MATHEMATICAL SCIENCES

STUDYING THE SPECTRUM AND RESOLVENT OF A SECOND ORDER INTEGRO-DIFFERENTIAL OPERATOR ON THE ENTIRE AXIS

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Abstract. Let us consider the integro-differential equation

$$Ly \equiv -y''(x) + p(x)y'(x) + \int_{-\infty}^{+\infty} K(x, s\rho)y'(s)ds - \lambda y(x) = f(x) \tag{2.1}$$

where $f(x) \in L^2(R_-^+), R_-^+ = (-\infty, \infty) \quad \lambda = \rho^2, \operatorname{Im}\rho > 0.$

Assume that $\lambda = \rho^2, \operatorname{Im}\rho > 0$ is not an eigen-value of the integro-differential operator L so that the resolvent R_λ exists. We find the explicit form of the resolvent R_λ of the integro-differential operator L and show that it is a bounded integral operator.

Let $p(x)$ be a complex-valued summable function in the interval R_-^+ and the kernel of the integro-differential operator (2.1) have the form:

$$K(x,s,\rho) = \begin{cases} e^{-\rho\omega_1(s-x)} P(s)K^*(x,s,\rho) & for \quad s < x \\ e^{-\rho\omega_2(s-x)} P(s)K^*(x,s,\rho) & for \quad s > x, \end{cases} \tag{2.2}$$

where ω_1, ω_2 are various roots of second degree from -1 while $K^*(x,s,\rho)$ is a complex-valued continuous bounded function on totality of (x,s,ρ) $-\infty < x, s < +\infty, |\operatorname{Im}\rho| < \frac{\varepsilon_0}{2}$ variables and analytic with respect to ρ in the domain $|\operatorname{Im}\rho| < \frac{\varepsilon_0}{2}$ for each fixed $(x,s), -\infty < x, s < +\infty$.

Furthermore, let the following condition be fulfilled:

$$|P(x)| \left[1 + \int_{-\infty}^{+\infty} |K^*(s,x,\rho)| ds\right] < ce^{-2\varepsilon_0|x|}, \ \varepsilon_0 > 0 \tag{2.3}$$

Then

$$\int_{-\infty}^{+\infty} |P(x)| \left[1 + \int_{-\infty}^{+\infty} |K^*(s,x,\rho)| ds\right] dx < B_1 < \infty, \tag{2.4}$$

where B_1 is independent of ρ.

If $y(x,\lambda)$ is an eigen-function of the integro-differential operator L corresponding to the eigen-value, λ then it is determined from the equation

$$y(x,\lambda) = \frac{1}{2\rho\omega_1} \int_{-\infty}^{+\infty} p(\tau)Q(x,\tau,\rho)y(\tau,\rho)d\tau, \ \operatorname{Im}\rho > 0 \tag{2.5}$$

where

$$Q(x,\tau,\rho) = e^{\rho\omega_1|x-\tau|}\left[1 + \int_{-\infty}^{+\infty} K^*(s,\tau,\rho)ds\right].$$

Therefore, the eigen-values are the zeros of the Fredholm determinant

$$D(\rho)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}D_n(\rho) \qquad (2.6)$$

We prove the following theorem.

Theorem 2.1. Let the condition (2.3) be fulfilled. Then the function $D(\rho)$ allows analytic continuation through the real positive semi-axis in the domain E, which is determined by the inequalities:

$E=\left\{|\rho|\geq r>0, \operatorname{Im}\rho>\frac{\varepsilon_0}{2}\right\}$, where r is a fixed number.

Theorem 2.2. The set of eigen-values of the integro-differential operator L is bounded, a unique possible concentration point of the integro-diffrential operator L is the point $\lambda=0.$

Theorem 3.2. If the condition (2.3) is fulfilled, then the integro-differential operator L may have only finitely many complex eigen-values.

It is proved that $|\lambda|\geq\left(\frac{B_1}{2L_0}\right)^2$ the integro-differential operator L has no eigen-values.

Keywords: integro-differential operator, theorem, condition, kernel, integral, entire axis, second order, eigen-function, values, complex-valued equation, variable, let, fixed.

