Computers and Mathematics with Applications 146 (2023) 213–224

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Shape optimization for the Stokes hemivariational inequality with slip boundary condition ☆

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ARTICLE INFO

Keywords:
Stokes equation
Hemivariational inequality
Stability analysis
Shape optimization
Finite element method

ABSTRACT

The paper is devoted to the analysis of a hemivariational inequality problem for the stationary Stokes equations in a bounded planar domain with a nonmonotone and multivalued slip boundary condition. First, a result on the stability of solutions of the hemivariational inequality on variations of the domain is established. Then we provide the existence of a solution to optimal shape design problems of the stationary Stokes hemivariational inequality. We investigate the convergence of shape optimization problems for the penalized inequality when the penalty parameter tends to zero. Finally, we prove a convergence result for a finite element approximation of the shape optimization problem.

1. Introduction

Since the 1960s, variational inequalities have been a focus of research in the fields of mathematical theories, numerical solutions, and applications ([34,18,23,22,58]). On the other hand, many theoretical results have appeared on the properties of solutions to Stokes problems, for instance, existence and uniqueness in [15,16], regularity in [53,52], and the continuous dependence on the data in [35]. In these references, the slip and leak boundary conditions of the friction type are expressed by monotone relations between physical quantities, and as a result the weak formulations of the corresponding problems have the form of variational inequalities. Since that time, many publications have appeared on variational inequalities for the Stokes equation, such as [37,40,17,38,31,9,2,10]. The finite element approximation for a variational inequality of the Stokes equation with a nonlinear slip boundary condition has been studied in [39,32]. Optimal control problems related to variational inequalities are considered in [4,56].

Hemivariational inequality represents a powerful tool in the study of a large number of nonlinear boundary value problems involving nonmonotone relations between physical quantities. The notion of a hemivariational inequality has been first proposed by Panagiotopoulos in the early 1980s ([49]), and its concept and development are closely related to the generalized gradient of a locally Lipschitz functional in the sense of Clarke ([7,8]). Some early comprehensive references on hemivariational inequalities include [50,47,46,27]. In these monographs, mathematical theories and numerical solutions of hemivariational inequalities have been studied systematically. Since then hemivariational inequalities have been widely exploited in a variety of subjects and have solved a large number of problems. For example, the study of hemivariational inequalities and their applications can be found in several works such as [5,54,36,21]. In recent years, numerical methods for solving hemivariational inequalities have been extensively studied. The finite element method is analyzed in [11] for solving a stationary Stokes hemivariational inequality with a slip boundary condition. The optimal order error estimates are given in [57] by using the linear virtual element solutions of elliptic hemivariational inequalities. With the development of science and technology, optimal control theory has a wide range of applications in the fields of mechanical system analysis, aviation and space technology, vehicle drive dynamics, and plant man-

☆ The project is supported the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, the Ministry of Science and Higher Education of Republic of Poland under Grant Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019, and the National Science Centre of Poland under Project No. 2021/41/B/ST1/01636, and Basic and Advanced Research Project of CQ CSTC No. cstc2020jcyj-msxmX0738, and National Natural Science Foundation of China No. 11771350.

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https://doi.org/10.1016/j.camwa.2023.06.039
Received 16 January 2023; Received in revised form 22 May 2023; Accepted 30 June 2023
Available online 18 July 2023
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agement. Research on optimal control problems for systems governed by hemivariational inequalities can be found in [12,42,43].

Shape optimization of nonlinear partial differential equations is a growing field in modern structural optimization design ([25,55]). Shape optimization of fluid models with slip boundary conditions as the state problem has great practical significance. For example, shape optimization of the interior of hydraulic elements can reduce the velocity gradient and thus achieve energy savings ([28]). Hämäläinen et al. in [20] have studied a shape optimization problem for the incompressible flows using the stabilized finite element method. The discretization and convergence analysis of the shape optimization for the Stokes problem with threshold slip boundary conditions are discussed in [30], see also [24]. Haslinger et al. in [26] have analyzed the existence of the optimal shape problem described by generalized Navier-Stokes equations. Further, the numerical analysis of the shape optimization problem for Navier-Stokes equations was presented in [29]. In all aforementioned references, the weak formulations of the problems are variational inequalities.

In this paper, we consider a hemivariational inequality problem for the stationary Stokes equations in a bounded domain $\mathrm{Q} \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\mathrm{Q}$. The boundary is divided into a slip boundary Γ_S and a non-slip boundary $\Gamma_D = \partial\mathrm{Q} \setminus \overline{\Gamma}_S$, where the measures $|\Gamma_S| > 0$, $|\Gamma_D| > 0$, and $\overline{\Gamma}_D \cap \overline{\Gamma}_S = \emptyset$. We consider the Stokes problem for steady flows of incompressible viscous fluids:

$$-\nu_0\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in Q}, \tag{1.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in Q}, \tag{1.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{1.1c}$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_S, \tag{1.1d}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, $p = p(\boldsymbol{x})$ and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$ are the flow velocity field, pressure and density of external forces, respectively. Let $\boldsymbol{\nu}$ and τ be the unit outward normal and tangential vectors on the boundary $\partial\mathrm{Q}$, respectively. If $\boldsymbol{v}$ is a vector-valued function on the boundary, its normal and tangential components are represented by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, respectively. With the fluid velocity $\boldsymbol{u}$ and the pressure p, we define the strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)$ and the Cauchy stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu_0\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ denotes the identity matrix and ν_0 is the kinematic viscosity. We denote the normal and the tangential components of $\boldsymbol{\sigma}$ on the boundary by $\sigma_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\sigma}\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively. We assume that the function $j : \Gamma_S \times \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz with respect to its second argument, and ∂j denotes the Clarke generalized subgradient of $j(\boldsymbol{x}, \cdot)$. To simplify notation, we shall write $j(\boldsymbol{x}, \boldsymbol{u}_\tau)$ as $j(\boldsymbol{u}_\tau)$. The condition (1.1d) is known as a slip boundary condition on Γ_S. It consists of the impermeability (no leak) condition $u_\nu = 0$ which means that the fluid cannot pass through this boundary outside the domain, and a multivalued friction condition between the friction force $\boldsymbol{\sigma}_\tau = 2\nu_0(\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{\nu})_\tau$ and the tangential velocity $\boldsymbol{u}_\tau$. If the potential j is a convex function, the problem of the Stokes equations leads to a variational inequality, see [14]. When the function j is nonconvex, the problem (1.1) corresponds to a hemivariational inequality.

We recall the definition of the generalized directional derivative and generalized subgradient in the Clarke sense for a locally Lipschitz continuous function ([8]).

Definition 1.1. Let V be a Banach space with the dual space V^*, and $f : V \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of f at $u \in V$ in the direction $v \in V$, denoted by $f^0(u; v)$, is defined by

$$f^0(u; v) = \limsup_{w \to u,\, \lambda \downarrow 0} \frac{f(w + \lambda v) - f(w)}{\lambda}.$$

The generalized subgradient or subdifferential of f at $u \in V$ is defined by

$$\partial f(u) = \{\zeta \in V^* \mid f^0(u; v) \geq \langle \zeta, v\rangle \ \text{ for all } \ v \in V\}.$$

We also need the following properties of the generalized directional derivative and the generalized subgradient, which can be found, for example, in [44].

Proposition 1.2. *Let V be a Banach space and $f : V \to \mathbb{R}$ be a locally Lipschitz function. Then the following holds.*

(i) *For every $u \in V$, the function $f^0(u; \cdot) : V \to \mathbb{R}$ is positively homogeneous and subadditive, i.e., $f^0(u; \lambda v) = \lambda f^0(u; v)$ for all $\lambda > 0$, $v \in V$ and $f^0(u; v_1 + v_2) \leq f^0(u; v_1) + f^0(u; v_2)$ for all $v_1, v_2 \in V$, respectively.*
(ii) *The function $f^0 : V \times V \to \mathbb{R}$ is upper semicontinuous, i.e., for all $\{u_n\}$, $\{v_n\} \subset V$, $u, v \in V$ such that $u_n \to u$ and $v_n \to v$ in V, we have $\limsup_{n\to\infty} f^0(u_n; v_n) \leq f^0(u; v)$.*
(iii) *For all $u, v \in V$, we have $f^0(u; v) = \max\{\langle \zeta, v\rangle \mid \zeta \in \partial f(u)\}$.*

The organization of this paper is as follows: In the next section, we introduce the weak formulation of the Stokes problem (1.1). In Section 3, we present the shape optimization problem related to the Stokes hemivariational inequality and recall some auxiliary material. The existence result for the shape optimization problem is established in Section 4. In Section 5, we discuss the shape optimization problem related to the Stokes hemivariational inequality problem with penalized terms, and analyze the stability of the shape optimization problem under consideration. In Section 6, we apply the finite element method to the convergence analysis of the shape optimization problem for the penalized Stokes problem.

2. Weak formulation

To give a weak formulation of the problem (1.1) we first introduce the following function spaces:

$$\boldsymbol{V}(\mathrm{Q}) = \{\boldsymbol{v} \in (H^1(\mathrm{Q}))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ v_\nu = 0 \text{ on } \Gamma_S\},$$

$$\boldsymbol{V}_{\mathrm{div}}(\mathrm{Q}) = \{\boldsymbol{v} \in \boldsymbol{V}(\mathrm{Q}) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in Q}\},$$

$$\boldsymbol{V}_0(\mathrm{Q}) = (H_0^1(\mathrm{Q}))^2,$$

$$L_0^2(\mathrm{Q}) = \Big\{q \in L^2(\mathrm{Q}) \mid \int_{\mathrm{Q}} q\, dx = 0\Big\}.$$

Let d be a positive integer and $\mathbb{S}^d$ be the space of symmetric $d \times d$ matrices. The canonical inner products on $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted by "$\cdot$" and "$:$", respectively.

The standard norms in $(L^2(\mathrm{Q}))^d$ and $(H^1(\mathrm{Q}))^d$ are denoted by $\|\cdot\|_{0,\mathrm{Q}}$ and $\|\cdot\|_{1,\mathrm{Q}}$, respectively. Since the measure $|\Gamma_D| > 0$, it is well known that the following Korn inequality, see [33, Lemma 6.2], is satisfied

$$C_K^{1/2}\|\boldsymbol{v}\|_{1,\mathrm{Q}} \leq \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \ \text{ for all } \ \boldsymbol{v} \in \boldsymbol{V}(\mathrm{Q}), \tag{2.1}$$

where $\mathcal{H} = L^2(\mathrm{Q}; \mathbb{S}^d)$ and the Korn constant $C_K > 0$ depends only on Q and Γ_D. Hence, we conclude that on $\boldsymbol{V}(\mathrm{Q})$ the norms $\|\cdot\|_{1,\mathrm{Q}}$ and $|\cdot|_{1,\mathrm{Q}} := \|\boldsymbol{\varepsilon}(\cdot)\|_{\mathcal{H}}$ are equivalent, and $(\boldsymbol{V}(\mathrm{Q}), |\cdot|_{1,\mathrm{Q}})$ is a Hilbert space. By the Sobolev trace theorem, we know that the tangential trace operator $\gamma : \boldsymbol{V}(\mathrm{Q}) \to L^2(\Gamma_S)^2$ given by $\gamma\boldsymbol{v} = \boldsymbol{v}_\tau$ on Γ_S for $\boldsymbol{v} \in \boldsymbol{V}(\mathrm{Q})$ is linear and continuous. We still denote its operator norm by $\|\gamma\|$.

The mixed variational formulation of the problem (1.1) involves a hemivariational inequality and it can be derived analogously as in [11, Section 3].

Problem ($\mathcal{M}$). Find $(\boldsymbol{u}, p) \in \boldsymbol{V}(\mathrm{Q}) \times L_0^2(\mathrm{Q})$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq (\boldsymbol{f}, \boldsymbol{v})_{0,\mathrm{Q}} \ \text{ for all } \ \boldsymbol{v} \in \boldsymbol{V}(\mathrm{Q}), \tag{2.2}$$

$$b(\boldsymbol{u}, q) = 0 \ \text{ for all } \ q \in L_0^2(\mathrm{Q}). \tag{2.3}$$

Here $a : \boldsymbol{V}(\mathrm{Q}) \times \boldsymbol{V}(\mathrm{Q}) \to \mathbb{R}$ and $b : \boldsymbol{V}(\mathrm{Q}) \times L_0^2(\mathrm{Q}) \to \mathbb{R}$ are defined by

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu_0 \int_Q \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \quad \text{for} \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}(Q), \tag{2.4}$$

$$b(\boldsymbol{v},q) = \int_Q q \operatorname{div} \boldsymbol{v}\,dx \quad \text{for} \quad \boldsymbol{v} \in \boldsymbol{V}(Q),\ q \in L^2_0(Q), \tag{2.5}$$

and

$$(\boldsymbol{f},\boldsymbol{v})_{0,Q} = \int_Q \boldsymbol{f}\cdot\boldsymbol{v}\,dx \quad \text{for} \quad \boldsymbol{v} \in \boldsymbol{V}(Q),\ \boldsymbol{f} \in L^2(Q)^2. \tag{2.6}$$

Note that the incompressibility constraint (1.1b) implies $\Delta\boldsymbol{u} = 2\operatorname{Div}\boldsymbol{\varepsilon}(\boldsymbol{u})$, see, e.g., [28, Remark 2.1], and, therefore, we have

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu_0 \int_Q \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \nu_0 \int_Q \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,dx \quad \text{for} \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}(Q).$$

We observe that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive, see [41],

$$|a(\boldsymbol{u},\boldsymbol{v})| \le 2\nu_0\, |\boldsymbol{u}|_{1,Q} |\boldsymbol{v}|_{1,Q} \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}(Q), \tag{2.7}$$

$$a(\boldsymbol{v},\boldsymbol{v}) = 2\nu_0\, |\boldsymbol{v}|^2_{1,Q} \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(Q). \tag{2.8}$$

The bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the inf-sup condition: there are constants $\rho_0,\ \rho_1 > 0$ such that

$$|b(\boldsymbol{v},q)| \le \rho_0\, |\boldsymbol{v}|_{1,Q} \|q\|_{0,Q} \quad \text{for all} \quad (\boldsymbol{v},q) \in \boldsymbol{V}(Q)\times L^2_0(Q), \tag{2.9}$$

$$\rho_1\, \|q\|_{0,Q} \le \sup_{\boldsymbol{v}\in \boldsymbol{V}_0(Q)} \frac{b(\boldsymbol{v},q)}{|\boldsymbol{v}|_{1,Q}} \quad \text{for all} \quad q \in L^2_0(Q). \tag{2.10}$$

A detailed proof of (2.10) can be found [19, (5.14), page 81].

