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The Baer Radical of Rings in Term of Prime and Semiprime Generalized Bi-ideals

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Abstract—Using the idea of prime and semiprime bi-ideals of rings, the concept of prime and semiprime generalized bi-ideals of rings is introduced, which is an extension of the concept of prime and semiprime bi-ideals of rings and some interesting characterizations of prime and semiprime generalized bi-ideals are obtained. Also, we give the relationship between the Baer radical and prime and semiprime generalized bi-ideals of rings in the same way as of bi-ideals of rings which was studied by Roux.

Keywords—ring, prime and semiprime (generalized) bi-ideal, Baer radical.

I. Introduction and Preliminaries

The notion of generalized bi-ideals which is a generalization of bi-ideals of rings introduced by Szász [5], [6] in 1970. In 1971, Lajos and Szász [3] studied bi-ideals in associative rings. In 1983, Walt [7] studied prime and semiprime bi-ideals of associative rings with unity. In 1995, Roux [4] extended the results of prime and semiprime bi-ideals of associative rings with unity to associative rings without unity. Moreover, Roux proved that the Baer radical of rings is the intersection of all semiprime bi-ideals. The concept of bi-ideals play an important role in studying the structure of rings. Now, the notion of generalized bi-ideals is an important and useful generalization of bi-ideals of rings. Therefore, we will study generalized bi-ideals of rings in the same way as of bi-ideals of rings which was studied by Roux.

Our aim in this paper is threefold.

1) To introduce the concept of prime and semiprime generalized bi-ideals of rings.
2) To characterize the properties of prime and semiprime generalized bi-ideals of rings.
3) To characterize the relationship between the Baer radical and prime and semiprime generalized bi-ideals of rings.

To present the main results we discuss some elementary definitions that we use later. Throughout this paper, A will represent a ring. A subset I of A is called a *left(right) ideal* of A if

(1) I is a subgroup of $\langle A,+\rangle$,
(2) $ax \in I (xa \in I)$ for all $a \in A$ and $x \in I$.

A subset I of A is called an *ideal* of A if it is both a left and a right ideal of A. Let X be a subset of A and support that $\{A_j \mid j \in J\}$ be a family of all (left, right) ideals of A containing X. Then $\bigcap_{j\in J} A_j$ is called the *(left, right) ideal of A generated by X* [2] and denoted by $((X)_l, (X)_r)(X)$. If $X = \{x\}$, then $((X)_l, (X)_r)$ (X) is usually denoted by (x) $((x)_l, (x)_r)$. From [2], we have

$$(x)_r = \{nx + \sum_{i=1}^{m} xs_i \mid s_i \in A, m \in \mathbb{Z}^+, n \in \mathbb{Z}\}$$

and

$$(x)_l = \{nx + \sum_{i=1}^{m} s_i x \mid s_i \in A, m \in \mathbb{Z}^+, n \in \mathbb{Z}\}.$$

Let I be an ideal of A. Then

(1) I is called a *prime ideal* of A if
$$XY \subseteq I \text{ implies } X \subseteq I \text{ or } Y \subseteq I$$
for any ideals X and Y of A. Equivalently,
$$xAy \subseteq I \text{ implies } x \in I \text{ or } y \in I$$
for any $x, y \in A$ [1].

(2) I is called a *semiprime ideal* of A if
$$X^2 \subseteq I \text{ implies } X \subseteq I$$
for any ideal X of A. Equivalently,
$$xAx \subseteq I \text{ implies } x \in I$$
for any $x \in A$ [1].

From [1], a semiprime ideal of A is an intersection of prime ideals of A. If I is a left(right) ideal of A, then I is a subgroup of $\langle A,+\rangle$. Since $II \subseteq AI \subseteq I$, we have I is a subsemigroup of $\langle A,\cdot\rangle$. Hence I is a subring of A. A subset B of A is called a *bi-ideal* [4] of A if

(1) B is a subring of A,
(2) $b_1ab_2 \in B$ for all $b_1, b_2 \in B$ and $a \in A$.

We can easily prove that bi-ideals are a generalization of left(right) ideals. A subset B of A is called a *generalized bi-ideal* [5] of A if

(1) B is a subgroup of $\langle A,+\rangle$,
(2) $b_1ab_2 \in B$ for all $b_1, b_2 \in B$ and $a \in A$.