Introduction.

Denote by L_0 an operator $-y''$ in $L^2(R_-^+).$

This is a positive-definite self-adjoint operator, therefore for all $\rho=\rho_1+\omega_1\rho_2$ except the semi-axis $\rho_1\geq 0,$ there exists a resolvent $R_\lambda^0,$ as a bounder integral operator.

Therefore, if $f(x)\in L^2(R_-^+)$ and $\lambda=\rho^2, \operatorname{Im}\rho>0,$ then it is known that the solutions of the equations

$$-y''(x)-\rho^2 y|(x)=f(x)$$

are given by the formula

$$y(x,\rho)=R_\lambda^0 f=-\int_{-\infty}^{x}\frac{e^{\rho\omega_1(x-\tau)}}{2\rho\omega_1}f(\tau)d\tau-\int_{x}^{\infty}\frac{e^{\rho\omega_1(\tau-x)}}{2\rho\omega_1}f(\tau)d\tau. \qquad (2.7)$$

Therefore, the integro-differential equation (2.1) is equivalent to the following one:

$$y(x,\rho)=-\int_{-\infty}^{x}\frac{e^{\rho\omega_1(\tau-x)}}{2\rho\omega_1}\left[f(\tau)-p(\tau)y'(\tau,\rho)-\int_{-\infty}^{\infty}K(\tau,s,\rho)y'(s,\rho)ds\right]d\tau-$$

$$-\int_{x}^{\infty}\frac{e^{\rho\omega_1(\tau-x)}}{2\rho\omega_1}\left[f(\tau)-p(\tau)y'(\tau,\rho)-\int_{-\infty}^{\infty}K(\tau,s,\rho)y'(s,\rho)ds\right]d\tau.$$

Taking into account formula (2.2), we obtain

$$y(x,\rho)=-\int_{-\infty}^{+\infty}M_1(x,\tau,\rho)f(\tau)+\int_{-\infty}^{+\infty}N_1(x,\tau,\rho)y'(\tau,\rho)d\tau \qquad (2.8)$$

where

$$M_1(x,\tau,\rho)=\begin{cases}-\dfrac{e^{\rho\omega_1(x-\tau)}}{2\rho\omega_1} & for \quad \tau<x,\\ -\dfrac{e^{\rho\omega_1(\tau-x)}}{2\rho\omega_1} & for \quad \tau>x,\end{cases} \qquad (2.9)$$

and

$$N_1(x,\tau,\rho)=\begin{cases}\dfrac{e^{\rho\omega_1(x-\tau)}}{2\rho\omega_1}p(\tau)\left[1+\int\limits_{-\infty}^{+\infty}K^*(s,\tau,\rho)ds\right] & for \quad \tau<x,\\ \dfrac{e^{\rho\omega_1(x-\tau)}}{2\rho\omega_1}p(\tau)\left[1+\int\limits_{-\infty}^{+\infty}K^*(s,\tau,\rho)ds\right] & for \quad \tau>x.\end{cases} \tag{2.10}$$

Since $\lambda=\rho^2, \operatorname{Im}\rho>0$ is not an eigen-value of the integro-differntial operator L, then (2.8) is solvable and we have

$$y(x,\rho)=\int\limits_{-\infty}^{+\infty}\left[M_1(x,\tau,\rho)+\int\limits_{-\infty}^{+\infty}\Gamma_1(x,\tau,\rho)M_1(t,\tau,\rho)d\tau\right]f(\tau)d\tau=R_\lambda f, \tag{2.11}$$

where $\Gamma_1(x,\tau,\rho)$ is a solvent kernel for the kernel $N_1(x,\tau,\rho)$.

Consequently, the resolvent R_λ is an integral operator with the kernel

$$T(x,\tau,\rho)=M_1(x,\tau,\rho)+\int\limits_{-\infty}^{\infty}\Gamma_1(x,\tau,\rho)M_1(t,\tau,\rho)dt.$$

We now prove that this kernel in fact sets an integral operator bounded and determined in the entire space $L^2(R_-^+)$.

