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Approximate Calculation of Functionals on Multidimensional Wiener Process

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This paper proposes a formula for the approximate calculation of functionals depending on the multidimensional Wiener process. The formula is exact for third-order monomials, including not only products of Wiener processes, but also for products containing time as one of the factors. Examples of using the formula in calculating mathematical expectations for some nonlinear functionals are given.

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1. Introduction

Due to the complex structure of the simulated processes, it is quite difficult to take into account all the factors influencing their evolution. Therefore, models with a stochastic component are increasingly used. This approach allows us to significantly reduce the number of factors involved in describing the modeled phenomenon. It is important to note that even the factors taken into account in the model are often random in nature. As an example, consider various financial and economic problems. Often in financial problems one of the factors is the base rate, which is random.

The appearance of one random process in a model, even a relatively simple one, leads, as a rule, to the impossibility of explicitly obtaining a solution and, thus, to a problem of assessing the behavior of the model as a whole. In models describing real processes, as a rule, there are several factors that are non-deterministic. At the moment, financial models often have only one chaotic component (see, e.g. [1–3, 5, 8, 9]). When modeling economic processes, even if chaotic behavior is taken into account (see, for example, [4, 11, 12]), often only one process is used, although in reality each equation entered into the modeled system is random in nature, and even if there is a correlation between the processes described by the equations, it is only partial.

One of the most universal approaches to calculating functionals depending on random processes is the use of simulation approaches based on the use of so-called strong methods (see, for example, [7, 10] for the case when the functional contains a solution to the stochastic differential equation), but this is associated with significant computational costs. The proposed formula refers to weak methods for calculating functionals and continues the research presented in [6]. A distinctive feature of this approach is a significant reduction in calculation time.

2. Approximate formula

The above-mentioned aspects of modeling systems with chaotic behavior lead to the need to take into account the influence of many random influences, correlated to some extent or independent. Thus, the actual task is to calculate the mathematical expectation of a functional of the form

$$\mathbb{E}[G[\cdot, W_1(\cdot), \ldots, W_n(\cdot)]].$$

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Since the random effect in the systems under study usually depends on a certain part of the trajectory of the system solution, this paper proposes a formula that allows us to take into account such a dependence.

In the designation of the functional G, the symbol ‘’·“ is used to indicate that the functional may depend on whole or part of the trajectory of processes W_j or time (for example, an integral may be present). Moreover, in each position where this symbol is used, we assume that this dependence is different, i.e. a symbol in W_j and W_k, $j \neq k$ or in some position of the functional indicates dependence on different points in time or different time intervals for different places where it is mentioned. Also we suppose that G is a smooth functional and the processes $W_j(t), j = 1, n$ pairwise independent, $t \in [0,1]$. The random processes $W_j(\cdot), j = \overline{1,n}$ are independent ones. In this case, we do not consider correlated processes, since when defining systems, the mutual dependence can be described using additional equations.

Theorem 1. *The approximate value $J(G)$ given by the formula (2.1) is exact for mathematical expectations of monomials of the form $G(\cdot, W_1(\cdot), \overset{(n)}{\dots}, W_n(\cdot)) \equiv \prod_{i=1}^{n} \xi_i, n = \overline{1,3}$, where $\xi_i \in \{\cdot, W_1(\cdot), \overset{(n)}{\dots}, W_n(\cdot)\}, i = 1,2,3$ except the monomial of the form $\xi_1\xi_2\xi_3 = t_1t_2t_3$, $t_1, t_2, t_3, s \in [0,1]$.*

$$\mathbb{E}[G(\cdot, W_1(\cdot), \overset{(n)}{\dots}, W_n(\cdot))] \approx J(G) = \frac{1}{2^n}\sum_{j=1}^{2} A_j \int_0^1\int_0^1\int_{-1}^1 \overset{(n)}{\dots} \int_{-1}^1 du_1 du_2 dv_1 \overset{(n)}{\dots} dv_n$$
$$\times G\left[\rho_{j1}(\cdot, u_1) + \rho_{j2}(\cdot, u_2), \rho_1(\cdot, v_1), \overset{(n)}{\dots}, \rho_n(\cdot, v_n)\right] \quad (2.1)$$