Hence generalized bi-ideals are a generalization of bi-ideals. Let B be a generalized bi-ideal of A. Then

(1) B is called a *prime generalized bi-ideal* of A if
$$xAy \subseteq B \text{ implies } x \in B \text{ or } y \in B$$
for any $x, y \in A$.

(2) B is called a *semiprime generalized bi-ideal* of A if
$$xAx \subseteq B \text{ implies } x \in B$$
for any $x \in A$.

For any generalized bi-ideal B of A, let

$$L(B) = \{x \in B \mid Ax \subseteq B\}$$

and

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$$H(B) = \{y \in L(B) \mid yA \subseteq L(B)\}.$$

Let $\{P_i \mid i \in I\}$ be a family of all prime ideals of A. Then $\bigcap_{i \in I} P_i$ is called the *Baer radical* [1] of A and denoted by $\beta(A)$. From [1], we have $\beta(A)$ is the smallest semiprime ideal of A. A ring A is called *regular* [4] if for any $a \in A$, there exists $x \in A$ such that $a = axa$.

II. Lemmas

Before the characterizations of prime and semiprime generalized bi-ideals of rings for the main results, we give some auxiliary results which are necessary in what follows. The following two lemmas are easy to verify.

Lemma II.1. *For all $x \in A$, xA is a right ideal and Ax is a left ideal of A.*

Lemma II.2. *For all $x \in A$, xAx is a bi-ideal of A.*

Lemma II.3. *Let B be a generalized bi-ideal of A. Then $L(B)$ is a left ideal of A such that $L(B) \subseteq B$.*

Proof: By definition, it is clear that $\emptyset \neq L(B) \subseteq B$. Let $x, y \in L(B)$. Then $x, y \in B$ and $Ax \subseteq B$ and $Ay \subseteq B$, so $x - y \in B$ and $A(x-y) \subseteq Ax - Ay \subseteq B$. Thus $x - y \in L(B)$, so $L(B)$ is a subgroup of $\langle A, + \rangle$. Let $x \in L(B)$ and $z \in A$. Since $zx \in Ax \subseteq B$, we have $zx \in B$ and $Azx \subseteq AAx \subseteq Ax \subseteq B$. Hence $zx \in L(B)$, so $L(B)$ is a left ideal of A and $L(B) \subseteq B$. ■

Lemma II.4. *Let B be a generalized bi-ideal of A. Then $H(B)$ is a subgroup of $\langle A, + \rangle$.*

Proof: Let $x, y \in H(B)$. Then $x, y \in L(B), xA \subseteq L(B)$ and $yA \subseteq L(B)$. Since $x \in L(B)$, $x \in B$ and $Ax \subseteq B$. Since $y \in L(B)$, $y \in B$ and $Ay \subseteq B$. Since $x, y \in B$ and B is a subgroup of $\langle A, + \rangle$, we have $x - y \in B$. Thus $A(x - y) \subseteq Ax - Ay \subseteq B$, so $x - y \in L(B)$. Now, $(x-y)A \subseteq xA - yA \subseteq L(B) - L(B) \subseteq L(B)$, so $x - y \in H(B)$. Hence $H(B)$ is a subgroup of $\langle A, + \rangle$. ■

Lemma II.5. *Let B be a left ideal of A. Then $L(B) = B$.*

Proof: Clearly, $L(B) \subseteq B$. Conversely, let $x \in B$. Since B is a left ideal A, we have $Ax \subseteq B$. Thus $x \in L(B)$, so $L(B) = B$. ■

III. Main Results

In this section, give some characterizations of prime and semiprime generalized bi-ideals of rings. Finally, we prove that the Baer radical of rings is the intersection of all prime and semiprime bi-ideals.

Proposition III.1. *Let B be a generalized bi-ideal of A. Then B is a prime generalized bi-ideal of A if and only if for any right ideal R and left ideal L of A, $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof: Assume that B is a prime generalized bi-ideal of A. Let R be a right ideal of A and L a left ideal of A such that $RL \subseteq B$. Suppose that $R \nsubseteq B$, let $x \in L$ and $r \in R \backslash B$. Then $rAx \subseteq RL \subseteq B$. Since B is a prime generalized bi-ideal of A and $r \notin B$, we have $x \in B$. Hence $L \subseteq B$.