At first using

$N_1(x,\tau,\rho)=N^*(x,\tau,\rho)$ and $N_n^*(x,\tau,\rho)=\int\limits_{-\infty}^{\infty}N^*(x,\tau,\rho)N_{n-1}^*(\tau_1,\tau,\rho)d\tau_1$ we show that

$$\Gamma_1(x,\tau,\rho)=\sum_{n=1}^{\infty}N_n^*(x,\tau,\rho)$$

converges if $|\rho|>\dfrac{B}{2q}$.

For $\operatorname{Im}\rho>0,$ we have $e^{-\rho_2|x-\tau|}\le 1,$ then from (2.10) it follows that

$$\left|N_1^*(x,\tau,\rho)\right|\le\frac{e^{-\rho_2|x-\tau|}}{2|\rho|}|p(\tau)|\left[1+\int\limits_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]\le$$

$$\le\frac{|p(\tau)|}{2|\rho|}\left[1+\int\limits_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right];$$

$$\left|N_2^*(x,\tau,\rho)\right|\le\frac{1}{(2|\rho|)^2}|p(\tau)|\left[1+\int\limits_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]\int\limits_{-\infty}^{+\infty}|p(\tau_1)|\left[1+\int\limits_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]d\tau_1;$$

by induction

$$\left|N_n^*(x,\tau,\rho)\right|\le\frac{|p(\tau)|}{(2|\rho|)^n}\left[1+\int\limits_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]\int\limits_{-\infty}^{+\infty}\dots\int\limits_{-\infty}^{+\infty}\prod_{j=1}^{n}|p(\tau_j)|\left[1+\int\limits_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]d\tau_j.$$

Then

$$\left|\Gamma_1(x,\tau,\rho)\right| \le \sum_{n=1}^{\infty}\left|N_n^*(x,\tau,\rho)\right| \le \sum_{n=1}^{\infty} \frac{|p(\tau)|}{(2|\rho|)^n}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}\prod_{j=1}^{n-1}\left|p(\tau_j)\right|\times$$

$$\times\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right]d\tau_j \le \sum_{n=1}^{\infty}\frac{B^{n-1}}{(2|\rho|)^n}|p(\tau)|\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]=$$

$$=\frac{|p(\tau)|}{(2|\rho|)^n}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]\sum_{n=1}^{\infty}\frac{B^{n-1}}{(2|\rho|)^{n-1}}.$$

The series $\sum_{n=1}^{\infty}\left(\frac{B}{2|\rho|}\right)^{n-1}$ converges if $\frac{B}{2|\rho|}<q<1$ or $|\rho|>\frac{B}{2q}$, i.e. in the domain

$$\frac{1}{2q}\int_{-\infty}^{+\infty}\left|p(\tau_j)\right|\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]d\tau_j<|\rho|$$

these is no a spectrum of the integro-differential operator L . The sum of this series equals $\frac{2|\rho|}{2|\rho|-B}$.

Therefore, the series $\sum_{n=1}^{\infty}N_n^*(x,\tau,\rho)$ is majorized by the convergent number series $\sum_{n=1}^{\infty}\left(\frac{B}{2|\rho|}\right)^{n-1}$ and so its sum $\Gamma_1(x,\tau,\rho)$ will be analytic with respect to ρ in the domain $\operatorname{Im}\rho>0$ for each fixed x. . Then the term by term differentiation of the series $\sum_{n=1}^{\infty}N_n^*(x,\tau,\rho)$ is admissible.

Thus,

$$\left|\int_{-\infty}^{\infty}T(x,\tau,\rho)f(\tau)d\tau\right|^2 \le \int\frac{e^{-\rho_2|x-\tau|}}{2|\rho|}\left|f(\tau)d\tau\right|+\frac{2|\rho|}{2|\rho|-B}|p(\tau)|\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]\times$$

$$\times\left\{\int_{-\infty}^{+\infty}\frac{e^{-\rho_2|x-\tau|}}{2|\rho|}\left|f(\tau)dt\right|\right\}d\tau.$$