Concerning the superpotential j, we assume the following hypothesis.

<u>$H(j)$</u>: $j : \Gamma_S \times \mathbb{R}^2 \to \mathbb{R}$ is such that

(i) $j(\cdot,\boldsymbol{\xi})$ is measurable on Γ_S for all $\boldsymbol{\xi}\in\mathbb{R}^2$ and $j(\cdot,\boldsymbol{0}) \in L^1(\Gamma_S)$,
(ii) $j(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}^2$ for a.e. $\boldsymbol{x}\in\Gamma_S$,
(iii) $\|\boldsymbol{\eta}\|_{\mathbb{R}^2} \le c_0 + c_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^2}$ for all $\boldsymbol{\xi}\in\mathbb{R}^2$, $\boldsymbol{\eta}\in\partial j(\boldsymbol{x},\boldsymbol{\xi})$, a.e. $\boldsymbol{x}\in\Gamma_S$ with c_0, $c_1 \ge 0$,
(iv) $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -m_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{\mathbb{R}^2}$ for all $\boldsymbol{\xi}_i \in \mathbb{R}^2$, $\boldsymbol{\eta}_i \in \partial j(\boldsymbol{x},\boldsymbol{\xi}_i)$, $i = 1,\ 2$, a.e. $\boldsymbol{x}\in\Gamma_S$ with a constant $m_\tau \ge 0$.

Remark 2.1. The inequality in $H(j)$(iv) is called the relaxed monotonicity condition for the subgradient $\partial j(\boldsymbol{x},\cdot)$ for a.e. $\boldsymbol{x}\in\Gamma_S$, and it is known from [44] that it is equivalent to

$$j^0(\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1) + j^0(\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \le m_\tau\, \|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2_{\mathbb{R}^2} \quad \text{for all} \quad \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{R}^2.$$

Moreover, if, in addition, $j(\boldsymbol{x},\cdot)$ for a.e. $\boldsymbol{x}\in\Gamma_S$ is a convex function, then the relaxed monotonicity condition, or equivalently $H(j)$(iv) holds, with $m_\tau = 0$, due to the monotonicity of the convex subdifferential.

Example 2.2. We provide an example of the potential function in the slip boundary condition (1.1d) such that the Stokes problem (1.1) does not lead to a variational inequality, and results of [14] are not applicable. Here, we are motivated by [11, Section 5]. Let $j : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$j(\boldsymbol{\xi}) = \int_0^{\|\boldsymbol{\xi}\|} \mu(r)\,dr \quad \text{for} \quad \boldsymbol{\xi}\in\mathbb{R}^2, \tag{2.11}$$

where the integrand satisfies

<u>$H(\mu)$</u>: $\mu : [0,+\infty) \to \mathbb{R}$ is such that

(1) μ is continuous,
(2) $|\mu(r)| \le \mu_0 + \mu_1 |r|$ for $r \ge 0$ with $\mu_0, \mu_1 \ge 0$,
(3) $\mu(r_1) - \mu(r_2) \ge -m_\mu (r_1 - r_2)$ for $r_1 > r_2 \ge 0$ with $m_\mu > 0$.

Under $H(\mu)$(1) and (2), the function given by (2.11) is locally Lipschitz, Clarke regular, and satisfies $H(j)$(ii) and (iii). If, in addition, $H(\mu)$(3) holds, then $H(j)$ is satisfied with $m_\tau = m_\mu$. The condition $H(\mu)$(3) is the so-called one-sided Lipschitz condition, which allows the function to decrease at a rate not faster than m_μ. The concrete function $\mu(r) = (a-b)e^{-\alpha r} + b$ with $a > b$ and α nonnegative constants, satisfies $H(\mu)$ with $m_\mu = \alpha(a-b)$. If $H(\mu)$(1) and (2) hold, then $\partial j(\boldsymbol{\xi}) = \mu(\|\boldsymbol{\xi}\|)\partial\|\boldsymbol{\xi}\|$ for $\boldsymbol{\xi}\in\mathbb{R}^2$ and the friction in (1.1d) is modeled by a nonmonotone law which is equivalent to the threshold slip boundary condition of the form

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \le \mu(0) & \text{if} \ \ \boldsymbol{u}_\tau = 0 \\ -\boldsymbol{\sigma}_\tau = \mu(\|\boldsymbol{u}_\tau\|)\dfrac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} & \text{if} \ \ \boldsymbol{u}_\tau \ne 0. \end{cases} \tag{2.12}$$

In this way we recover the slip boundary condition of frictional type with the threshold function μ depending on the fluid tangential speed $\|\boldsymbol{u}_\tau\|$, see [2,9,24,28]. The coefficient of friction μ represents the magnitude of the slip bound (or limiting friction bound) at which the slip begins.

Theorem 2.3. *Assume $H(j)$, $\boldsymbol{f} \in (L^2(Q))^2$, and*

$$2\nu_0 > m_\tau \|\gamma\|^2, \quad 2\nu_0 C_K > c_1 \|\gamma\|^2. \tag{2.13}$$

Then Problem $(\mathcal{M})$ has the unique solution $(\boldsymbol{u},p) \in \boldsymbol{V}(Q)\times L^2_0(Q)$ and

$$\|\boldsymbol{u}\|_{1,Q} + \|p\|_{0,Q} \le c\,(1 + \|\boldsymbol{f}\|_{0,Q}), \tag{2.14}$$

where a positive constant c depends on Q, Γ_S, $\|\gamma\|$, C_K, c_0, c_1, ν_0, and is independent of $\boldsymbol{f}$.

Proof. For the existence and uniqueness part, we use [11, Theorems 3.4 and 3.5] which is summarized as follows. Under $H(j)$ and the first smallness condition in (2.13), for any $\boldsymbol{f}\in L^2(Q)$, Problem $(\mathcal{M})$ has a unique solution $(\boldsymbol{u},p) \in \boldsymbol{V}(Q)\times L^2_0(Q)$.

Next, we establish (2.14). To this end, we use the second smallness condition in (2.13). Let us take $\boldsymbol{v} = -\boldsymbol{u}$ in (2.2), and from (2.3), we obtain

$$a(\boldsymbol{u},\boldsymbol{u}) \le \int_{\Gamma_S} j^0(\boldsymbol{u}_\tau;-\boldsymbol{u}_\tau)\,ds + (\boldsymbol{f},\boldsymbol{u})_{0,Q}. \tag{2.15}$$

By Proposition 1.2 (iii) and $H(j)$ (iii), we obtain

$$\begin{aligned} j^0(\boldsymbol{u}_\tau;-\boldsymbol{u}_\tau) &= \max\{\boldsymbol{\eta}\cdot(-\boldsymbol{u}_\tau) \mid \boldsymbol{\eta}\in\partial j(\boldsymbol{u}_\tau)\} \\ &\le \|\boldsymbol{\eta}\|_{\mathbb{R}^2}\|\boldsymbol{u}_\tau\|_{\mathbb{R}^2} \le (c_0 + c_1\|\boldsymbol{u}_\tau\|_{\mathbb{R}^2})\|\boldsymbol{u}_\tau\|_{\mathbb{R}^2}. \end{aligned} \tag{2.16}$$

From the Korn inequality (2.1), (2.8), (2.15), and (2.16), we have

$$\begin{aligned} 2\nu_0 C_K \|\boldsymbol{u}\|^2_{1,Q} &\le 2\nu_0|\boldsymbol{u}|^2_{1,Q} = a(\boldsymbol{v},\boldsymbol{v}) \\ &\le \int_{\Gamma_S} (c_0 + c_1\|\boldsymbol{u}_\tau\|_{\mathbb{R}^2})\|\boldsymbol{u}_\tau\|_{\mathbb{R}^2}\,ds + \|\boldsymbol{f}\|_{0,Q}\|\boldsymbol{u}\|_{0,Q} \\ &\le c_0 \int_{\Gamma_S} \|\boldsymbol{u}_\tau\|_{\mathbb{R}^2}\,ds + c_1 \int_{\Gamma_S} \|\boldsymbol{u}_\tau\|^2_{\mathbb{R}^2}\,ds + \|\boldsymbol{f}\|_{0,Q}\|\boldsymbol{u}\|_{1,Q} \\ &= c_0\sqrt{|\Gamma_S|}\|\boldsymbol{u}_\tau\|_{0,\Gamma_S} + c_1\,\|\boldsymbol{u}_\tau\|^2_{0,\Gamma_S} + \|\boldsymbol{f}\|_{0,Q}\|\boldsymbol{u}\|_{1,Q} \\ &\le (c_0\sqrt{|\Gamma_S|} + c_1\,\|\gamma\|\|\boldsymbol{u}\|_{1,Q})\|\gamma\|\|\boldsymbol{u}\|_{1,Q} + \|\boldsymbol{f}\|_{0,Q}\|\boldsymbol{u}\|_{1,Q}, \end{aligned} \tag{2.17}$$

where we have used the Hölder inequality and continuity of the tangential trace operator. Hence

$$(2\nu_0 C_K - c_1\|\gamma\|^2)\|\boldsymbol{u}\|_{1,Q} \le c_0\sqrt{|\Gamma_S|}\|\gamma\| + \|\boldsymbol{f}\|_{0,Q}. \tag{2.18}$$

It follows from (2.18) and the smallness condition (2.13) that

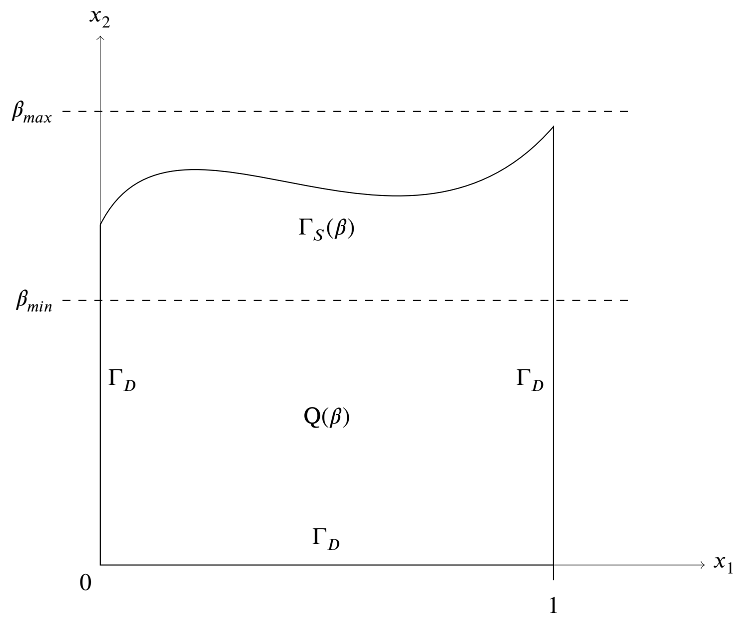

Fig. 1. Geometry of admissible domains.

$$\|\boldsymbol{u}\|_{1,Q} \leq (2\nu_0 C_K - c_1\|\gamma\|^2)^{-1}(c_0\sqrt{|\Gamma_S|}\|\gamma\| + \|\boldsymbol{f}\|_{0,Q}). \tag{2.19}$$

Now, we show the boundedness of the norm of the pressure. Using the test function $\boldsymbol{v} \in \boldsymbol{V}_0(Q)$, from (2.1), (2.2), (2.7) and (2.18), we deduce

$$b(\boldsymbol{v}, p) \leq a(\boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_{0,Q} \leq c\,(1 + \|\boldsymbol{f}\|_{0,Q})|\boldsymbol{v}|_{1,Q},$$

where $c > 0$ depends on Q, Γ_S, $\|\gamma\|$, C_K, c_0, c_1, ν_0, and it is independent of $\boldsymbol{f}$. Thus, it follows from (2.10) that

$$\rho_1\|p\|_{0,Q} \leq \sup_{\boldsymbol{v}\in \boldsymbol{V}_0(Q)} \frac{b(\boldsymbol{v}, p)}{|\boldsymbol{v}|_{1,Q}} \leq c\,(1 + \|\boldsymbol{f}\|_{0,Q}).$$

This completes the proof. □

Remark 2.4.

(a) The first smallness condition in (2.13) means that if the loss of monotonicity of $j(\boldsymbol{x}, \cdot)$ is not "too large" and is compensated by the coercivity constant of the form a, then Problem $(\mathcal{M})$ has a unique solution. A particular form of the problem (1.1) with the boundary conditions $u_\nu = 0$ and (2.12) has been studied under an analogous smallness condition in [28, Theorem 2.4, see (2.16), and Remark 2.5] by a fixed point argument.
(b) Remark that if, in addition, $j(\boldsymbol{x}, \cdot)$ for a.e. $\boldsymbol{x} \in \Gamma_S$ is convex, then $H(j)$(iv) and the first smallness condition in (2.13) hold trivially, see Remark 2.1. Note also that in Theorem 2.3, the pressure is unique up to a constant in $L^2(Q)$, while in $L^2_0(Q)$ it is unique. For a more general existence and uniqueness results for a class of variational-hemivariational inequalities, we refer to [45, Theorem 18].

3. Formulation of shape optimization problem

In this section we shall formulate a shape optimization problem for the Stokes model (1.1) and provide some preparatory material needed in the next sections. In the shape optimization problem the goal is to find a domain from an admissible class of sets which minimizes a cost functional depending on the solution to (1.1).

