that the PGMTS methods converge faster than the GMTS method, whenever the GMTS method is convergent.

This paper is organized as follows. In Section II, we give some important definitions and the known results as the preliminaries of the paper. In Section III, we propose three preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. In Section IV, we give an example to confirm our theoretical results.

II. Preliminaries

Definition 1[2] $A \in R^{n\times n}$ is called a Z-matrix if $a_{ij} \le 0$ for $i, j = 1,2,...,n(i \ne j)$.

Definition 2[2] Let A be a Z-matrix with positive diagonal elements. Then the matrix A is called an M-matrix if A is nonsingular and $A^{-1} \ge 0$.

Definition 3[3] The splitting $A = M - N$ is called

(1) a regular splitting of A if $M^{-1} \ge 0$ and $N \ge 0$;

(2) a nonnegative splitting of A if $M^{-1} \ge 0$, $M^{-1}N \ge 0$ and $NM^{-1} \ge 0$;

(3) a weak nonnegative splitting of A if $M^{-1} \ge 0$ and either $M^{-1}N \ge 0$ (the first type) or $NM^{-1} \ge 0$ (the second type);

(4) a convergent splitting of A if $\rho(M^{-1}N) < 1$.

Lemma 1 [1] Let A be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.

Lemma 2 [4] Let $A = M - N$ be a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative.

Lemma 3 [5] Let matrix $A = (a_{ij})_{n\times n}$ be given such that

(1) $a_{ij} \le 0$ for $i, j = 1,2,...,n(i \ne j)$

(2) A is nonsingular.

(3) $A^{-1} \ge 0$.

Then,

(4) $a_{ii} > 0$ for $i = 1,2,...,n$, i.e., A is an M-matrix.

(5) $\rho(B) < 1$ where $B = I - D^{-1}A$, where $D = diag\{a_{11},...,a_{nn}\}$.

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Lemma 4 [3] Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak nonnegative splittings of A, where $A^{-1} \geq 0$, if $M_1^{-1} \geq M_2^{-1}$ then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.

III. The Preconditioned GMTS Method and the Comparison Results

Consider the linear system (1), the generalized mixed-type splitting (GMTS) iterative method is given as follows:

$$(\hat{D} + D_1 + L_1 - \hat{L})y^{(k+1)} = (D_1 + L_1 + \hat{U})y^{(k)} + f \quad (3)$$

where $\hat{D}$, $\hat{L}$ and $\hat{U}$ are defined by (2), and D_1 is an auxiliary nonnegative block diagonal matrix, L_1 is an auxiliary strictly nonnegative block lower triangular matrix such that $0 \leq D_1 \leq \hat{D}$ and $0 \leq L_1 \leq \hat{L}$. Evidently, the iteration matrix of the GMTS iterative method is given as follow:

$$T = (\hat{D} + D_1 + L_1 - \hat{L})^{-1}(D_1 + L_1 + \hat{U}).$$

In this paper, we propose the preconditioners as follow,

$$P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \; i = 1, 2, \quad (4)$$

where

$$S_1 = \begin{pmatrix} 0 & \alpha_2 + b_{12} & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & \alpha_p + b_{p-1,p} \\ 0 & 0 & ... & 0 & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ \beta_2 + b_{21} & 0 & ... & 0 & 0 \\ 0 & \beta_3 + b_{32} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \beta_p + b_{p,p-1} & 0 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & \gamma_2 + c_{12} & 0 & ... & 0 \\ 0 & 0 & \gamma_3 + c_{23} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \gamma_q + c_{q-1,q} \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ \delta_2 + c_{21} & 0 & ... & 0 & 0 \\ 0 & \delta_3 + c_{32} & ... & 0 & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & ... & \delta_q + c_{q,q-1} & 0 \end{pmatrix}.$$