Applying the Bunyakovsky inequality, it is easy to be convinced that

$$\left(\int_{-\infty}^{+\infty}\frac{e^{-\rho_2|x-\tau|}}{2|\rho|}\left|f(\tau)dt\right|\right)^2 \le \|f\|^2\int_{-\infty}^{+\infty}\frac{e^{-\rho_2|x-\tau|}}{4|\rho|^2}d\tau=\frac{\|f\|^2}{4\rho_2|\rho|^2}.$$

Therefore

$$|R_\lambda f| \le \frac{\|f\|^2}{4\rho_2|\rho|^2}\left[1+\left|\frac{2|\rho|}{2|\rho|-B}\right|^2 B^2\right]$$

So, for any function $f(x)\in L^2(R_-^+)$ the formula (2.11) determines the function $y(x)\in L^2(R_-^+)$, it is easy to verify that $y(x,\rho)$ satisfies the equation (2.8) and consequently the integro-differential equation (2.1).

So, the closed operator R_λ is defined on the entire space $L^2(R_-^+)$.

Let now $y(x,\lambda)$ be an eigen function of the integro-differential operator L corresponding to the eigen-value λ.

Then it is defined from the equation

$$y(x,\lambda)=\frac{1}{2\rho\omega_1}\int_{-\infty}^{+\infty}p(\tau)Q(x,\tau,\rho)y(\tau,\rho)d\tau, \operatorname{Im}\rho>0, \tag{2.12}$$

where

$$Q(x,\tau,\rho)=e^{\rho w_1|x-\tau|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right].$$

Therefore, the eigen-values are the zeros of the Fredholm determinant

$$D(\rho)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}D_n(\rho), \tag{2.13}$$

where

$$D_n(\rho)=\frac{1}{(2\rho\omega_1)^n}\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty}\det\left\|Q_\rho(\tau_i,\tau_j)\right\|\prod_{k=1}^{n}p(\tau_k)d\tau_k, \quad i,j=\overline{1,n} \tag{2.14}$$

Here we introduce the following denotations:

$$\det\left\|Q_\rho(\tau_i,\tau_j)\right\|=Q_\rho\begin{pmatrix}\tau_1, & \tau_2 & ,..., & \tau_n\\ \tau_1, & \tau_2 & ,..., & \tau_n\end{pmatrix}$$

and

$$Q_\rho(\tau_i,\tau_j)=e^{\rho\omega_1|\tau_i-\tau_j|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right], \quad i,j=\overline{1,n}.$$

Therefore, the operator R_λ does not exist only for those ρ for which $D(\rho)=0.$

We now prove the following theorem.

Theorem 2.1. Let the condition (2.3) be fulfilled. Then the function $D(\rho)$ allows analytic continuation through the real positive semi-axis in the domain $E,$ which is determined by the inequalities:

$E=\left\{|\rho|\ge r>0, \operatorname{Im}\rho>-\frac{\varepsilon_0}{2}\right\},$ where r is a fixed number.

Proof. In the domain E the quantity $Q_\rho(\tau_i,\tau_j), i,j=\overline{1,n}$ is determined for all ρ and for the fixed values τ_i,τ_j is an analytic function of ρ.

In the domain E the following estimation is valid:

$$Q_\rho(\tau_i,\tau_j)=e^{\rho_2|\tau_i-\tau_j|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right]\le e^{\frac{\varepsilon_0}{2}(|\tau_i|+|\tau_j|)}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau,\rho)\right|ds\right]. \tag{2.15}$$

In what follows,

$$\det\left\|Q_\rho(\tau_i,\tau_j)\right\|\prod_{j=1}^{n}p(\tau_j)=\det\left\|Q_\rho(\tau_i,\tau_j)\right\|\prod_{i=1}^{n}p(\tau_j)e^{\frac{\varepsilon_0}{2}\sum_{j=1}^{n}|\tau_j|}e^{-\frac{\varepsilon_0}{2}|\tau_j|}=$$

$$=\prod_{i=1}^{n}p(\tau_j)e^{\frac{\varepsilon_0}{2}\sum_{j=1}^{n}|\tau_j|}\det\left\|Q_\rho(\tau_i,\tau_j)e^{-\frac{\varepsilon_0}{2}|\tau_j|}\right\|.$$