where

$$\rho_{jk}(s, u_k) = a_{jk}1_{[u_k,1]}(s), k = 1,2; \quad \rho_l(s, v_l) = \mathrm{sign}(v_l)1_{[|v_l|,1]}(s),$$
$$A_1 + A_2 = 1,$$
$$a_{11} = \tfrac{1}{2}\left(1 - \sqrt{-\tfrac{A_2}{A_1}}\right), \quad a_{12} = \tfrac{1}{2}\left(1 + \sqrt{-\tfrac{A_2}{A_1}}\right),$$
$$a_{21} = \tfrac{1}{2}\left(1 - \sqrt{-\tfrac{A_1}{A_2}}\right), \quad a_{22} = \tfrac{1}{2}\left(1 + \sqrt{-\tfrac{A_1}{A_2}}\right).$$

PROOF

Since the formula must combine random and deterministic components, in contrast to the formulas proposed earlier (see [6]), it is necessary to take into account the interaction of these components within the functional that needs to be calculated. At the first stage we will show that the formula is accurate for the mathematical expectation of the functional form $G(\cdot, W_1(\cdot), \overset{(n)}{\dots}, W_n(\cdot)) = t$:

$$J(G) = \frac{1}{2^n}\sum_{j=1}^{2} A_j \int_0^1\int_0^1\int_{-1}^1 \cdots \int_{-1}^1 \rho_{j1}(t, u_1) + \rho_{j2}(t, u_2) du_1 du_2 dv_1 \dots dv_n$$

$$= \sum_{j=1}^{2} A_j \left\{\int_0^1 a_{j1}1_{[u_1,1]}(t)du_1 + \int_0^1 a_{j2}1_{[u_2,1]}(t)du_2\right\} = t\sum_{j=1}^{2} A_j(a_{j1} + a_{j2}) = t(A_1 + A_2) = t.$$

Let us consider the case $G(\cdot, W_1(\cdot), \overset{(n)}{\ldots}, W_n(\cdot)) = tW_i(t)$, then the approximate formula for $\mathbb{E}[G]$ has the following form:

$$J(G) = \frac{1}{2^n} \sum_{j=1}^{2} A_j \int_0^1 \int_0^1 \int_{-1}^1 \cdots \int_{-1}^1 \left(\rho_{j1}(t, u_1) + \rho_{j2}(t, u_2)\right) \rho_i(t, v_i) du_1 du_2 dv_1 \ldots dv_n$$

$$= \sum_{j=1}^{2} A_j \int_0^1 \int_0^1 \left(\rho_{j1}(t, u_1) + \rho_{j2}(t, u_2)\right) du_1 du_2 \times \frac{1}{2} \int_{-1}^1 \operatorname{sign}(v_i) 1_{[|v_i|,1]}(t) dv_i = 0$$

due to the oddness of the function $\operatorname{sign}(v_i)$.

Next, we consider the case when the functional contains products of independent Wiener processes $G(\cdot, W_1(\cdot), \overset{(n)}{\ldots}, W_n(\cdot)) = W_i(t)W_l(s)$ where $i \neq l$. Due to the independence of $W_i(t)$ and $W_l(s)$, the mathematical expectation of this functional is equal to zero. Let us show that the approximate formula also gives zero:

$$J(G) = \frac{1}{2^n} \sum_{j=1}^{2} A_j \int_0^1 \int_0^1 \int_{-1}^1 \cdots \int_{-1}^1 \rho_i(t, v_i) \rho_l(s, v_l) du_1 du_2 dv_1 \ldots dv_n$$

$$= \sum_{j=1}^{2} A_j \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \rho_i(t, v_i) \rho_l(s, v_l) dv_i dv_l = \frac{1}{4} \int_{-1}^1 \operatorname{sign}(v_i) 1_{[|v_i|,1]}(t) dv_i \int_{-1}^1 \operatorname{sign}(v_l) 1_{[|v_l|,1]}(s) dv_l = 0.$$

The last expression is equal to zero due to the oddness of the function sign also.