In Section 2 we have treated the Stokes problem in a fixed domain. For an optimization problem we will consider a family of admissible domains, parametrized by suitable functions. We suppose that only a part of the boundary with the prescribed slip condition is the object of optimization.

Let us consider a specific family of sets whose part is represented by the graph of a function (see Fig. 1):

$$\mathcal{O} = \{Q(\beta) \mid \beta \in \Sigma_{ad}\},$$

where

$$\Sigma_{ad} = \{\beta \in C^{1,1}([0,1]) \mid \beta_{\min} \leq \beta \leq \beta_{\max} \text{ in } [0,1],$$
$$|\beta^{(i)}| \leq C_i,\ i = 1, 2, \text{ a.e. in } (0,1)\},$$
$$Q(\beta) = \{\boldsymbol{x} = (x_1, x_2) \mid x_1 \in (0,1),\ x_2 \in (0, \beta(x_1))\} \quad \text{for} \quad \beta \in \Sigma_{ad},$$

see [24]. Here, $C^{1,1}([0,1])$ is the set of all $(1,1)$-Hölder continuous functions on $[0,1]$, $\beta^{(i)}$ is the ith-order derivative of β with respect to x_1, and the positive constants $\beta_{\min}$, $\beta_{\max}$, C_1, C_2 are such that $\Sigma_{ad} \neq \emptyset$. In what follows, the functions $\beta \in \Sigma_{ad}$ will play the role of the control variables from the admissible set of controls Σ_{ad}.

Let the slip boundary $\Gamma_S(\beta)$ be the graph of a function β, i.e.,

$$\Gamma_S(\beta) = \{(x_1, x_2) \mid x_1 \in (0,1),\ x_2 = \beta(x_1)\},\ \beta \in \Sigma_{ad}.$$

To underline the dependence of the Stokes problem on a parameter by β, we shall write $\boldsymbol{V}(\beta) := \boldsymbol{V}(Q(\beta))$, $\boldsymbol{V}_{\text{div}}(\beta) := \boldsymbol{V}_{\text{div}}(Q(\beta))$, $\boldsymbol{V}_0(\beta) := \boldsymbol{V}_0(Q(\beta))$, and $L^2_0(\beta) := L^2_0(Q(\beta))$. When the domain Q and the boundary part Γ_S are replaced by $Q(\beta)$ and $\Gamma_S(\beta)$, respectively, then the bilinear forms a_β, b_β and the function j_β represent a in (2.4), b in (2.5), and j in $H(j)$, respectively. Further, we assume that $\boldsymbol{f} \in (L^2_{\text{loc}}(\mathbb{R}^2))^2$, for simplicity.

The weak formulation of Problem $(\mathcal{M})$ on a set $Q(\beta)$ reads as follows.

Problem $(\mathcal{M}(\beta))$. Find $(\boldsymbol{u}(\beta), p(\beta)) \in \boldsymbol{V}(\beta) \times L^2_0(\beta)$ such that

$$a_\beta(\boldsymbol{u}(\beta), \boldsymbol{v}) - b_\beta(\boldsymbol{v}, p(\beta)) + \int_{\Gamma_S(\beta)} j^0_\beta(\boldsymbol{u}_\tau(\beta); \boldsymbol{v}_\tau)\,ds \geq (\boldsymbol{f}, \boldsymbol{v})_{0,Q(\beta)}$$
$$\text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(\beta), \tag{3.1}$$

$$b_\beta(\boldsymbol{u}(\beta), q) = 0 \quad \text{for all} \quad q \in L^2_0(\beta). \tag{3.2}$$

In Problem $(\mathcal{M}(\beta))$ various functions have different domains of definition, and to compare them, we need their extension to the common domain $\hat{\mathrm{Q}}$ defined by $\hat{\mathrm{Q}} = (0,1) \times (0, 2\beta_{\max})$. From now on, we suppose that $\mathrm{Q}(\rho) \subseteq \hat{\mathrm{Q}}$ for all $\rho \in \Sigma_{ad}$.

The analysis of Problem $(\mathcal{M}(\beta))$ as well as of the shape optimization problem introduced below will heavily rely on several facts which hold uniformly, for all admissible domains $\mathrm{Q}(\beta)$ or equivalently for all functions $\beta \in \Sigma_{ad}$. These facts are collected in the following.

Remark 3.1.

(a) Let $\psi_\beta : V(\beta) \to (H^1(\hat{\mathrm{Q}}))^2$ be a linear and bounded extension operator for $\beta \in \Sigma_{ad}$. The family $\mathcal{O}$ consists of domains with Lipschitz boundaries, and therefore all domains $\mathrm{Q}(\beta)$, $\beta \in \Sigma_{ad}$ satisfy the uniform cone property, see [6]. Hence, by [6, Theorem II.1], see also [25, Theorem A.8], there exists an extension operator ψ_β whose norm is estimated independently of $\beta \in \Sigma_{ad}$.
(b) The norm $\|\gamma\|$ of the tangential trace operator $\gamma : \boldsymbol{V}(\beta) \to L^2(\Gamma_S(\beta))^2$ can be bounded from above uniformly with respect to $\beta \in \Sigma_{ad}$, see [25, Lemma 2.19, p. 62].
(c) The constant in the Korn inequality can be chosen uniformly with respect to a class of domains with the uniform cone property, see [48]. Hence, the constant $C_K > 0$ in (2.1) can be selected independently of $\beta \in \Sigma_{ad}$.
(d) The constant in the inf-sup condition for pressure can be chosen independently of $\beta \in \Sigma_{ad}$, see [30, Lemma 2].

Let $W : \Delta \to \mathbb{R}$ be a prescribed cost functional, where

$$\Delta = \left\{ (\beta, \boldsymbol{w}, q) \mid \beta \in \Sigma_{ad},\ \boldsymbol{w} \in \boldsymbol{V}(\beta),\ q \in L_0^2(\beta) \right\}.$$

For simplicity, we write $\mathcal{W}(\beta) := W(\beta, \boldsymbol{u}(\beta), p(\beta))$. Now we introduce the following shape optimization problem.

Problem $(\mathbb{M})$. Find $\beta^* \in \Sigma_{ad}$ such that

$$\mathcal{W}(\beta^*) \le \mathcal{W}(\beta) \ \text{ for all } \ \beta \in \Sigma_{ad},$$

where $(\boldsymbol{u}(\beta), p(\beta))$ is the unique solution of Problem $(\mathcal{M}(\beta))$, see Lemma 4.1 in Section 4.

In order to prove the existence of the solutions to Problem $(\mathbb{M})$, an auxiliary material is needed. First, we recall the definitions of convergence of domains in $\mathcal{O}$, and convergence of functions defined on different domains, see [25, Definitions 1-3].

Definition 3.2. Let $\mathrm{Q}(\beta_n)$, $\mathrm{Q}(\beta) \subset \mathcal{O}$, where β_n, $\beta \in \Sigma_{ad}$, $n \in \mathbb{N}$. We say that $\mathrm{Q}(\beta_n) \to \mathrm{Q}(\beta)$, if

$$\beta_n \to \beta \ \text{ in } \ C^1([0,1]). \tag{3.3}$$

Definition 3.3. Let $\boldsymbol{w}_n \in \boldsymbol{V}(\beta_n)$, $\boldsymbol{w} \in \boldsymbol{V}(\beta)$, where β_n, $\beta \in \Sigma_{ad}$, $n \in \mathbb{N}$. We say that $\boldsymbol{w}_n \rightharpoonup \boldsymbol{w}$ (and $\boldsymbol{w}_n \to \boldsymbol{w}$, *respectively*), if

$$\psi_{\beta_n} \boldsymbol{w}_n \rightharpoonup \psi_\beta \boldsymbol{w} \text{ (and } \psi_{\beta_n} \boldsymbol{w}_n \to \psi_\beta \boldsymbol{w}, \textit{respectively}) \ \text{ in } \ (H^1(\hat{\mathrm{Q}}))^2, \tag{3.4}$$

where ψ_{β_n} and ψ_β denote the bounded and linear extension operators, see Remark 3.1(a), whose operator norms are bounded independently of $\beta \in \Sigma_{ad}$.

Definition 3.4. Let $e_n \in H_0^1(\mathrm{Q}(\beta_n))$, $e \in H_0^1(\mathrm{Q}(\beta))$, where β_n, $\beta \in \Sigma_{ad}$, $n \in \mathbb{N}$. We say that $e_n \rightharpoonup e$ (and $e_n \to e$, *respectively*), if

$$e_n^0 \rightharpoonup e^0 \text{ (and } e_n^0 \to e^0, \textit{respectively}) \ \text{ in } \ H_0^1(\hat{\mathrm{Q}}), \tag{3.5}$$

where the symbol "e^0" stands for the zero extension of a function $e \in H_0^1(\mathrm{Q}(\beta))$ to the set $\hat{\mathrm{Q}}$, i.e.,

$$e^0 = \begin{cases} e & \text{in } \mathrm{Q}(\beta), \\ 0 & \text{in } \hat{\mathrm{Q}} \setminus \overline{\mathrm{Q}}(\beta). \end{cases}$$

Similarly we define the convergence of a sequence $\{q_n\}$, where $q_n \in L_0^2(\beta_n)$.

We conclude this section by recalling three auxiliary lemmas which will be used in the next section. The first result can be easily proved based on the definition of Σ_{ad}, see also [30, Lemma 1].

Lemma 3.5. *Let χ_n, χ be the characteristic functions of the sets $\mathrm{Q}(\beta_n)$ and $\mathrm{Q}(\beta)$, respectively. Then*

(i) *the family of domains $\mathcal{O}$ is compact with respect to the convergence of sets introduced in Definition 3.2,*
(ii) *if $\mathrm{Q}(\beta_n) \to \mathrm{Q}(\beta)$ with β_n, $\beta \in \Sigma_{ad}$, then*

$$\chi_n \to \chi \ \text{ in } \ L^q(\hat{\mathrm{Q}}) \ \text{ for all } \ q \in [1, \infty). \tag{3.6}$$

The next lemma on the continuity type result for the trace operator, plays an important role in a proof of convergence of solutions to Problem $(\mathcal{M}(\beta_n))$, see [25, Lemma 2.21, p. 65].

Lemma 3.6. *Let $\{(\beta_n, \boldsymbol{w}_n)\}$, $\beta_n \in \Sigma_{ad}$, $\boldsymbol{w}_n \in (H^1(\hat{\mathrm{Q}}))^2$ be such that*

$$\beta_n \rightrightarrows \beta \ \text{ (uniformly) } \ \text{on } \ [0,1],$$

$$\boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \ \text{ in } \ (H^1(\hat{\mathrm{Q}}))^2,\ n \to \infty,$$

for some $\beta \in \Sigma_{ad}$ and $\boldsymbol{w} \in (H^1(\hat{\mathrm{Q}}))^2$. Then

$$\boldsymbol{w}_n|_{\Gamma_S(\beta_n)} \circ \beta_n \to \boldsymbol{w}|_{\Gamma_S(\beta)} \circ \beta \ \text{ in } \ (L^2(0,1))^2,\ n \to \infty,$$

where $\boldsymbol{w}_n|_{\Gamma_S(\beta_n)} \circ \beta_n = \boldsymbol{w}(x_1, \beta_n(x_1))$ and $\boldsymbol{w}|_{\Gamma_S(\beta)} \circ \beta = \boldsymbol{w}(x_1, \beta(x_1))$, $x_1 \in (0,1)$.

Finally, we also recall a result that can be found in [30, Lemma 3].

Lemma 3.7. *Let $\beta_n \to \beta$ in $C^1([0,1])$, β_n, $\beta \in \Sigma_{ad}$ and $\boldsymbol{v} \in \boldsymbol{V}(\beta)$ be given. Then there exist a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\hat{\mathrm{Q}}))^2$ and a function $\overline{\boldsymbol{v}} \in (H^1(\hat{\mathrm{Q}}))^2$ such that $\overline{\boldsymbol{v}}|_{\mathrm{Q}(\beta)} = \boldsymbol{v}$ and*

$$\boldsymbol{v}_k \to \overline{\boldsymbol{v}} \ \text{ in } \ (H^1(\hat{\mathrm{Q}}))^2,\ k \to \infty, \tag{3.7}$$

where for any $k \in \mathbb{N}$ and an appropriate $n_k \in \mathbb{N}$, we have

$$\boldsymbol{v}_k|_{\mathrm{Q}(\beta_{n_k})} \in \boldsymbol{V}(\beta_{n_k}). \tag{3.8}$$

4. Existence of an optimal shape

In what follows, we will establish the existence result for solutions to the shape optimization problem, Problem $(\mathbb{M})$.

We begin with the following result on a unique solvability and a stability estimate for Problem $(\mathcal{M}(\beta))$, for every fixed $\beta \in \Sigma_{ad}$. It can be proved similarly as Theorem 2.3 provided we admit hypotheses on the data which hold for all $\beta \in \Sigma_{ad}$, uniformly with respect to $\beta \in \Sigma_{ad}$.

Consider the following hypothesis.