Let us consider the corresponding splitting for the preconditioned GMTS (PGMTS) method, that is the generalized mixed-type splitting for the $\bar{H}_i = P_i^* H = \bar{M}_i - \bar{N}_i$, where

$$\bar{M}_i = \hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*, \; \bar{N}_i = \bar{D}_1 + \bar{L}_1 + \hat{U}_i^* \text{ and}$$

$$\hat{D}_i^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \; \hat{L}_i^* = \begin{pmatrix} 0 & 0 \\ -L_i^* & 0 \end{pmatrix}, \; \hat{U}_i^* = \begin{pmatrix} B_i^* & -U_i^* \\ 0 & C_i^* \end{pmatrix}, \; i = 1, 2.$$

The iterative matrix of the preconditioned GMTS method is

$$T_i^* = (\hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*)^{-1}(\bar{D}_1 + \bar{L}_1 + \hat{U}_i^*).$$

Now, we show that in the case that the GMTS converges, the preconditioned GMTS methods converge faster.

Lemma 5[1] Assume that $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$ and H in (1) is irreducible. If D_1 is nonsingular, then the iteration matrix of the GMTS method is irreducible.

Lemma 6[1] Assume that $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, then the corresponding splitting of GMTS method is a regular splitting for the matrix H.

Similar to the proof of Lemma 6, we can prove the following lemmas.

Lemma 7 Assume that $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$,

$$-b_{i-1,i} < \alpha_i < \frac{b_{i-1,i}}{1 - b_{ii}} - b_{i-1,i}, \quad -c_{j-1,j} < \gamma_j < \frac{c_{j-1,j}}{1 - c_{jj}} - c_{j-1,j},$$

for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q\}$. Then the corresponding splitting of PGMTS method is a regular splitting for the matrix $P_1^* H$.

Lemma 8 Assume that $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$,

$$-b_{i,i-1} < \beta_i < \frac{b_{i,i-1}}{1 - b_{i-1,i-1}} - b_{i,i-1}, \; -c_{j,j-1} < \delta_j < \frac{c_{j,j-1}}{1 - c_{j-1,j-1}} - c_{j,j-1},$$

for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q\}$. Then the corresponding splitting of PGMTS method is a regular splitting for the matrix $P_2^* H$.

Theorem 1 Let H be an M-matrix and

$$-b_{i-1,i} < \alpha_i < \frac{b_{i-1,i}}{1 - b_{ii}} - b_{i-1,i}, \quad -c_{j-1,j} < \gamma_j < \frac{c_{j-1,j}}{1 - c_{jj}} - c_{j-1,j},$$

for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., p-1\}$. Then $P_1^* H$ is an M-matrix.

Proof. Consider the following splitting for H, $H = M_1 - N_1$, where $M_1 = (P_1^*)^{-1}$ and $N_1 = (P_1^*)^{-1}(\hat{L}^* + \hat{U}^*)$, where $\hat{L}^* = \begin{pmatrix} 0 & 0 \\ -L_1^* & 0 \end{pmatrix}$, and $\hat{U}^* = \begin{pmatrix} B_1^* & -U_1^* \\ 0 & C_1^* \end{pmatrix}$.

It is easy to see that $M_1^{-1}N_1 = \hat{L}^* + \hat{U}^*$ and $M_1^{-1} \ge 0$. So we can get that $H = M_1 - N_1$ is a weak nonnegative splitting of the first type. By the assumption H is an M-matrix, hence Lemma 1 implies that $\rho(M_1^{-1}N_1) < 1$. Let us assume that $P_i^* H = I - \hat{L}^* - \hat{U}^*$, using the fact that $\rho(\hat{L}^* + \hat{U}^*) = \rho(M_1^{-1}N_1) < 1$, by Lemma 2 and Lemma 3, the result follows immediately.

Similar to the proof of Theorem 1, we can prove the following theorem.

Theorem 2 Let H be an M-matrix and

$$-b_{i,i-1} < \beta_i < \frac{b_{i,i-1}}{1-b_{i-1,i-1}} - b_{i,i-1},\ -c_{j.j-1} < \delta_j < \frac{c_{j,j-1}}{1-c_{j-1,j-1}} - c_{j,j-1},$$

for some $i \in \{2,3,\dots,p\}$, $j \in \{2,3,\dots,q\}$. Then $P_2^* H$ is an M-matrix.