Separately, it is necessary to consider the case when the functional contains correlating terms, namely the case described by a monomial of the form $G(\cdot, W_1(\cdot), \overset{(n)}{\ldots}, W_n(\cdot)) = W_i(t)W_i(s)$. The value of the mathematical expectation of this functional is well known and equals $\min(t, s)$:

$$J(G) = \frac{1}{2^n} \sum_{j=1}^{2} A_j \int_0^1 \int_0^1 \int_{-1}^1 \cdots \int_{-1}^1 \rho_i(t, v_i) \rho_i(s, v_i) du_1 du_2 dv_1 \ldots dv_n$$

$$= \sum_{j=1}^{2} A_j \frac{1}{2} \int_{-1}^1 \operatorname{sign}(v_i) 1_{[|v_i|,1]}(t) \operatorname{sign}(v_i) 1_{[|v_i|,1]}(s) dv_i = \frac{1}{2} 2 \int_0^1 1_{[|v_i|,1]}(t) 1_{[|v_i|,1]}(s) dv_i = \min(t, s).$$

Another problem in constructing the formula was represented by second-order monomials containing only deterministic terms, namely monomials of the form $G(\cdot, W_1(\cdot), \overset{(n)}{\ldots}, W_n(\cdot)) = ts$. Since this is a completely deterministic functional, the mathematical expectation in this case is equal to its value. Let us show that for this case the proposed formula also gives an exact value. Really, after opening the brackets we get that

$$J(G) = \frac{1}{2^n} \sum_{j=1}^{2} A_j \int_0^1 \int_0^1 \int_{-1}^1 \cdots \int_{-1}^1 (\rho_{j1}(t,u_1) + \rho_{j2}(t,u_2)) (\rho_{j2}(s,u_1) + \rho_{j2}(s,u_2)) \, du_1 du_2 dv_1 \ldots dv_n$$

$$= \sum_{j=1}^{2} A_j \int_0^1 \int_0^1 \left(a_{j1}^2 1_{[u_1,1]}(t) 1_{[u_1,1]}(s) + a_{j1} a_{j2} 1_{[u_1,1]}(t) 1_{[u_2,1]}(s) \right.$$

$$\left. + a_{j1} a_{j2} 1_{[u_1,1]}(s) 1_{[u_2,1]}(t) + a_{j2}^2 1_{[u_2,1]}(t) 1_{[u_2,1]}(s) \right) du_1 du_2$$

$$= \sum_{j=1}^{2} A_j \left(\int_0^1 a_{j1}^2 1_{[u_1,1]}(t) 1_{[u_1,1]}(s) du_1 + 2 a_{j1} a_{j2} ts + \int_0^1 a_{j2}^2 1_{[u_2,1]}(t) 1_{[u_2,1]}(s) du_2 \right)$$

$$= \min(t,s) \sum_{j=1}^{2} A_j (a_{j1}^2 + a_{j2}^2) + ts \sum_{j=1}^{2} 2 A_j a_{j1} a_{j2} = ts.$$

Here we took advantage of the fact that

$$\sum_{j=1}^{2} 2 A_j a_{j1} a_{j2} = 1,$$

$$\sum_{j=1}^{2} A_j (a_{j1}^2 + a_{j2}^2) = 0,$$

which is easy to show based on the previously defined conditions for the choice of coefficients.