$\underline{H(j_\beta)}$: $\quad j_\beta : \Gamma_S(\beta) \times \mathbb{R}^2 \to \mathbb{R}$ is such that

(i) $j_\beta(\cdot, \boldsymbol{\xi})$ is measurable on $\Gamma_S(\beta)$ for all $\boldsymbol{\xi} \in \mathbb{R}^2$ and $j_\beta(\cdot, \boldsymbol{0}) \in L^1(\Gamma_S(\beta))$, for all $\beta \in \Sigma_{ad}$,
(ii) $j_\beta(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^2$ for a.e. $\boldsymbol{x} \in \Gamma_S(\beta)$, all $\beta \in \Sigma_{ad}$,
(iii) there are constants c_0, $c_1 > 0$ independent of $\beta \in \Sigma_{ad}$ such that for all $\beta \in \Sigma_{ad}$, all $\boldsymbol{\xi} \in \mathbb{R}^2$, $\boldsymbol{\eta} \in \partial j_\beta(\boldsymbol{x}, \boldsymbol{\xi})$, a.e. $\boldsymbol{x} \in \Gamma_S(\beta)$, it holds $\|\boldsymbol{\eta}\|_{\mathbb{R}^2} \le c_0 + c_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^2}$,
(iv) there is a constant $m_\tau \ge 0$ independent of $\beta \in \Sigma_{ad}$ such that for all $\beta \in \Sigma_{ad}$, all $\boldsymbol{\xi}_i \in \mathbb{R}^2$, $\boldsymbol{\eta}_i \in \partial j_\beta(\boldsymbol{x}, \boldsymbol{\xi}_i)$, $i = 1, 2$, a.e. $\boldsymbol{x} \in \Gamma_S(\beta)$, we have

$$(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -m_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^2}^2.$$

(v) for every $\beta_n \to \beta$ in $C^1([0,1])$, $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$, $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ in $\mathbb{R}^2$, it holds

$$\limsup_{n\to\infty} j^0_{\beta_n}(\boldsymbol{\xi}_n;\boldsymbol{\eta}_n) \le j^0_\beta(\boldsymbol{\xi};\boldsymbol{\eta}).$$

For an example of functions satisfying $H(j_\beta)$(v), we refer to [13, Example 2.4]

Lemma 4.1. *Assume $H(j_\beta)$(i)-(iv), $\boldsymbol{f} \in (L^2_{loc}(\mathbb{R}^2))^2$, and*

$$2\nu_0 > m_\tau \|\gamma\|^2, \quad 2\nu_0 C_K > c_1\|\gamma\|^2. \tag{4.1}$$

Then, for any fixed $\beta \in \Sigma_{ad}$, Problem $(\mathcal{M}(\beta))$ has the unique solution $(\boldsymbol{u}(\beta), p(\beta)) \in \boldsymbol{V}(\beta) \times L^2_0(\beta)$, and there exists a positive constant $d := d(\boldsymbol{f}, c_0, c_1)$ independent of $\beta \in \Sigma_{ad}$ such that

$$\|\psi_\beta \boldsymbol{u}(\beta)\|_{1,\hat{\mathrm{Q}}} + \|p^0(\beta)\|_{0,\hat{\mathrm{Q}}} \le d. \tag{4.2}$$

Proof. The existence of a unique solution, for any fixed $\beta \in \Sigma_{ad}$, can be established as in the proof of Theorem 2.3.

The first term in (4.2) can be estimated as follows. Analogously as in the proof of (2.18) in Theorem 2.3, we have

$$(2\nu_0 C_K - c_1\|\gamma\|^2)\|\boldsymbol{u}(\beta)\|_{1,\mathrm{Q}(\beta)} \le c_0\sqrt{|\Gamma_S(\beta)|}\,\|\gamma\| + \|\boldsymbol{f}\|_{0,\mathrm{Q}(\beta)}.$$

On the other hand, since $\boldsymbol{u}(\beta) \in \boldsymbol{V}(\beta)$, by the boundedness of the extension operator, we know that there exists a constant $c_2 > 0$ independent of $\beta \in \Sigma_{ad}$ such that

$$\|\psi_\beta \boldsymbol{u}(\beta)\|_{1,\hat{\mathrm{Q}}} \le c_2\,\|\boldsymbol{u}(\beta)\|_{1,\mathrm{Q}(\beta)}.$$

We combine the last two inequalities to get

$$\|\psi_\beta \boldsymbol{u}(\beta)\|_{1,\hat{\mathrm{Q}}} \le c_2\,\frac{c_0\sqrt{|\Gamma_S(\beta)|}\,\|\gamma\| + \|\boldsymbol{f}\|_{0,\mathrm{Q}(\beta)}}{2\nu_0 C_K - c_1\|\gamma\|^2}. \tag{4.3}$$

It follows from Remark 3.1 (a), (b) and (c) that positive constants c_2, $\|\gamma\|$ and C_K, respectively, do not depend on $\beta \in \Sigma_{ad}$. Recall that also ν_0 and the constants c_0 and c_1, by $H(j_\beta)$ (iii) are independent of $\beta \in \Sigma_{ad}$. Therefore, the upper bound in (4.3) does not depend on $\beta \in \Sigma_{ad}$, and we obtain the uniform bound on the term $\|\psi_\beta \boldsymbol{u}(\beta)\|_{1,\hat{\mathrm{Q}}}$.

For the estimate of the second term in (4.2), let $\boldsymbol{v} \in \boldsymbol{V}_0(\mathrm{Q}(\beta))$. From (3.1), analogously as in Theorem 2.3, we have

$$b(\boldsymbol{v}, p(\beta)) \le a(\boldsymbol{u}(\beta), \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_{0,\mathrm{Q}(\beta)} \le c_3\,(1 + \|\boldsymbol{f}\|_{0,\mathrm{Q}(\beta)}\,|\boldsymbol{v}|_{1,\mathrm{Q}(\beta)}),$$

where $c_3 > 0$ is independent of $\beta \in \Sigma_{ad}$. Having in mind Remark 3.1 (d), from the inf-sup condition (2.10), it follows

$$\rho_1\|p(\beta)\|_{0,\mathrm{Q}(\beta)} \le \sup_{\boldsymbol{v}\in \boldsymbol{V}_0(\beta)} \frac{b(\boldsymbol{v}, p(\beta))}{|\boldsymbol{v}|_{1,\mathrm{Q}(\beta)}} \le d,$$

with $d > 0$ independent of $\beta \in \Sigma_{ad}$. This completes the proof of (4.2). □

The following convergence result is important in the forthcoming analysis.

Theorem 4.2. *Let $\beta_n,\ \beta \in \Sigma_{ad}$, and $(\boldsymbol{u}_n, p_n) := (\boldsymbol{u}(\beta_n), p(\beta_n)) \in \boldsymbol{V}(\beta_n) \times L^2_0(\beta_n)$, $n \in \mathbb{N}$, be a solution of Problem $(\mathcal{M}(\beta_n))$. Suppose that $H(j_\beta)$ and (3.3) hold, and that there exists an element $(\overline{\boldsymbol{u}}, \overline{p}) \in (H^1_0(\hat{\mathrm{Q}}))^2 \times L^2_0(\hat{\mathrm{Q}})$ such that*

$$\psi_{\beta_n}\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } (H^1(\hat{\mathrm{Q}}))^2, \tag{4.4}$$

$$p^0_n \rightharpoonup \overline{p} \quad \text{in } L^2_0(\hat{\mathrm{Q}}),\ n \to \infty. \tag{4.5}$$

Then the pair $(\boldsymbol{u}(\beta), p(\beta)) := (\overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)}, \overline{p}|_{\mathrm{Q}(\beta)})$ is a solution of Problem $(\mathcal{M}(\beta))$.

Proof. First we show that $(\overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)}, \overline{p}|_{\mathrm{Q}(\beta)}) \in \boldsymbol{V}_{\mathrm{div}}(\beta) \times L^2_0(\beta)$. Using the definition of $(\boldsymbol{u}_n, p_n)$, from (4.4) and (4.5), we have $\boldsymbol{u}(\beta) := \overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)} = 0$ on $\Gamma_D(\beta)$ and $p(\beta) := \overline{p}|_{\mathrm{Q}(\beta)} \in L^2_0(\beta)$. Let $\boldsymbol{\nu}^\beta := \boldsymbol{\nu}^\beta(x_1)$ and $\boldsymbol{\nu}^{\beta_n} := \boldsymbol{\nu}^{\beta_n}(x_1)$ be the unit outward normal vector of $\Gamma_S(\beta)$ and $\Gamma_S(\beta_n)$, respectively. We denote their natural extensions defined in $\hat{\mathrm{Q}}$ by the same notation, i.e., $\boldsymbol{\nu}^\beta(\boldsymbol{x}) := \boldsymbol{\nu}^\beta(x_1)$ and $\boldsymbol{\nu}^{\beta_n}(\boldsymbol{x}) := \boldsymbol{\nu}^{\beta_n}(x_1)$, where $\boldsymbol{x} = (x_1, x_2) \in \hat{\mathrm{Q}}$. Since $\beta_n \to \beta$ in $C^1([0,1])$, we have

$$\boldsymbol{\nu}^{\beta_n} \to \boldsymbol{\nu}^\beta \quad \text{in } C(\overline{\hat{\mathrm{Q}}}).$$

From Definition 3.3, we know that $\boldsymbol{u}(\beta_n) \rightharpoonup \boldsymbol{u}(\beta)$ in $(H^1(\hat{\mathrm{Q}}))^2$. Thus,

$$\boldsymbol{u}_n \cdot \boldsymbol{\nu}^{\beta_n} = (\boldsymbol{u}_n - \boldsymbol{u}) \cdot \boldsymbol{\nu}^{\beta_n} + \boldsymbol{u} \cdot \boldsymbol{\nu}^{\beta_n} \to \boldsymbol{u} \cdot \boldsymbol{\nu}^\beta,\ n \to \infty.$$

Therefore, $\boldsymbol{u} \cdot \boldsymbol{\nu}^\beta = 0$ on $\Gamma_S(\beta)$. Let $\phi \in H^1(\mathrm{Q}(\beta))$ be given such that $\phi = 0$ on $\Gamma_D(\beta)$. Now we denote the extension of ϕ on $\hat{\mathrm{Q}}$ by $\tilde{\phi} \in H^1(\hat{\mathrm{Q}})$ satisfying $\tilde{\phi} = 0$ on $\partial\hat{\mathrm{Q}} \setminus [0,1] \times \{0\}$. Hence, $\tilde{\phi} \in H^1(\mathrm{Q}(\beta_n))$ and $\tilde{\phi} = 0$ on $\Gamma_D(\beta_n)$. Let $\overline{\phi} \in L^2_0(\mathrm{Q}(\beta_n)) = L^2(\mathrm{Q}(\beta_n))/\mathbb{R}$. Note that $|\mathrm{Q}(\beta_n)| > 0$. Then there exists a constant $\kappa = (\int_{\mathrm{Q}(\beta_n)} \tilde{\phi}\,dx)/|\mathrm{Q}(\beta_n)|$ such that $\tilde{\phi} = \overline{\phi} + \kappa$. Since $\boldsymbol{u}_n \in \boldsymbol{V}(\beta_n)$ for all $n \in \mathbb{N}$, we deduce that

$$\begin{aligned}
\int_{\mathrm{Q}(\beta_n)} \tilde{\phi}\,\mathrm{div}\,\boldsymbol{u}_n\,dx &= \int_{\mathrm{Q}(\beta_n)} \overline{\phi}\,\mathrm{div}\,\boldsymbol{u}_n\,dx + \int_{\mathrm{Q}(\beta_n)} \kappa\,\mathrm{div}\,\boldsymbol{u}_n\,dx \\
&= \kappa \int_{\mathrm{Q}(\beta_n)} \mathrm{div}\,\boldsymbol{u}_n\,dx \\
&= \kappa \int_{\partial\mathrm{Q}(\beta_n)} \boldsymbol{u}_n \cdot \boldsymbol{\nu}^{\beta_n}\,ds - \kappa \int_{\mathrm{Q}(\beta_n)} \boldsymbol{u}_n \cdot 0\,dx \\
&= \kappa \int_{\Gamma_S(\beta_n)} \boldsymbol{u}_n \cdot \boldsymbol{\nu}^{\beta_n}\,ds \\
&= 0,
\end{aligned}$$

where the third equation is due to the Green formula. By applying the Green formula again, we get

$$\int_{\mathrm{Q}(\beta_n)} \boldsymbol{u}_n \cdot \nabla\tilde{\phi}\,dx = \int_{\partial\mathrm{Q}(\beta_n)} \tilde{\phi}(\boldsymbol{u}_n \cdot \boldsymbol{\nu}^{\beta_n})\,ds - \int_{\mathrm{Q}(\beta_n)} \tilde{\phi}\,\mathrm{div}\,\boldsymbol{u}_n\,dx = 0,$$

which means that

$$\int_{\hat{\mathrm{Q}}} \chi_n \psi_{\beta_n}\boldsymbol{u}_n \cdot \nabla\tilde{\phi}\,dx = 0,$$

where χ_n denotes the characteristic function of the set $\mathrm{Q}(\beta_n)$. Applying the Lebesgue-dominated convergence theorem, see [44, Theorem 1.65], Lemma 3.5 (ii) and (3.6), we obtain

$$\int_{\hat{\mathrm{Q}}} \chi_n \psi_{\beta_n}\boldsymbol{u}_n \cdot \nabla\tilde{\phi}\,dx \to \int_{\hat{\mathrm{Q}}} \chi\overline{\boldsymbol{u}} \cdot \nabla\tilde{\phi}\,dx = \int_{\mathrm{Q}(\beta)} \boldsymbol{u}(\beta) \cdot \nabla\phi\,dx = 0,$$

where χ is the characteristic function of the set $\mathrm{Q}(\beta)$. Subsequently, let $\phi \in C^\infty_0(\mathrm{Q}(\beta)) \subset H^1_0(\mathrm{Q}(\beta))$. Since

$$\begin{aligned}
\int_{\mathrm{Q}(\beta)} \boldsymbol{u}(\beta) \cdot \nabla\phi\,dx &= \int_{\partial\mathrm{Q}(\beta)} \phi(\boldsymbol{u}(\beta) \cdot \boldsymbol{\nu}^\beta)\,ds - \int_{\mathrm{Q}(\beta)} \phi\,\mathrm{div}\boldsymbol{u}(\beta)\,dx \\
&= -\int_{\mathrm{Q}(\beta)} \phi\,\mathrm{div}\boldsymbol{u}(\beta)\,dx = 0,
\end{aligned}$$

using the variational lemma [44, Lemma 2.9], we get that $\boldsymbol{u}(\beta) \in \boldsymbol{V}_{\mathrm{div}}(\beta)$.