Conversely, assume that for any right ideal R and left ideal L of A, $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$. Let $x, y \in A$ be such that $xAy \subseteq B$. Then

$$(xA)(Ay) \subseteq xA^2y \subseteq xAy \subseteq B.$$

By Lemma II.1, we have xA is a right ideal and Ay is a left ideal of A. By assumption, we have $xA \subseteq B$ or $Ay \subseteq B$. Suppose $xA \subseteq B$. Then $x^2 \in B$. Let $z \in (x)_r(x)_l$. Then, by I and I, we get

$$z = \sum_{i=1}^{n} (m_i x + x a_i)(k_i x + b_i x)$$

for some $a_i, b_i \in A$ and $m_i, k_i, n \in \mathbb{Z}^+$, so

$$z = \sum_{i=1}^{n} m_i k_i x^2 + m_i x b_i x + k_i x a_i x + x a_i b_i x.$$

Since $x^2 \in B, b_i x, a_i x, a_i b_i x \in A$ and $xA \subseteq B$, we have $z \in B$. Hence $(x)_r(x)_l \subseteq B$. By assumption, we have

$$(x)_r \subseteq B \text{ or } (x)_l \subseteq B.$$

Hence $x \in B$. We can prove in a similar manner that $y \in B$. Therefore B is a prime generalized bi-ideal of A. ■

Proposition III.2. *Let B be a prime generalized bi-ideal of A. Then B is a prime one-sided ideal of A.*

Proof: We have to show that B is a one-sided ideal of A. Now,

$$(BA)(AB) \subseteq BAB \subseteq B.$$

Since BA is a right ideal and AB is a left ideal of A and by Proposition III.1, we have $BA \subseteq B$ or $AB \subseteq B$. Hence B is a right ideal or a left ideal of A. ■

Proposition III.3. *Let B be a generalized bi-ideal of A. Then $H(B)$ is the largest ideal of A such that $H(B) \subseteq B$.*

Proof: Since $H(B) \subseteq L(B)$ and $L(B) \subseteq B$, $H(B) \subseteq B$. By Lemma II.4, we have $H(B)$ is a subgroup of $\langle A, + \rangle$. Let $x \in H(B)$ and $y \in A$. Then $x \in L(B)$, so $Ax \subseteq B$ and $xA \subseteq L(B)$. Thus $yx \in Ax \subseteq B$. Since $Ayx \subseteq Ax \subseteq B$, we have $yx \in L(B)$. By Lemma II.3, we have $yxA \subseteq AxA \subseteq AL(B) \subseteq L(B)$. Thus $yx \in H(B)$. Hence $H(B)$ is a left ideal of A. Similarly, $xy \in xA \subseteq L(B)$. Thus $xyA \subseteq xA \subseteq L(B)$, so $xy \in H(B)$. Hence $H(B)$ is a right ideal of A. Therefore $H(B)$ is an ideal A such that $H(B) \subseteq B$. Assume that S is an ideal of A such that $S \subseteq B$ and let $s \in S$. Then $s \in B$ and $As \subseteq AS \subseteq S \subseteq B$, so $s \in L(B)$. Hence $S \subseteq L(B)$. Now, $sA \subseteq SA \subseteq S \subseteq L(B)$, so $s \in H(B)$. Hence $S \subseteq H(B)$. Therefore $H(B)$ is the largest ideal of A such that $H(B) \subseteq B$. ■

Proposition III.4. *Let B be a generalized bi-ideal of A. Then $H(B)$ is a prime ideal of A.*

Proof: Let X and Y be ideals of A such that $XY \subseteq H(B)$. Since $H(B) \subseteq B$, $XY \subseteq B$. By Proposition III.1, we have $X \subseteq B$ or $Y \subseteq B$. By Proposition III.3, we have $H(B)$ is the largest ideal of A such that $H(B) \subseteq B$. Thus

$X \subseteq H(B)$ or $Y \subseteq H(B)$. Hence $H(B)$ is a prime ideal of A. ■

Corollary III.5. *The Baer radical $\beta(A)$ is the intersection of all prime generalized bi-ideals of A.*