In the domain E by virtue of the estimation (2.15) we obtain :

$$\left|Q_\rho(\tau_i,\tau_j)e^{-\frac{\varepsilon_0}{2}|\tau_i|}\right|=e^{\frac{\varepsilon_0}{2}|\tau_i|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right].$$

Applying the Hadamard inequality, we obtain:

$$\left|\det\left\|Q_\rho(\tau_i,\tau_j)e^{-\frac{\varepsilon_0}{2}|\tau_j|}\right\|\right| \le n^{\frac{n}{2}}\prod_{j=1}^{n} e^{\frac{\varepsilon_0}{2}|\tau_j|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right]$$

Hence

$$\left|\det\left\|Q_\rho(\tau_i,\tau_j)\right\|\prod_{j=1}^{n}p(\tau_j)\right| \le n^{\frac{n}{2}}\prod_{j=1}^{n}\left|p(\tau_j)\right|e^{\frac{\varepsilon_0}{2}|\tau_j|}e^{\frac{\varepsilon_0}{2}\sum_{j=1}^{n}|\tau_j|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right]=$$
$$= n^{\frac{n}{2}}\prod_{j=1}^{n}\left|p(\tau_j)\right|e^{\varepsilon_0|\tau_j|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right]. \qquad (2.16)$$

Using the conditions (2.3) and (2.16) , we obtain:

$$\left|D_n(\rho)\right| \le \frac{n^{\frac{n}{2}}}{(2r)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{j=1}^{n}\left|p(\tau_j)\right|e^{2\varepsilon_0|\tau_j|}\left[1+\int_{-\infty}^{\infty}\left|K^*(s,\tau_j,\rho)\right|ds\right]d\tau \le \left(\frac{\sqrt{n}B_1}{2r}\right)^n$$

for all $\rho \in E.$

Therefore, in the domain $E, \left|D_n(\rho)\right| \le 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\sqrt{n}B_1}{2r}\right)^n.$

By the D'Alambent test, a number series converges, therefore the series of analytic in the domain E functions $D_n(\rho)$ converges uniformly, i.e. $D(\rho)$ is a function analytic in the domain E . Theorem 2.1. proved.

Let us consider the function of L

$$f(L)=\sum_{n=1}^{\infty}\frac{\left(\sqrt{n}L\right)^n}{n!},(L\ge 0).$$

$f(L)$ is determined for all $L \ge 0$ and monitonically increases in this domain $f(0)=0,$ therefore, there exists a unique L_0 determined by the condition $f(L_0)=1.$

Now, we choose such a number r that $\frac{B_1}{2r}<L_0,$ i.e. for

$$r>\frac{B_1}{2L_0}=\frac{\int_{-\infty}^{+\infty}e^{2\varepsilon|x|}\left|p\,|\,(x)\right|\left[1+\int_{-\infty}^{\infty}\left|K^*(s,x,\rho)\right|ds\right]dx}{2L_0} \qquad (2.17)$$

Then for

$$|\rho| \ge r > \frac{B_0}{2L_0},$$

$$\left|D(\rho)\right| \ge 1-\left|\sum_{n=1}^{\infty}\left(\frac{\sqrt{n}B_1}{2r}\right)^n\frac{1}{n!}\right| \ne 0$$

the i.e. for rather large $|\rho|$ $D(\rho)\ne 0.$

So, $D(\rho)\ne 0$ for rather large $|\rho|$ in the domain $E.$ Therefore, it follows from the Fredholm theory that complex-valued values of the integro-diffrential operator L form at most a denumerable set of points with a unique possible concentration point in zero .

Theorem 2.2. The set of eigen-values of the integro-diffrential operator L is bounded, a unique possible concentration point of the integro-diferential operator L is the point $\lambda = 0.$

The formula (2.17) estimates also the radius of the circle in the complex λ plane $(\lambda = \rho^2),$ outside of which there are knowingly no eigen-values of the integro-diffrential operator L. More exactly for $|\lambda| \geq R,$ where $R = \left(\dfrac{B_1}{2L_0}\right)^2$ the integro-differential operator L has no eigen values.