It remains to prove that $(\boldsymbol{u}(\beta), p(\beta))$ solves Problem $(\mathcal{M}(\beta))$. Let $\boldsymbol{v} \in \boldsymbol{V}(\beta)$. Then there exists a sequence $\{\boldsymbol{v}_k\}$ with $\boldsymbol{v}_k \in (H^1(\hat{\mathrm{Q}}))^2$ satisfying (3.7) and (3.8). Using $\boldsymbol{v}_k$ as a test function in Problem $(\mathcal{M}(\beta_{n_k}))$, we get

$$a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k) - b_{n_k}(\boldsymbol{v}_k, p_{n_k}) + \int_{\Gamma_S(\beta_{n_k})} j^0_{n_k}(\boldsymbol{u}_{n_k\tau}; \boldsymbol{v}_{k\tau})\,ds \ge (\boldsymbol{f}, \boldsymbol{v}_k)_{0,\mathrm{Q}(\beta_{n_k})}, \tag{4.6}$$

where, for simplicity of notation, we write $a_{n_k} := a_{\beta_{n_k}}$, $b_{n_k} := b_{\beta_{n_k}}$ and $j_{n_k} := j_{\beta_{n_k}}$. Next, by (4.4), (3.7), and Lemma 3.5 (ii), it follows that

$$\lim_{k\to\infty}\int_{\hat{Q}} \chi_{n_k}\nabla\psi_{\beta_{n_k}}\boldsymbol{u}_{n_k}\cdot\nabla\boldsymbol{v}_k\,dx$$
$$=\lim_{k\to\infty}\int_{\hat{Q}} \chi_{n_k}\left[\nabla\psi_{\beta_{n_k}}\boldsymbol{u}_{n_k}\cdot(\nabla\boldsymbol{v}_k-\nabla\overline{\boldsymbol{v}})+\nabla\psi_{\beta_{n_k}}\boldsymbol{u}_{n_k}\cdot\nabla\overline{\boldsymbol{v}}\right]dx$$
$$=\int_{\hat{Q}}\chi\nabla\overline{\boldsymbol{u}}\cdot\nabla\overline{\boldsymbol{v}}\,dx.$$

Therefore, it yields

$$\begin{aligned}\lim_{k\to\infty} a_{n_k}(\boldsymbol{u}_{n_k},\boldsymbol{v}_k) &= \nu_0\lim_{k\to\infty}\int_{Q(\beta_{n_k})}\nabla\boldsymbol{u}_{n_k}\cdot\nabla\boldsymbol{v}_k\,dx\\ &=\nu_0\lim_{k\to\infty}\int_{\hat{Q}}\chi_{n_k}\nabla\psi_{\beta_{n_k}}\boldsymbol{u}_{n_k}\cdot\nabla\boldsymbol{v}_k\,dx\\ &=\nu_0\int_{\hat{Q}}\chi\nabla\overline{\boldsymbol{u}}\cdot\nabla\overline{\boldsymbol{v}}\,dx\\ &=\nu_0\int_{Q(\beta)}\nabla\boldsymbol{u}(\beta)\cdot\nabla\boldsymbol{v}\,dx\\ &=a_\beta(\boldsymbol{u}(\beta),\boldsymbol{v}).\end{aligned}\tag{4.7}$$

From (4.5), (3.7) and Lemma 3.5(ii), we obtain

$$\lim_{k\to\infty}\int_{\hat{Q}}\chi_{n_k}p^0_{n_k}\,\mathrm{div}\boldsymbol{v}_k\,dx=\int_{\hat{Q}}\chi\,\overline{p}\,\mathrm{div}\overline{\boldsymbol{v}}\,dx,$$

and subsequently

$$\begin{aligned}\lim_{k\to\infty} b_{n_k}(\boldsymbol{v}_k,p_{n_k}) &=\lim_{k\to\infty}\int_{Q(\beta_{n_k})}p_{n_k}\cdot\mathrm{div}\boldsymbol{v}_k\,dx\\ &=\int_{Q(\beta)}p(\beta)\cdot\mathrm{div}\boldsymbol{v}\,dx\\ &=b_\beta(\boldsymbol{v},p(\beta)).\end{aligned}\tag{4.8}$$

In view of (3.7), we have

$$\lim_{k\to\infty}\int_{\hat{Q}}\chi_{n_k}\boldsymbol{f}\cdot\boldsymbol{v}_k\,dx=\int_{\hat{Q}}\chi\boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,dx,$$

and

$$\begin{aligned}\lim_{k\to\infty}(\boldsymbol{f},\boldsymbol{v}_k)_{0,Q(\beta_{n_k})} &=\lim_{k\to\infty}\int_{Q(\beta_{n_k})}\boldsymbol{f}\cdot\boldsymbol{v}_k\,dx\\ &=\int_{Q(\beta)}\boldsymbol{f}\cdot\boldsymbol{v}\,dx\\ &=(\boldsymbol{f},\boldsymbol{v})_{0,Q(\beta)}.\end{aligned}\tag{4.9}$$

Since $\boldsymbol{u}_{n_k}(\boldsymbol{x})=\boldsymbol{u}(x_1,\beta_{n_k}(x_1))$ on $\Gamma_S(\beta_{n_k})$ and $\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{u}(x_1,\beta(x_1))$ on $\Gamma_S(\beta)$, from Lemma 3.6 we have

$$\boldsymbol{u}(x_1,\beta_{n_k}(x_1))\to\boldsymbol{u}(x_1,\beta(x_1))\text{ in }(L^2(0,1))^2,\ k\to\infty.$$

Since $\beta'_{n_k}\rightrightarrows\beta'$ uniformly in $[0,1]$, applying $H(j_\beta)$ and the Fatou lemma ([44, Theorem 1.64]), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\int_{\Gamma_S(\beta_{n_k})} j^0_{\beta_{n_k}}(\boldsymbol{u}_{n_k\tau};\boldsymbol{v}_{k\tau})\,ds &=\limsup_{k\to\infty}\int_0^1 j^0_{\beta_{n_k}}(\boldsymbol{u}_{n_k\tau};\boldsymbol{v}_{k\tau})\sqrt{1+|\beta'_{n_k}|^2}\,dx_1\\ &\le\int_0^1\limsup_{k\to\infty} j^0_{\beta_{n_k}}(\boldsymbol{u}_{n_k\tau};\boldsymbol{v}_{k\tau})\sqrt{1+|\beta'_{n_k}|^2}\,dx_1\\ &\le\int_0^1 j^0_\beta(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\sqrt{1+|\beta'|^2}\,dx_1\\ &=\int_{\Gamma_S(\beta)} j^0_\beta(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,ds.\end{aligned}\tag{4.10}$$

It follows from (4.7)-(4.10) that $(\boldsymbol{u}(\beta),p(\beta))$ is a solution of Problem $(\mathcal{M}(\beta))$ which completes the proof. □

We conclude this section with an existence result for the optimal shape design problem, Problem $(\mathbb{M})$.

Theorem 4.3. *Suppose the hypotheses of Theorem 4.2. If the cost functional $W:\Delta\to\mathbb{R}$ satisfies the following lower-semicontinuity condition:*

$$\begin{cases}\beta_n\to\beta & \text{in } C^1([0,1]),\ \beta_n,\ \beta\in\Sigma_{ad}\\ \boldsymbol{w}_n\rightharpoonup\boldsymbol{w} & \text{in } (H^1(\hat{Q}))^2,\ \boldsymbol{w}_n,\ \boldsymbol{w}\in(H^1_0(\hat{Q}))^2\\ q_n\rightharpoonup q & \text{in } L^2(\hat{Q}),\ q_n,\ q\in L^2_0(\hat{Q})\end{cases}$$
$$\Longrightarrow\ \liminf_{n\to\infty}W(\beta_n,\boldsymbol{w}_n|_{Q(\beta_n)},q_n|_{Q(\beta_n)})\ge W(\beta,\boldsymbol{w}|_{Q(\beta)},q|_{Q(\beta)}),\tag{4.11}$$

then Problem $(\mathbb{M})$ has a solution.

Proof. Let $\{(\boldsymbol{u}_n,p_n)\}$ be a minimizing sequence for Problem $(\mathbb{M})$, i.e.,

$$\lim_{n\to\infty}W(\beta_n,\boldsymbol{u}(\beta_n),p(\beta_n))=\inf_{\beta\in\Sigma_{ad}}W(\beta,\boldsymbol{u}(\beta),p(\beta)),$$

where $(\boldsymbol{u}_n,p_n)$ solves Problem $(\mathcal{M}(\beta_n))$. Let $(\psi_{\beta_n}\boldsymbol{u}_n,p^0_n)\in(H^1(\hat{Q}))^2\times L^2_0(\hat{Q})$. Then, from Lemma 4.1 and Theorem 4.2, we know that there exists an element $(\overline{\boldsymbol{u}},\overline{p})\in(H^1_0(\hat{Q}))^2\times L^2_0(\hat{Q})$ satisfying

$$\begin{aligned}\beta_n&\to\beta^*\quad\text{in } C^1([0,1]),\\ \psi_{\beta_n}\boldsymbol{u}_n&\rightharpoonup\overline{\boldsymbol{u}}\quad\text{in } (H^1(\hat{Q}))^2,\\ p^0_n&\rightharpoonup\overline{p}\quad\text{in } L^2_0(\hat{Q}),\end{aligned}$$

where $\beta_n,\ \beta^*\in\Sigma_{ad}$. The lower-semicontinuity property (4.11) of W yields

$$\begin{aligned}\liminf_{n\to\infty}W(\beta_n,\psi_{\beta_n}\boldsymbol{u}_n|_{Q(\beta_n)},p^0_n|_{Q(\beta_n)}) &\ge W(\beta^*,\overline{\boldsymbol{u}}|_{Q(\beta^*)},\overline{p}|_{Q(\beta^*)})\\ &=W(\beta^*,\boldsymbol{u}(\beta^*),p(\beta^*)).\end{aligned}$$

Thus

$$W(\beta^*,\boldsymbol{u}(\beta^*),p(\beta^*))\le W(\beta,\boldsymbol{u}(\beta),p(\beta))$$

and the proof is completed. □

5. Penalized problem

In this section we shall establish the existence and uniqueness result for the Stokes hemivariational inequality problem with a penalty term. Moreover, we will discuss the shape optimization problem for the penalized Stokes hemivariational inequality. To this end, we introduce the following functional spaces.

$$\widetilde{\boldsymbol{V}}(\beta)=\{\boldsymbol{v}\in(H^1(Q(\beta)))^2\mid\boldsymbol{v}=\boldsymbol{0}\ \text{ on }\ \Gamma_D(\beta)\},$$
$$\widetilde{\boldsymbol{V}}_{\mathrm{div}}(\beta)=\{\boldsymbol{v}\in\widetilde{\boldsymbol{V}}(\beta)\mid b_\beta(\boldsymbol{v},q)=0\ \text{ for all }\ q\in L^2_0(\beta)\},\ \beta\in\Sigma_{ad}.$$

The penalty term $c_\beta(\boldsymbol{u},\boldsymbol{v})$ is defined as follows:

$$c_\beta(\boldsymbol{u},\boldsymbol{v})=\int_0^1(\boldsymbol{u}\circ\beta\cdot\boldsymbol{\nu}^\beta)(\boldsymbol{v}\circ\beta\cdot\boldsymbol{\nu}^\beta)\,dx_1,\tag{5.1}$$

where $\boldsymbol{u}\circ\beta\cdot\boldsymbol{\nu}^\beta=\boldsymbol{u}(x_1,\beta(x_1))\cdot\boldsymbol{\nu}^\beta(x_1)$, $x_1\in(0,1)$. We formulate a hemivariational inequality with a penalty term.

Problem $(\mathcal{M}_\varepsilon(\beta))$. Find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \widetilde{\boldsymbol{V}}(\beta) \times L_0^2(\beta)$ such that

$$a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - b_\beta(\boldsymbol{v}, p_\varepsilon) + \int_{\Gamma_S(\beta)} j_\beta^0(\boldsymbol{u}_{\varepsilon\tau}; \boldsymbol{v}_\tau)\, ds + \frac{1}{\varepsilon} c_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{v})$$

$$\geq (\boldsymbol{f}, \boldsymbol{v})_{0,\mathrm{Q}(\beta)} \quad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}}(\beta), \tag{5.2}$$

$$b_\beta(\boldsymbol{u}_\varepsilon, q) = 0 \quad \forall q \in L_0^2(\beta), \tag{5.3}$$

where $\beta \in \Sigma_{ad}$ and $\varepsilon > 0$ is a penalty parameter.

First, we will present the existence and uniqueness result for Problem $(\mathcal{M}_\varepsilon(\beta))$.

Theorem 5.1. *Assume $H(j_\beta)$, $\boldsymbol{f} \in (L^2_{loc}(\mathbb{R}^2))^2$ and (2.13). Then, for any fixed $\varepsilon > 0$, Problem $(\mathcal{M}_\varepsilon(\beta))$ has the unique solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$, and*

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \ \text{ in } \ (H^1(\mathrm{Q}(\beta)))^2, \tag{5.4}$$

$$p_\varepsilon \rightharpoonup p \ \text{ in } \ L_0^2(\beta), \ \varepsilon \to 0, \tag{5.5}$$

where $(\boldsymbol{u}, p)$ solves Problem $(\mathcal{M}(\beta))$.