Proof: Let

$$\begin{aligned}\mathscr{B} &= \{B \mid B \text{ is a prime generalized bi-ideal of } A\},\\ \mathscr{H} &= \{H(B) \mid B \text{ is a prime generalized bi-ideal of } A\},\\ \mathscr{P} &= \{P \mid P \text{ is a prime ideal of } A\}.\end{aligned}$$

Since every prime ideal of A is a prime generalized bi-ideal, we have $\mathscr{P} \subseteq \mathscr{B}$. Thus

$$\bigcap \mathscr{B} \subseteq \bigcap \mathscr{P} = \beta(A).$$

Since $H(B) \subseteq B$ and by Proposition III.4, we have

$$\beta(A) = \bigcap \mathscr{P} \subseteq \bigcap \mathscr{H} \subseteq \bigcap \mathscr{B}.$$

From III and III, we have $\beta(A) = \bigcap \mathscr{B}$. This completes the proof. ■

Proposition III.6. *Let B be a semiprime generalized bi-ideal and L (R) a left(right) ideal of A. If $L^2 \subseteq B(R^2 \subseteq B)$, then $L \subseteq B(R \subseteq B)$.*

Proof: Assume $L^2 \subseteq B$ and suppose that $L \nsubseteq B$. Then there exists $x \in L$ but $x \notin B$. Now, $xAx \subseteq LAL \subseteq LL \subseteq B$. Since B is a semiprime generalized bi-ideal of A, we have $x \in B$ that is a contradiction. Hence $L \subseteq B$. In a similar way, we can prove that if $R^2 \subseteq B$, then $R \subseteq B$. ■

Proposition III.7. *Let B be a semiprime generalized bi-ideal of A. Then $H(B)$ is a semiprime ideal of A.*

Proof: By Proposition III.3, we have $H(B)$ is an ideal of A. Let X be an ideal of A such that $X^2 \subseteq H(B)$. Since $H(B) \subseteq B$, $X^2 \subseteq B$. By Proposition III.6, we have $X \subseteq B$. By Proposition III.3 again, we have $X \subseteq H(B)$. Hence $H(B)$ is a semiprime ideal of A. ■

Corollary III.8. *The Baer radical $\beta(A)$ is the intersection of all semiprime generalized bi-ideals of A.*

Proof: Let

$$\begin{aligned}\mathscr{S} &= \{S \mid S \text{ is a semiprime ideal of } A\},\\ \mathscr{C} &= \{C \mid C \text{ is a semiprime generalized bi-ideal of } A\},\\ \mathscr{H} &= \{H(C) \mid C \text{ is a semiprime generalized bi-ideal of } A\}.\end{aligned}$$

Since every semiprime ideal of A is a semiprime generalized bi-ideal, we have $\mathscr{S} \subseteq \mathscr{C}$. Since $\beta(A)$ is the smallest semiprime ideal of A, we have

$$\bigcap \mathscr{C} \subseteq \bigcap \mathscr{S} = \beta(A).$$

By Proposition III.7, we have $H(C)$ is a semiprime ideal of A and $H(C) \subseteq C$. Thus

$$\beta(A) = \bigcap \mathscr{S} \subseteq \bigcap \mathscr{H} \subseteq \bigcap \mathscr{C}.$$

From III and III, we have $\beta(A) = \bigcap \mathscr{C}$. The proof is then completed. ■

Proposition III.9. *A ring A is regular if and only if every generalized bi-ideal of A is a semiprime generalized bi-ideal.*

Proof: Assume that A is regular and let B be a generalized bi-ideal of A. Let $a \in A$ be such that $aAa \subseteq B$. Since A is regular, there exists $x \in A$ such that $a = axa$. Thua $a = axa \in aAa \subseteq B$. Hence B is a semiprime generalized bi-ideal of A.

Conversely, assume that every generalized bi-ideal of A is a semiprime generalized bi-ideal. Let $a \in A$. Then, by Lemma II.2, we have aAa is a generalized bi-ideal of A. By assumption, we have aAa is a semiprime generalized bi-ideal of A. Now, $aAa \subseteq aAa$, we get $a \in aAa$. Thus $a = axa$ for some $x \in A$. Hence A is regular, and so the proof is completed. ■

IV. Conclusion

In comparison our above results with results of bi-ideals of rings, we see that the Baer Radical is the intersection of all prime and semiprime generalized bi-ideals of A which is an analogous result of bi-ideals of rings.

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