Now, we will show that in the case that the GMTS converges; the preconditioned GMTS methods converge faster.

Theorem 3 Let T and T_1^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix H is irreducible, $L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 \le D_1 \le \hat{D}$, $0 \le \bar{D}_1 \le \hat{D}_1^*$, $0 \le L_1 \le \hat{L}$, $0 \le \bar{L}_1 \le \hat{L}_1^*$, $b_{i-1,i} > 0$, $c_{j-1,j} > 0$, for some $i \in \{2,3,\dots,p\}$, $j \in \{2,3,\dots,q\}$,

$$-b_{i-1,i} < \alpha_i < \frac{b_{i-1,i}}{1-b_{ii}} - b_{i-1,i},\quad -c_{j-1,j} < \gamma_j < \frac{c_{j-1,j}}{1-c_{jj}} - c_{j-1,j},$$

for some $i \in \{2,3,\dots,p\}$, $j \in \{2,3,\dots,q-1\}$.

If $\rho(T) < 1$, $\bar{D}_1 \le D_1$ and $\bar{L}_1 \le L_1$, then $\rho(T_1^*) \le \rho(T)$.

Proof. As the matrix H is irreducible, so the $P_1^* H$ is irreducible. So by Lemma 5, the matrix T and $\hat{T}_1^*$ are irreducible. Consider the GMTS splitting for the matrix H, $H = M - N$, where

$$M = \hat{D} + D_1 + L_1 - \hat{L},\ N = D_1 + L_1 + \hat{U}.$$

Obviously, $H = M - N$ is a regular splitting and by the assumption $\rho(M^{-1}N) < 1$, we know that H is an M-matrix. From Theorem 1, we can get that $P_1^* H$ is also an M-matrix. Thus, from Lemma 7, we know that $\bar{H}_1 = \bar{M}_1 - \bar{N}_1$ is a regular splitting. Therefore, as H is an M-matrix, we can get

$$\rho(T_1^*) = \rho(\bar{M}_1^{-1}\bar{N}_1) < 1.$$

Now, we define the following splitting for the matrix H, $H = M_1^* - N_1^*$, in which $M_1^* = (I + \bar{S}_1)^{-1}\bar{M}$, $N_1^* = (I + \bar{S}_1)^{-1}\bar{N}$ and

$$\bar{S}_1 = \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}.$$

Consider the iteration matrix of the GMTS method $T = M^{-1}N$, it is easy to see that

$$M - \bar{M}_1 = \begin{pmatrix} D_{11} - D_{11}^* & 0 \\ L_{21} + L - L_{21}^* - L_1^* & D_{22} - D_{22}^* \end{pmatrix},$$

where

$$D_1 = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \le \hat{D},\ \bar{D}_1 = \begin{pmatrix} D_{11}^* & 0 \\ 0 & D_{22}^* \end{pmatrix} \le \hat{D}_1^*,$$

$$L_1 = \begin{pmatrix} 0 & 0 \\ L_{21} & 0 \end{pmatrix} \le \hat{L} \text{ and } \bar{L}_1 = \begin{pmatrix} 0 & 0 \\ L_{21}^* & 0 \end{pmatrix} \le \hat{L}_1.$$

It is known that $L_1^* = (I + V_1)L$, hence $L_1^* - L = V_1 L \le 0$.

By computations, we know that $\bar{M}_1 \le M$, so $\bar{M}_1^{-1} \ge M^{-1}$. Consequently,

$$M^{-1} \le \bar{M}_1^{-1} \le \bar{M}_1(I + S_1) = (M_1^*)^{-1}.$$

From Lemma 4, we deduce that

$$\rho(\bar{M}_1^{-1}\bar{N}_1) = \rho((M_1^*)^{-1}N_1^*) \le \rho(M^{-1}N),$$

so $\rho(T_1^*) \le \rho(T)$.

Similar to the proof of Theorem 3, we can prove the following theorem.