Proof. First, we show that Problem $(\mathcal{M}_\varepsilon(\beta))$ is uniquely solvable. Let $a_{\beta\varepsilon}(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) = a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + \frac{1}{\varepsilon} c_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{v})$ in (5.2). Since $|a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{v})| \leq |\boldsymbol{u}_\varepsilon|_{1,\mathrm{Q}(\beta)} |\boldsymbol{v}|_{1,\mathrm{Q}(\beta)}$ and $a_\beta(\boldsymbol{v}, \boldsymbol{v}) = |\boldsymbol{v}|^2_{1,\mathrm{Q}(\beta)}$, from (5.1), we have

$$|a_{\beta\varepsilon}(\boldsymbol{u}_\varepsilon, \boldsymbol{v})| \leq c_1(\varepsilon) |\boldsymbol{u}_\varepsilon|_{1,\mathrm{Q}(\beta)} |\boldsymbol{v}|_{1,\mathrm{Q}(\beta)}$$

and

$$a_{\beta\varepsilon}(\boldsymbol{v}, \boldsymbol{v}) \geq c_2(\varepsilon) |\boldsymbol{v}|^2_{1,\mathrm{Q}(\beta)},$$

where $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are positive constants. Hence, the form $a_{\beta\varepsilon}(\cdot,\cdot)$ is continuous and coercive. Thus, similarly as in the proof of Theorem 2.3, we deduce that Problem $(\mathcal{M}_\varepsilon(\beta))$ has a unique solution.

Next, we will prove (5.4) and (5.5) in three steps.

Step 1. We claim that $\{\|\boldsymbol{u}_\varepsilon\|_{1,\mathrm{Q}(\beta)}\}$ and $\{\|p_\varepsilon\|_{0,\mathrm{Q}(\beta)}\}$ are bounded.

Setting $\boldsymbol{v} = -\boldsymbol{u}_\varepsilon$ in (5.2) and using $b_\beta(\boldsymbol{u}_\varepsilon, p_\varepsilon) = 0$, we obtain

$$a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) + \frac{1}{\varepsilon} c_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) \leq \int_{\Gamma_S(\beta)} j_\beta^0(\boldsymbol{u}_{\varepsilon\tau}; -\boldsymbol{u}_{\varepsilon\tau})\, ds + (\boldsymbol{f}, \boldsymbol{u}_\varepsilon)_{0,\mathrm{Q}(\beta)}.$$

Then the boundedness of $\{\|\boldsymbol{u}_\varepsilon\|_{1,\mathrm{Q}(\beta)}\}$ is obtained from (2.1), (2.8) and (2.13). Using a proof similar to that of Lemma 4.1 and the inf-sup condition, we also know that $\{\|p_\varepsilon\|_{0,\mathrm{Q}(\beta)}\}$ is bounded.

Step 2. We show the weak convergences

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \quad \text{in } (H^1(\mathrm{Q}(\beta)))^2, \ \ p_\varepsilon \rightharpoonup p \quad \text{in } L_0^2(\beta), \ \varepsilon \to 0.$$

Since $\|\boldsymbol{u}_\varepsilon\|_{1,\mathrm{Q}(\beta)}$ and $\|p_\varepsilon\|_{0,\mathrm{Q}(\beta)}$ are bounded by a constant independent of ε, there exist elements $\boldsymbol{u}^* \in (H^1(\mathrm{Q}(\beta)))^2$ and $p^* \in L_0^2(\beta)$ such that

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}^* \quad \text{in } (H^1(\mathrm{Q}(\beta)))^2, \ \ p_\varepsilon \rightharpoonup p^* \quad \text{in } L_0^2(\beta), \ \varepsilon \to 0.$$

By passing to a subsequence if necessary, we may assume that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}^*$ a.e. on $\Gamma_S(\beta)$. By Proposition 1.2(ii), we have

$$\limsup_{\varepsilon \to 0} j_\beta^0(\boldsymbol{u}_{\varepsilon\tau}; \boldsymbol{v}_\tau) \leq j_\beta^0(\boldsymbol{u}^*_\tau; \boldsymbol{v}_\tau).$$

Taking upper limit in (5.2), by Fatou's lemma, we get

$$(\boldsymbol{f}, \boldsymbol{v})_{0,\mathrm{Q}(\beta)} \leq a_\beta(\boldsymbol{u}^*, \boldsymbol{v}) - b_\beta(\boldsymbol{v}, p^*) + \int_{\Gamma_S(\beta)} j_\beta^0(\boldsymbol{u}^*_\tau; \boldsymbol{v}_\tau)\, ds \tag{5.6}$$

for any $\boldsymbol{v} \in \boldsymbol{V}(\beta)$. Letting $\varepsilon \to 0$ in (5.3), we obtain

$$b_\beta(\boldsymbol{u}^*, q) = 0 \quad \forall q \in L_0^2(\beta). \tag{5.7}$$

From (5.6), (5.7) and Theorem 2.3, we see that $(\boldsymbol{u}^*, p^*)$ is the unique solution of Problem $(\mathcal{M}(\beta))$, i.e., $\boldsymbol{u} = \boldsymbol{u}^*$ and $p = p^*$.

Step 3. We prove the strong convergence

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \ \text{ in } \ (H^1(\mathrm{Q}(\beta)))^2 \text{ as } \varepsilon \to 0.$$

Setting $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\varepsilon$ in (5.2), we get

$$a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u} - \boldsymbol{u}_\varepsilon) - b_\beta(\boldsymbol{u} - \boldsymbol{u}_\varepsilon, p_\varepsilon) + \int_{\Gamma_S(\beta)} j_\beta^0(\boldsymbol{u}_{\varepsilon\tau}(\beta); \boldsymbol{u}_\tau - \boldsymbol{u}_{\varepsilon\tau})\, ds + \frac{1}{\varepsilon} c_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u} - \boldsymbol{u}_\varepsilon)$$

$$\geq (\boldsymbol{f}, \boldsymbol{u} - \boldsymbol{u}_\varepsilon)_{0,\mathrm{Q}(\beta)}. \tag{5.8}$$

Since $b_\beta(\boldsymbol{u}, p_\varepsilon) = 0$ and $b_\beta(\boldsymbol{u}_\varepsilon, p_\varepsilon) = 0$, taking upper limit in (5.8) and applying Proposition 1.2 (ii), we have

$$\begin{aligned}\limsup_{\varepsilon \to 0} a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) &\leq \limsup_{\varepsilon \to 0} \int_{\Gamma_S(\beta)} j_\beta^0(\boldsymbol{u}_{\varepsilon\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{\varepsilon\tau})\, ds \\ &\leq \int_{\Gamma_S(\beta)} \limsup_{\varepsilon \to 0} j_\beta^0(\boldsymbol{u}_{\varepsilon\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{\varepsilon\tau})\, ds \\ &\leq \int_{\Gamma_S(\beta)} j_\beta^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau - \boldsymbol{u}_\tau)\, ds \\ &= 0.\end{aligned}$$

Since

$$\begin{aligned}c\, \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\mathrm{Q}(\beta)} \leq |\boldsymbol{u}_\varepsilon - \boldsymbol{u}|^2_{1,\mathrm{Q}(\beta)} &= a_\beta(\boldsymbol{u}_\varepsilon - \boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) \\ &= a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) - a_\beta(\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u}),\end{aligned}$$

where a positive constant c is independent of ε, we obtain

$$c \limsup_{\varepsilon \to 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\mathrm{Q}(\beta)} \leq \limsup_{\varepsilon \to 0} a_\beta(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon - \boldsymbol{u}) \leq 0.$$

Hence,

$$\limsup_{\varepsilon \to 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|^2_{1,\mathrm{Q}(\beta)} = 0,$$

and $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ in $(H^1(\mathrm{Q}(\beta)))^2$ as $\varepsilon \to 0$. This completes the proof. □

Next, we introduce a class of optimal shape design problems for which we use the solution of Problem $(\mathcal{M}_\varepsilon(\beta))$ as the state variable. For any $\varepsilon > 0$ fixed, the optimization problem reads as follows:

Problem $(\mathbb{M}_\varepsilon)$. Find $\beta^*_\varepsilon \in \Sigma_{ad}$ such that

$$\mathcal{W}_\varepsilon(\beta^*_\varepsilon) \leq \mathcal{W}_\varepsilon(\beta) \quad \forall \beta \in \Sigma_{ad},$$

where $\mathcal{W}_\varepsilon(\beta) = W(\beta, \boldsymbol{u}_\varepsilon(\beta), p_\varepsilon(\beta))$ and $(\boldsymbol{u}_\varepsilon(\beta), p_\varepsilon(\beta))$ is the solution of Problem $(\mathcal{M}_\varepsilon(\beta))$. As before, we can also get the following result.

Theorem 5.2. *Suppose the hypotheses of Theorem 4.3. Then Problem $(\mathbb{M}_\varepsilon)$ has a solution for any $\varepsilon > 0$.*

Next, we will analyze the correlation between the solutions of Problems $(\mathbb{M})$ and $(\mathbb{M}_\varepsilon)$, as $\varepsilon \to 0$. Using similar arguments as in Section 4, we have

Lemma 5.3. *Assume that $H(j_\beta)$(i)-(iv), $\boldsymbol{f} \in (L^2_{loc}(\mathbb{R}^2))^2$ and (2.13) hold, and let $(\boldsymbol{u}_\varepsilon(\beta), p_\varepsilon(\beta))$ be a solution of Problem $(\mathcal{M}_\varepsilon(\beta))$ for any $\varepsilon > 0$. Then, we have the estimate*

$$\|\psi_\beta \boldsymbol{u}_\varepsilon(\beta)\|_{1,\widehat{\mathrm{Q}}} + \frac{1}{\varepsilon} c_\beta(\boldsymbol{u}_\varepsilon(\beta), \boldsymbol{u}_\varepsilon(\beta)) + \|p^0_\varepsilon(\beta)\|_{0,\widehat{\mathrm{Q}}} \leq c, \tag{5.9}$$

with a constant $c := c(\|\boldsymbol{f}\|_{0,\widehat{\mathrm{Q}}}) > 0$ independent of $\beta \in \Sigma_{ad}$.

Proof. First, we prove the boundedness of the first two terms in (5.9). Setting $\boldsymbol{v} = -\boldsymbol{u}_\varepsilon(\beta) \in \widetilde{\boldsymbol{V}}_{\mathrm{div}}(\beta)$ in Problem $(\mathcal{M}_\varepsilon(\beta))$, we have

$$a_\beta(\boldsymbol{u}_\varepsilon,\boldsymbol{u}_\varepsilon)+\frac{1}{\varepsilon}c_\beta(\boldsymbol{u}_\varepsilon,\boldsymbol{u}_\varepsilon)\le\int_{\Gamma_S(\beta)}j^0_\beta(\boldsymbol{u}_{\varepsilon\tau};-\boldsymbol{u}_{\varepsilon\tau})\,ds+(\boldsymbol{f},\boldsymbol{u}_\varepsilon)_{0,\mathrm{Q}(\beta)}.$$

Since $\frac{1}{\varepsilon}c_\beta(\boldsymbol{u}_\varepsilon,\boldsymbol{u}_\varepsilon)\ge 0$, the boundedness of the first two terms in (5.9) can be proved by using a similar approach as in Lemma 4.1. Next, we show the boundedness of $\{p_\varepsilon(\beta)\}$. From (5.2), we have

$$b_\beta(\boldsymbol{v},p_\varepsilon(\beta))\le a_\beta(\boldsymbol{u}_\varepsilon(\beta),\boldsymbol{v})-(\boldsymbol{f},\boldsymbol{v})_{0,\mathrm{Q}(\beta)}\le c|\boldsymbol{v}|_{1,\mathrm{Q}(\beta)},\quad\forall\boldsymbol{v}\in\boldsymbol{V}_0(\beta),$$

where the constant $c:=c(\|\boldsymbol{f}\|_{0,\hat{\mathrm{Q}}})>0$ does not depend on $\beta\in\Sigma_{ad}$. Thus,

$$\rho_1\|p_\varepsilon(\beta)\|_{0,\mathrm{Q}(\beta)}\le\sup_{\boldsymbol{v}\in\boldsymbol{V}_0(\beta)}\frac{b_\beta(\boldsymbol{v},p_\varepsilon(\beta))}{|\boldsymbol{v}|_{1,\mathrm{Q}(\beta)}}\le c.$$

Hence, we arrive at the estimate (5.9). □

The following lemma is crucial in proving the convergence of solutions to Problem $(\mathbb{M}_\varepsilon)$ as $\varepsilon\to 0$.

Lemma 5.4. *Let $H(j_\beta)$ and (3.3) hold, and $\{(\boldsymbol{u}_n,p_n)\}$ be the sequence of solutions to Problem $(\mathcal{M}_{\varepsilon_n}(\beta_n))$, $\varepsilon_n\to 0$. If there exists a pair $(\overline{\boldsymbol{u}},\overline{p})\in(H^1_0(\hat{\mathrm{Q}}))^2\times L^2_0(\hat{\mathrm{Q}})$ such that*

$$\psi_{\beta_n}\boldsymbol{u}_n\rightharpoonup\overline{\boldsymbol{u}}\quad\text{in }(H^1(\hat{\mathrm{Q}}))^2,\tag{5.10}$$

$$p^0_n\rightharpoonup\overline{p}\quad\text{in }L^2_0(\hat{\mathrm{Q}}),\ n\to\infty,\tag{5.11}$$

then $(\overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)},\overline{p}|_{\mathrm{Q}(\beta)})$ is a solution of Problem $(\mathcal{M}(\beta))$.

Proof. The existence of a subsequence satisfying (5.10) and (5.11) follows from Lemma 5.3. Since $\boldsymbol{u}_n\in\widetilde{\boldsymbol{V}}_{\mathrm{div}}(\beta_n)$, from (5.10) we have $\boldsymbol{u}:=\overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)}\in\widetilde{\boldsymbol{V}}_{\mathrm{div}}(\beta)$. Using Lemma 3.6, we obtain

$$c_n(\boldsymbol{u}_n,\boldsymbol{u}_n)\to c_\beta(\boldsymbol{u},\boldsymbol{u}),\ n\to\infty,\tag{5.12}$$

where, for brevity, we write $c_n:=c_{\beta_n}$. By the estimate (5.9), we know that

$$0\le c_n(\boldsymbol{u}_n,\boldsymbol{u}_n)\le c\,\varepsilon_n\to 0,\ n\to\infty.\tag{5.13}$$

Therefore, it follows from (5.12) and (5.13) that $\boldsymbol{u}\cdot\boldsymbol{\nu}^\beta=0$ on $\Gamma_S(\beta)$.