Show that $\lambda = 0$ is not a concentration point of eigen-values of the integro-differential operator L if condition (2.3) is fulfilled.

Indeed, according to theorem 2.1 in this case the function $D(\rho)$ will be analytic on the entire real axis: therefore, the zeros of the function $D(\rho)$ can not have limiting points on the real axis.

Theorem 2.3. The point $\lambda = 0$ is not an eigen-value of the integro-differential operator L if the condition (2.3) is fulfilled.

Proof. For $\rho = 0$ the integro-differential equation

$$y''(x) + \rho^2 y(x) = p(x)y'(x) + \int_{-\infty}^{\infty} K(x,s,\rho)y'(s)ds \tag{2.18}$$

and condition (2.2) takes the form

$$-y''(x) + p(x)y'(x) + \int_{-\infty}^{\infty} K(x,s)y'(s)ds = 0 \tag{2.19}$$

$$K(x,s) = \rho(s)K^*(x,s), \tag{2.20}$$

where $K(x,s,0) = K(x,s)$ and $K^*(x,s,0) = K^*(x,s)$.

To prove the theorem, it suffices to show that the integro-differential equation (2.19) has no solutions belonging to $L^2(R_-^+).$

Consider the integral equation for $x \geq 0$:

$$y_k(x) = X^{k-1} + \int_x^{\infty}\left[(\tau - x)p(\tau)y_k'(\tau) + \int_{-\infty}^{\infty}(s-x)K(\tau,s)y_k'(s)ds\right]d\tau, k=1,2 \tag{2.21}$$

Taking into account the condition (2.20), we obtain:

$$y_k(x) = x^{k-1}Hy_k', \ k=1,2, \tag{2.22}$$

where

$$Hy_k' = \int_x^{\infty}(\tau - x)p(\tau)\left[1 + \int_{-\infty}^{\infty}\left|K^*(s,\tau)\right|ds\right]y_k'(\tau)d\tau.$$

Show that each of these equations have solutions that as $x \to \infty$ have the following asympics:

$$y_k(x) = x^{k-1} + O\left(e^{-2\varepsilon_0 x}\right), k=1,2. \tag{2.23}$$

Now we provide a proof for $y_1(x).$

We write the solution of the equation (2.22) in the form of the Neumann series

$$y_1(x) = 1 + H1 + H^2 1 + \ldots \tag{2.24}$$

Repeating the arguments of theorem 1.1 [4], we can easily show by induction that

$$\left|H^n 1\right| \leq \frac{ce^{-2n\varepsilon_0 x}}{2(2n)!!\varepsilon_0^{n+2}}.$$

Therefore, the series (2.24) converges uniformly for $x \geq 0,$ and $\left|y_1^2(x)\right| \leq A_1$ for $x \in R^+ = (0,\infty).$

We now estimate the integral terms and in (2.22) for $k=1$

$$\left|Hy_1'\right| \le A_1\left|H1\right| \le \frac{A_1 c e^{-2\varepsilon_0 x}}{(2\varepsilon_0)^2} \to O\left(e^{-2\varepsilon_0 x}\right)$$

as $x \to +\infty$.

By the same taken, the relation (2.22) is proved for $k=1.$

It is clear that $y_1(x)$ will be the solution of the integro-differential equation (2.19) for $x \ge 0.$

In that follows, $y_1(x)$ does not belong to $L^2(R_-^+).$

We continue $y_1(x)$ for $x \le 0$ as a solution for the integro-differential equation (2.19)

After substituting $y_1(x) = xu(x),$ we carry out analytic construction for $y_2(x)$.

It is seen from (2.23) that $y_1(x)$, $y_2(x)$ are linearly independent. Therefore, since all $y_k(x) \notin L^2(R_-^+), k=1,2,$ then any solution of (2.18) is $\notin L^2(R_-^+)$ as well.

The theorem is proved.

Summarizing, we can formulate the following main theorem.

Theorem 2.4. It the condition (2.3) is fulfilled , then the integro-diffrential operator L may have only finitely many complex eigen-values.

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