Theorem 4 Let T and T_2^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix H is irreducible,

$L \le 0$, $U \le 0$, $B \ge 0$, $C \ge 0$, $0 \le D_2 \le \hat{D}$, $0 \le \bar{D}_2 \le \hat{D}_2^*$, $0 \le L_2 \le \hat{L}$, $0 \le \bar{L}_2 \le \hat{L}_2^*$, $b_{i,i-1} > 0$, $c_{j,j-1} > 0$, for some $i \in \{2,3,\dots,p\}$, $j \in \{2,3,\dots,q\}$,

$$-b_{i,i-1} < \beta_i < \frac{b_{i,i-1}}{1-b_{i-1,i-1}} - b_{i,i-1},\ -c_{j.j-1} < \delta_j < \frac{c_{j,j-1}}{1-c_{j-1,j-1}} - c_{j,j-1},$$

for some $i \in \{2,3,\dots,p\}$, $j \in \{2,3,\dots,q\}$.

If $\rho(T) < 1$, $\bar{D}_2 \le D_2$ and $\bar{L}_2 \le L_2$, then $\rho(T_2^*) \le \rho(T)$.

IV. Example

Consider

$$H = \begin{pmatrix} I - B & U \\ L & I - C \end{pmatrix},$$

where $B=\left(b_{ij}\right)_{p\times p}$, $C=\left(c_{ij}\right)_{(n-p)\times(n-p)}$, $L=\left(l_{ij}\right)_{(n-p)\times p}$, and $U=\left(u_{ij}\right)_{p\times(n-p)}$ with

$$b_{ii}=\frac{1}{10\times(i+1)},\ i=1,2,\cdots,p,$$

$$b_{ij}=\frac{1}{30}-\frac{1}{30\times j+i},\ i<j,\ i=1,2,\cdots,p-1,$$
$$j=2,\cdots,p,$$

$$b_{ij}=\frac{1}{30}-\frac{1}{30\times(i-j+1)+i},\ i>j,\ i=2,\cdots,p,$$
$$j=1,2,\cdots,p-1,$$

$$c_{ii}=\frac{1}{10\times(p+i+1)},\ i=1,2,\cdots,n-p,$$

$$c_{ij}=\frac{1}{30}-\frac{1}{30\times(p+j)+p+i},\ i<j,$$
$$i=1,2,\cdots,n-p+1,\ j=2,\cdots,n-p,$$

$$c_{ij}=\frac{1}{30}-\frac{1}{30\times(i-j+1)+p+i},\ i>j,$$
$$i=,2,\cdots,n-p,\ j=1,2,\cdots,n-p-1,$$

$$k_{ij}=\frac{1}{30\times(p+i-j+1)+p+i}-\frac{1}{30},$$
$$i=1,2,\cdots,n-p,\ j=1,2,\cdots,p,$$

$$h_{ij}=\frac{1}{30\times(p+j)+i}-\frac{1}{30},\ i=1,2,\cdots,p,$$
$$j=1,2,\cdots,n-p.$$

In the experiments, the auxiliary are chosen such that

$$D_1=0.5(\frac{1}{\omega}-1)I,\overline{D}_1=0.5(\frac{1}{\omega}-1)I,$$

$$L_1=0.5(1-\frac{\gamma}{\omega})\widehat{L}_i,\overline{L}_1=0.5(1-\frac{\gamma}{\omega})\widehat{L}_i^{*}.$$

TABLE I
THE SPECTRAL RADII OF THE GMTS AND PRECONDITIONED GMTS ITERATION MATRICES

n	ω	r	p	$\alpha=\beta$	$\gamma=\delta$	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$
10	0.9	0.8	5	-0.01	-0.01	0.2352	0.2246	0.2296
20	0.8	0.6	5	-0.01	-0.001	0.5736	0.5622	0.5664
20	0.8	0.6	15	-0.001	-0.001	0.5725	0.5607	0.5647
30	0.9	0.7	10	-0.001	0.0005	0.8680	0.8635	0.8650

From Table I, we see that these results accord with Theorems 3-4.

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