Next we show that $\boldsymbol{u}$ solves Problem $(\mathcal{M}(\beta))$. Let $\overline{\boldsymbol{v}}\in\boldsymbol{V}(\beta)$ be given. In view of Lemma 3.7, there exists a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k\in(H^1(\hat{\mathrm{Q}}))^2$ satisfying (3.7) and (3.8). Thus,

$$\frac{1}{\varepsilon_{n_k}}c_{n_k}(\boldsymbol{u}_{n_k},\boldsymbol{v}_k)=0.$$

Using $\boldsymbol{v}_k|_{\mathrm{Q}(\beta_{n_k})}$ as a test function in Problem $(\mathcal{M}_{\varepsilon_{n_k}}(\beta_{n_k}))$, we have

$$a_{n_k}(\boldsymbol{u}_{n_k},\boldsymbol{v}_k)-b_{n_k}(\boldsymbol{v}_k,p_{n_k})+\int_{\Gamma_S(\beta_{n_k})}j^0_{\beta_{n_k}}(\boldsymbol{u}_{n_k\tau};\boldsymbol{v}_{k\tau})\,ds\ge(\boldsymbol{f},\boldsymbol{v}_k)_{0,\mathrm{Q}(\beta_{n_k})}.$$

The rest of the proof is similar to that of Theorem 4.2 and we omit the details. This completes the proof. □

As a consequence, we obtain the following convergence result, which states the relation between solutions of Problems $(\mathbb{M})$ and $(\mathbb{M}_\varepsilon)$ for $\varepsilon\to 0$.

Theorem 5.5. *Suppose that $H(j_\beta)$, $\boldsymbol{f}\in(L^2_{loc}(\mathbb{R}^2))^2$ and (4.11) hold. Let $W:\Delta\to\mathbb{R}$ satisfy the following condition:*

$$\begin{cases}\beta_n\to\beta & \text{in } C^1([0,1]),\ \beta_n,\ \beta\in\Sigma_{ad}\\ \boldsymbol{w}_n\to\boldsymbol{w} & \text{in } (H^1(\hat{\mathrm{Q}}))^2,\ \boldsymbol{w}_n,\ \boldsymbol{w}\in(H^1_0(\hat{\mathrm{Q}}))^2\\ q_n\rightharpoonup q & \text{in } L^2(\hat{\mathrm{Q}}),\ q_n,\ q\in L^2_0(\hat{\mathrm{Q}})\end{cases}$$
$$\Longrightarrow\ \lim_{n\to\infty}W(\beta_n,\boldsymbol{w}_n|_{\mathrm{Q}(\beta_n)},q_n|_{\mathrm{Q}(\beta_n)})=W(\beta,\boldsymbol{w}|_{\mathrm{Q}(\beta)},q|_{\mathrm{Q}(\beta)}).\tag{5.14}$$

Then, for any sequence $\{\beta^*_\varepsilon\}$ of solutions to Problem $(\mathbb{M}_\varepsilon)$, $\varepsilon\to 0$, there exists its subsequence (denoted in the same way) and a triplet $(\beta^*,\boldsymbol{u}^*,p^*)\in\Sigma_{ad}\times(H^1_0(\hat{\mathrm{Q}}))^2\times L^2_0(\hat{\mathrm{Q}})$ such that

$$\beta^*_\varepsilon\to\beta^*\quad\text{in }C^1([0,1]),\tag{5.15}$$

$$\psi_{\beta^*_\varepsilon}\boldsymbol{u}_\varepsilon(\beta^*_\varepsilon)\rightharpoonup\boldsymbol{u}^*\quad\text{in }(H^1(\hat{\mathrm{Q}}))^2,\tag{5.16}$$

$$p^0_\varepsilon(\beta^*_\varepsilon)\rightharpoonup p^*\quad\text{in }L^2_0(\hat{\mathrm{Q}}),\ \varepsilon\to 0.\tag{5.17}$$

In addition, β^* is a solution of Problem $(\mathbb{M})$ and $(\boldsymbol{u}^*|_{\mathrm{Q}(\beta^*)},p^*|_{\mathrm{Q}(\beta^*)})$ solves Problem $(\mathcal{M}(\beta^*))$. Note that any accumulation point of $\{(\beta^*_\varepsilon,\boldsymbol{u}_\varepsilon(\beta^*_\varepsilon),p_\varepsilon(\beta^*_\varepsilon))\}$ in the sense of (5.15)-(5.17) is a solution of Problem $(\mathbb{M})$.

Proof. Since $\beta^*_\varepsilon\in\Sigma_{ad}$, by using the Arzela-Ascoli theorem, see [1, Theorem 1.6.3], we are able to find a subsequence $\{\beta^*_\varepsilon\}$ satisfying (5.15). In view of Lemmas 5.3 and 5.4, we know that (5.16) and (5.17) hold, and that $(\boldsymbol{u}^*|_{\mathrm{Q}(\beta^*)},p^*|_{\mathrm{Q}(\beta^*)})$ solves Problem $(\mathcal{M}(\beta^*))$. Next, we need to show that β^* solves Problem $(\mathbb{M})$. Let $\overline{\beta}\in\Sigma_{ad}$ be arbitrary and $(\boldsymbol{u}(\overline{\beta}),p(\overline{\beta}))$ be the solution pair of Problem $(\mathcal{M}(\overline{\beta}))$. In view of Definitions 3.3 and 3.4, from (5.4) and (5.5), we get

$$\boldsymbol{u}_\varepsilon(\overline{\beta})\to\boldsymbol{u}(\overline{\beta})\quad\text{in }(H^1(\mathrm{Q}(\overline{\beta})))^2,$$

$$p_\varepsilon(\overline{\beta})\rightharpoonup p(\overline{\beta})\quad\text{in }L^2_0(\mathrm{Q}(\overline{\beta})),\ \varepsilon\to 0,$$

and

$$\psi_{\overline{\beta}}\boldsymbol{u}_\varepsilon(\overline{\beta})\to\psi_{\overline{\beta}}\boldsymbol{u}(\overline{\beta})\quad\text{in }(H^1(\hat{\mathrm{Q}}))^2,\tag{5.18}$$

$$p^0_\varepsilon(\overline{\beta})\rightharpoonup p^0(\overline{\beta})\quad\text{in }L^2_0(\hat{\mathrm{Q}}),\ \varepsilon\to 0.\tag{5.19}$$

From the definition of Problem $(\mathbb{M}_\varepsilon)$, we have

$$W(\beta^*_\varepsilon,\boldsymbol{u}_\varepsilon(\beta^*_\varepsilon),p_\varepsilon(\beta^*_\varepsilon))\le W(\overline{\beta},\boldsymbol{u}_\varepsilon(\overline{\beta}),p_\varepsilon(\overline{\beta})),\tag{5.20}$$

while using (4.11) from (5.15)-(5.17), we obtain

$$W(\beta^*,\boldsymbol{u}^*|_{\mathrm{Q}(\beta^*)},p^*|_{\mathrm{Q}(\beta^*)})\le\liminf_{\varepsilon\to 0}W(\beta^*_\varepsilon,\psi_{\beta^*_\varepsilon}\boldsymbol{u}_\varepsilon(\beta^*_\varepsilon)|_{\mathrm{Q}(\beta^*_\varepsilon)},p^0_\varepsilon(\beta^*_\varepsilon)|_{\mathrm{Q}(\beta^*_\varepsilon)}).$$

Similarly, from (5.14) and (5.18)-(5.19), we deduce

$$\begin{aligned}\lim_{\varepsilon\to 0}W(\overline{\beta},\psi_{\overline{\beta}}\boldsymbol{u}_\varepsilon(\overline{\beta})|_{\mathrm{Q}(\overline{\beta})},p^0_\varepsilon(\overline{\beta})|_{\mathrm{Q}(\overline{\beta})})&=W(\overline{\beta},\psi_{\overline{\beta}}\boldsymbol{u}(\overline{\beta})|_{\mathrm{Q}(\overline{\beta})},p^0(\overline{\beta})|_{\mathrm{Q}(\overline{\beta})})\\&=W(\overline{\beta},\boldsymbol{u}(\overline{\beta}),p(\overline{\beta})).\end{aligned}$$

Since $W(\beta^*_\varepsilon,\boldsymbol{u}_\varepsilon(\beta^*_\varepsilon),p_\varepsilon(\beta^*_\varepsilon))=W(\beta^*_\varepsilon,\psi_{\beta^*_\varepsilon}\boldsymbol{u}_\varepsilon(\beta^*_\varepsilon)|_{\mathrm{Q}(\beta^*_\varepsilon)},p^0_\varepsilon(\beta^*_\varepsilon)|_{\mathrm{Q}(\beta^*_\varepsilon)})$, by (5.20), we have

$$W(\beta^*,\boldsymbol{u}^*|_{\mathrm{Q}(\beta^*)},p^*|_{\mathrm{Q}(\beta^*)})\le W(\overline{\beta},\boldsymbol{u}(\overline{\beta}),p(\overline{\beta}))\quad\forall\overline{\beta}\in\Sigma_{ad}.$$

This completes the proof. □

6. Finite element approximation of Problem $(\mathbb{M}_\varepsilon)$

We will focus on the finite-dimensional approximation of Problem $(\mathbb{M}_\varepsilon)$. Let $\varepsilon>0$ be fixed. We introduce a discretization of Problem $(\mathcal{M}_\varepsilon(\beta))$ by using a discretization of the admissible set Σ_{ad} and a finite element approximation of the state problem. Next, we will analyze the solution of the discrete shape optimization problem and its convergence when the discrete parameter $h\to 0$.

We start with piecewise linear approximations of the admissible set Σ_{ad}. Let t be a positive integer and set $h=1/t$. We denote the equidistant partition of $[0,1]$ by δ_h, that is

$$\delta_h:\ 0=a_0<a_1<\cdots<a_t=1,$$

where

$$a_j=jh,\ j=0,\ 1,\dots,t.$$

The set of discrete admissible shapes Σ^h_{ad} consists of continuous, piecewise linear functions on δ_h which satisfy constraints analogous to those imposed in Σ_{ad}. Let

$$\Sigma_{ad}^h=\{\beta_h\in C([0,1])\mid \beta_{h|[a_{i-1},a_i]}\in P_1([a_{i-1},a_i]),\ \forall i=1,\dots,t;$$
$$\beta_{\min}\le\beta_h(a_i)\le\beta_{\max},\ \forall i=0,\dots,t;$$
$$|\beta_h(a_i)-\beta_h(a_{i-1})|\le C_1h,\ \forall i=1,\dots,t;$$
$$|\beta_h(a_{i+1})-2\beta_h(a_i)+\beta_h(a_{i-1})|\le C_2h^2,$$
$$\forall i=1,\dots,t-1\},$$

where the positive constants $\beta_{\min}$, $\beta_{\max}$, C_1 and C_2 are the same as in the definition of Σ_{ad}. We denote the set of discrete admissible shapes by

$$\mathcal{O}_h=\{\mathrm{Q}(\beta_h)\mid\beta_h\in\Sigma_{ad}^h\}.$$

Let $\{\mathcal{T}_h(\beta_h)\}$ be a regular family of triangular partitions of $\overline{\mathrm{Q}}(\beta_h)$ into triangles. With the partition $\mathcal{T}_h(\beta_h)$, we introduce the corresponding finite element spaces

$$\widetilde{\boldsymbol{V}}_h(\beta_h)=\{\boldsymbol{v}_h\in(C(\overline{\mathrm{Q}}(\beta_h)))^2\mid \boldsymbol{v}_{h|T}\in(P_2(T))^2\ \ \forall T\in\mathcal{T}_h(\beta_h),$$
$$\boldsymbol{v}_h=\boldsymbol{0}\text{ on }\Gamma_D(\beta_h)\},$$
$$L_h(\beta_h)=\{q_h\in C(\overline{\mathrm{Q}}(\beta_h))\mid q_{h|T}\in P_1(T)\ \forall T\in\mathcal{T}_h(\beta_h),\ \int_{\mathrm{Q}(\beta_h)}q_h\,dx=0\},$$

where $(P_k(T))^l$ (k and l are positive integers) represents the space of polynomials in which the total degree of the corresponding l-dimensional vector-valued polynomial space in T is less than or equal to k.

Let $\beta_h\in\Sigma_{ad}^h$, $h>0$ and $\varepsilon>0$ be given. We introduce the following discrete approximation of the penalized state problem $(\mathcal{M}_\varepsilon(\beta))$:

Problem $(\mathcal{M}_{h\varepsilon}(\beta_h))$. Find $(\boldsymbol{u}_{h\varepsilon},p_{h\varepsilon}):=(\boldsymbol{u}_{h\varepsilon}(\beta_h),p_{h\varepsilon}(\beta_h))\in\widetilde{\boldsymbol{V}}_h(\beta_h)\times L_h(\beta_h)$ such that

$$a_{\beta_h}(\boldsymbol{u}_{h\varepsilon},\boldsymbol{v}_h)-b_{\beta_h}(\boldsymbol{v}_h,p_{h\varepsilon})+\int_{\Gamma_S(\beta_h)}j^0_{\beta_h}(\boldsymbol{u}_{h\varepsilon\tau};\boldsymbol{v}_{h\tau})\,ds$$
$$+\frac{1}{\varepsilon}c_{\beta_h}(\boldsymbol{u}_{h\varepsilon},\boldsymbol{v}_h)\ge(\boldsymbol{f},\boldsymbol{v}_h)_{0,\mathrm{Q}(\beta_h)}\quad\forall\boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h(\beta_h),\tag{6.1}$$
$$b_{\beta_h}(\boldsymbol{u}_{h\varepsilon},q_h)=0\quad\forall q_h\in L_h(\beta_h).\tag{6.2}$$

Denote $\boldsymbol{V}_{h0}(\beta_h)=\widetilde{\boldsymbol{V}}_h(\beta_h)\cap\boldsymbol{V}_0(\beta_h)$. We assume that the discrete inf-sup condition is satisfied: for a constant $\rho_2>0$ independent of h, it holds

$$\rho_2\|q_h\|_{0,\mathrm{Q}(\beta_h)}\le\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}(\beta_h)}\frac{b_{\beta_h}(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_{1,\mathrm{Q}(\beta_h)}}\quad\forall q_h\in L_h(\beta_h).\tag{6.3}$$

Similarly to Theorem 5.1, we can show that Problem $(\mathcal{M}_{h\varepsilon}(\beta_h))$ has the unique solution under the assumptions stated in Theorem 5.1 and (6.3).

Lemma 6.1. *Assume that $H(j_\beta)$(i)-(iv), $\boldsymbol{f}\in(L^2_{loc}(\mathbb{R}^2))^2$ and (2.13) hold. Let $(\boldsymbol{u}_{h\varepsilon},p_{h\varepsilon})$ be a solution of Problem $(\mathcal{M}_{h\varepsilon}(\beta_h))$. Then there exists a constant $c:=c(\|\boldsymbol{f}\|_{0,\hat{\mathrm{Q}}})>0$ independent of $\varepsilon>0$, $h>0$ and $\beta_h\in\Sigma_{ad}^h$ such that*

$$\|\psi_{\beta_h}\boldsymbol{u}_{h\varepsilon}\|_{1,\hat{\mathrm{Q}}}+\frac{1}{\varepsilon}c_{\beta_h}(\boldsymbol{u}_{h\varepsilon},\boldsymbol{u}_{h\varepsilon})+\|p^0_{h\varepsilon}\|_{0,\hat{\mathrm{Q}}}\le c.\tag{6.4}$$

Proof. The proof is similar to that in Lemma 5.3, and hence will be omitted. □

We define the graph in the discrete case:

$$\mathcal{G}_{h\varepsilon}=\{(\beta_h,\boldsymbol{u}_{h\varepsilon}(\beta_h),p_{h\varepsilon}(\beta_h))\mid\beta_h\in\Sigma_{ad}^h,\ (\boldsymbol{u}_{h\varepsilon}(\beta_h),p_{h\varepsilon}(\beta_h))\text{ solves }(\mathcal{M}_{h\varepsilon}(\beta_h))\},$$

which is, in general, a multivalued control-to-state mapping. For any $\varepsilon>0$ and $h>0$, the discrete shape optimization problem is formulated by:

Problem $(\mathbb{M}_{h\varepsilon})$. Find $\beta^*_{h\varepsilon}\in\Sigma_{ad}^h$ such that

$$\mathcal{W}_{h\varepsilon}(\beta^*_{h\varepsilon})\le\mathcal{W}_{h\varepsilon}(\beta_h)\ \forall\beta_h\in\Sigma_{ad}^h,$$

where $\mathcal{W}_{h\varepsilon}(\beta_h)=W(\beta_h,\boldsymbol{u}_{h\varepsilon}(\beta_h),p_{h\varepsilon}(\beta_h))$ and $(\boldsymbol{u}_{h\varepsilon}(\beta_h),p_{h\varepsilon}(\beta_h))$ is the solution of Problem $(\mathcal{M}_{h\varepsilon}(\beta_h))$.

Similarly as in the proof in [51, Proposition 3.1], we can show that $\mathcal{G}_{h\varepsilon}$ is compact in the discrete case for each $\varepsilon>0$ and $h>0$. Hence, we can easily get the following result.

Theorem 6.2. *Let h, $\varepsilon>0$ be fixed and $\mathcal{W}_{h\varepsilon}$ be the lower semicontinuous functional on Σ_{ad}^h. Then Problem $(\mathbb{M}_{h\varepsilon})$ has a solution.*

In order to show that the solution of Problem $(\mathbb{M}_{h\varepsilon})$ converges to the corresponding one of Problem $(\mathbb{M}_\varepsilon)$ as $h\to0$, we need the following two results concerning the relation between Σ_{ad}^h and Σ_{ad} as $h\to0$.

Lemma 6.3. ([51]) *For any $\beta\in\Sigma_{ad}$, there exists a sequence $\{\beta_h\}\subseteq\Sigma_{ad}^h$ such that $\beta_h\to\beta$ in $C([0,1])$ as $h\to0$.*

Lemma 6.4. ([51]) *Let $\{\beta_h\}\subseteq\Sigma_{ad}^h$ be such that $\beta_h\to\beta$ in $C([0,1])$ as $h\to0$. Then $\beta\in\Sigma_{ad}$ and there exists a subsequence $\{\beta_{h_m}\}\subset\{\beta_h\}$ satisfying*

$$\beta'_{h_m}\to\beta'\quad\text{in }L^\infty(0,1)\text{ as }h_m\to0.\tag{6.5}$$

Next we will analyze the convergence properties of solutions to Problem $(\mathcal{M}_{h\varepsilon}(\beta_h))$ as $h\to0$.

Lemma 6.5. *Assume that $H(j_\beta)$ holds. Let $\{\beta_h\}\subseteq\Sigma_{ad}^h$ with $h\to0$ be an arbitrary sequence. Then there exists a subsequence (denoted by the same symbol) and a triplet $(\beta,\overline{\boldsymbol{u}},\overline{p})\in\Sigma_{ad}\times(H^1_0(\hat{\mathrm{Q}}))^2\times L^2_0(\hat{\mathrm{Q}})$ such that*

$$\beta_h\to\beta\quad\text{in }C([0,1]),\tag{6.6}$$
$$\psi_{\beta_h}\boldsymbol{u}_{h\varepsilon}(\beta_h)\rightharpoonup\overline{\boldsymbol{u}}\quad\text{in }(H^1(\hat{\mathrm{Q}}))^2,\tag{6.7}$$
$$p^0_{h\varepsilon}(\beta_h)\rightharpoonup\overline{p}\quad\text{in }L^2_0(\hat{\mathrm{Q}}),\ h\to0.\tag{6.8}$$

In addition, the pair $(\overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)},\overline{p}|_{\mathrm{Q}(\beta)})$ solves Problem $(\mathcal{M}(\beta)_\varepsilon)$.

Proof. From Lemmas 6.1 and 6.4, and the Arzela-Ascoli theorem, it follows that there exists a convergent subsequence of $\{(\beta_h,\psi_{\beta_h}\boldsymbol{u}_{h\varepsilon}(\beta_h),p^0_{h\varepsilon}(\beta_h))\}$ satisfying (6.6)-(6.8). We show that $(\overline{\boldsymbol{u}}|_{\mathrm{Q}(\beta)},\overline{p}|_{\mathrm{Q}(\beta)})$ is a solution to Problem $(\mathcal{M}(\beta)_\varepsilon)$. Let $\boldsymbol{v}\in\widetilde{\boldsymbol{V}}(\beta)$. Then there exists $\overline{\boldsymbol{v}}\in(H^1(\hat{\mathrm{Q}}))^2$ and a sequence $\{\boldsymbol{v}_h\}$ with $\boldsymbol{v}_h\in(H^1(\hat{\mathrm{Q}}))^2$ such that $\overline{\boldsymbol{v}}|_{\mathrm{Q}(\beta)}=\boldsymbol{v}$, $\boldsymbol{v}_h|_{\mathrm{Q}(\beta_h)}\in\widetilde{\boldsymbol{V}}_h(\beta_h)$, and

$$\boldsymbol{v}_h\to\overline{\boldsymbol{v}}\quad\text{in }(H^1(\hat{\mathrm{Q}}))^2,\ h\to0,\tag{6.9}$$

see [30, Lemma 9]. The passage to the limit in Problem $\mathcal{M}_{h\varepsilon}(\beta_h)$, as $h\to0$, can be done as in the proof of Theorem 4.2 by using (6.5). □

Finally, we obtain the following convergence result.

Theorem 6.6. *Assume that $H(j_\beta)$ holds, and $W:\ \Delta\to\mathbb{R}$ satisfy the following continuity property:*

$$\begin{cases}\beta_h\to\beta & \text{in }C([0,1]),\ \beta_h\in\Sigma_{ad}^h,\ \beta\in\Sigma_{ad}\\ \psi_{\beta_h}\boldsymbol{w}_h\rightharpoonup\boldsymbol{w} & \text{in }(H^1(\hat{\mathrm{Q}}))^2,\ \boldsymbol{w}_h\in\widetilde{\boldsymbol{V}}_h(\beta_h),\ \boldsymbol{w}\in(H^1_0(\hat{\mathrm{Q}}))^2\\ q^0_h\rightharpoonup q & \text{in }L^2(\hat{\mathrm{Q}}),\ q_h\in L_h(\beta_h),\ q\in L^2_0(\hat{\mathrm{Q}})\end{cases}$$
$$\Longrightarrow\ \lim_{h\to0}W(\beta_h,\boldsymbol{w}_h,q_h)=W(\beta,\boldsymbol{w}|_{\mathrm{Q}(\beta)},q|_{\mathrm{Q}(\beta)}).\tag{6.10}$$

Let $h\to0$ and $\{\beta^*_{h\varepsilon}\}\subseteq\Sigma_{ad}^h$ be a sequence of solutions to Problem $(\mathbb{M}_{h\varepsilon})$. Then, one can choose a subsequence of $\{\beta^*_{h\varepsilon}\}$ (denoted in the same way) and find a triplet $(\beta^*_\varepsilon,\boldsymbol{u}^*_\varepsilon,p^*_\varepsilon)\in\Sigma_{ad}\times(H^1_0(\hat{\mathrm{Q}}))^2\times L^2_0(\hat{\mathrm{Q}})$ such that

$$\beta^*_{h\varepsilon}\to\beta^*_\varepsilon\quad\text{in }C([0,1]),\tag{6.11}$$
$$\psi_{\beta^*_{h\varepsilon}}\boldsymbol{u}_{h\varepsilon}(\beta^*_{h\varepsilon})\rightharpoonup\boldsymbol{u}^*_\varepsilon\quad\text{in }(H^1(\hat{\mathrm{Q}}))^2,\tag{6.12}$$
$$p^0_{h\varepsilon}(\beta^*_{h\varepsilon})\rightharpoonup p^*_\varepsilon\quad\text{in }L^2_0(\hat{\mathrm{Q}}),\ h\to0.\tag{6.13}$$

Moreover, β_ε^* is a solution of Problem $(\mathbb{M}_\varepsilon)$ *and* $(\boldsymbol{u}_\varepsilon^*|_{Q(\beta_\varepsilon^*)}, p_\varepsilon^*|_{Q(\beta_\varepsilon^*)})$ *is a solution of Problem* $(\mathcal{M}_\varepsilon(\beta_\varepsilon^*))$.

Proof. The fact that $(\boldsymbol{u}_\varepsilon^*|_{Q(\beta_\varepsilon^*)}, p_\varepsilon^*|_{Q(\beta_\varepsilon^*)})$ solves Problem $(\mathcal{M}_\varepsilon(\beta_\varepsilon^*))$ is proved in Lemma 6.5. Since $\beta_{h\varepsilon}^* \in \Sigma_{ad}^h$ be a solution of Problem $(\mathbb{M}_\varepsilon)$, we have

$$W(\beta_{h\varepsilon}^*, \boldsymbol{u}_{h\varepsilon}(\beta_{h\varepsilon}^*), p_{h\varepsilon}(\beta_{h\varepsilon}^*)) \le W(\beta_h, \boldsymbol{u}_{h\varepsilon}(\beta_h), p_{h\varepsilon}(\beta_h)).$$

Subsequently, from (6.6)-(6.8) and (6.10), we have

$$\lim_{h\to 0} W(\beta_h, \boldsymbol{u}_{h\varepsilon}(\beta_h), p_{h\varepsilon}(\beta_h)) = W(\beta, \overline{\boldsymbol{u}}|_{Q(\beta)}, \overline{p}|_{Q(\beta)}),$$

where $(\overline{\boldsymbol{u}}|_{Q(\beta)}, \overline{p}|_{Q(\beta)})$ solves Problem $(\mathcal{M}_\varepsilon(\beta))$. Similarly, by (6.10)-(6.13), we obtain

$$\lim_{h\to 0} W(\beta_{h\varepsilon}^*, \boldsymbol{u}_{h\varepsilon}(\beta_{h\varepsilon}^*), p_{h\varepsilon}(\beta_{h\varepsilon}^*)) = W(\beta_\varepsilon^*, \boldsymbol{u}_\varepsilon^*|_{Q(\beta_\varepsilon^*)}, p_\varepsilon^*|_{Q(\beta_\varepsilon^*)}).$$

Hence, we get

$$W(\beta_\varepsilon^*, \boldsymbol{u}_\varepsilon^*|_{Q(\beta_\varepsilon^*)}, p_\varepsilon^*|_{Q(\beta_\varepsilon^*)}) \le W(\beta, \overline{\boldsymbol{u}}|_{Q(\beta)}, \overline{p}|_{Q(\beta)}),$$

which completed the proof. □

Final comments

We established the existence result for the optimal shape design problems of the stationary Stokes hemivariational inequality. We showed the convergence of shape optimization problems for the penalized inequality, provided that the penalty parameter tends to zero. Moreover, we applied the finite element method for the convergence analysis of the shape optimization problem for the penalized Stokes problem.

We note that in the future project we will provide numerical analysis, implementation and simulations for the problems studied in this paper. We also plan to investigate shape optimization problems for more complicated fluid flow models with various conditions on different parts of the boundary, for instance, the generalized Stokes problem with an additional implicit obstacle constraint set depending on the solution, see, e.g., [59,60]. Furthermore, it would be interesting to study the necessary conditions of optimality for optimal shape design problems of the stationary Stokes hemivariational inequality. Finally, as it is known, subgradient methods have been widely adopted to solve the unconstrained minimization problem; see, e.g., [3] and the references therein. Therefore, it will be an interesting challenge to examine the numerical methods, for example, subgradient methods for solving nonsmooth optimal shape design problem.

Data availability

No data was used for the research described in the article.

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