Since the mathematical expectations of third-order monomials can be represented in the form

$$\mathbb{E}[tW_i(t)W_l(s)] = t\mathbb{E}[W_i(t)]\mathbb{E}[W_l(s)],$$

$$\mathbb{E}[tW_i(t)W_i(s)] = t\mathbb{E}[W_i(t)W_i(s)],$$

$$\mathbb{E}[tsW_i(s)] = ts\mathbb{E}[W_i(s)],$$

$$\mathbb{E}[W_i(t)W_l(s)W_k(z)] = \mathbb{E}[W_i(t)]\mathbb{E}[W_l(s)]\mathbb{E}[W_k(z)],$$

$$\mathbb{E}[W_i(t)W_l(s)W_l(z)] = \mathbb{E}[W_i(t)]\mathbb{E}[W_l(s)W_l(z)],$$

where $t, s, z \in [0,1]$, $i, l, k = \overline{1,n}$, $i \neq l \neq k$, the equality of their values and the values obtained using the formula 2.1 follows from the previous calculations for monomials of lower orders.

The mathematical expectation of a monomial of the form $W_i(t)W_i(s)W_i(z)$ is equal to zero. The calculations are similar to the case of $W_i(t)W_i(s)$, and the equality to zero of the value obtained by the approximate formula follows from the oddness of the integrand function. ■

3. Numerical experiment

Let us consider the application of the resulting formula (2.1) to the calculation of nonlinear functionals containing multiple random components. In this example and further we will assume that $A_1 = 4/3, A_2 = -1/3$ $a_{11} = 0.25, a_{12} = 0.75, a_{21} = -0.5, a_{22} = 1.5$

Let us consider a functional of the form

$$G[t, W_1(t), W_2(t)] = \sin(t + W_1(t) + W_2(t)).$$

The mathematical expectation of such a functional can be calculated exactly and is equal to

$$\mathbb{E}[G] = \mathrm{e}^{-t}\sin t.$$

For this functional the approximate formula has the following form:

$$J(G) = \frac{1}{4}\sum_{j=1}^{2} A_j \int_0^1\int_0^1\int_{-1}^1\int_{-1}^1 du_1 du_2 dv_1 dv_2 \sin\left(\rho_{j1}(t,u_1) + \rho_{j2}(t,u_2) + \rho_1(t,v_1) + \rho_2(t,v_2)\right).$$

After simple, but massive calculations we get that

$$J(G) = 1.06603t - 1.20467t^2$$
$$+0.431738t^3 - 0.0474551t^4.$$

The calculation results for mathematical expectation of the function $\sin(t+W_1(t)+W_2(t))$ are shown in the table 1.

Table 1. Results o calculation of $\mathbb{E}[\sin(t + W_1(t) + W_2(t))]$.

t	$\mathbb{E}[G]$	$J(G)$	$\lvert\mathbb{E}[G] - J(G)\rvert$
0.001	0.000999	0.00106483	0.0000658292
0.01	0.00990033	0.0105403	0.000639969
0.1	0.090333	0.0949837	0.00465067
0.5	0.290786	0.282851	0.00793537

Next, let us consider an example when

$$G[t, W_1(t), W_2(t)] = \cos(t + W_1(t) + W_2(t)).$$

The mathematical expectation in this case can also be calculated exactly:

$$\mathbb{E}[G] = \mathrm{e}^{-t}\cos t.$$

The corresponding approximate formula has the form

$$J(G) = \frac{1}{4}\sum_{j=1}^{2} A_j \int_0^1\int_0^1\int_{-1}^1\int_{-1}^1 du_1 du_2 dv_1 dv_2 \cos\left(\rho_{j1}(t,u_1) + \rho_{j2}(t,u_2) + \rho_1(t,v_1) + \rho_2(t,v2)\right),$$

which after simple massive calculations can be represented in the following form:

$$J(G) = 1. - 0.968034t - 0.15502t^2 \\ +0.367648t^3 - 0.0868659t^4.$$

A comparison of the results for the functional of the form $\cos(t+W_1(t)+W_2(t))$ obtained using the expression for the exact value of the mathematical expectation and using the formula proposed in the work is given in the table 2 below.

Table 2. Results o calculation of $\mathbb{E}[\cos(t + W_1(t) + W_2(t))]$.

t	$\mathbb{E}[G]$	$J(G)$	$\lvert\mathbb{E}[G] - J(G)\rvert$
0.001	0.999	0.999032	0.0000318114
0.01	0.99	0.990305	0.000304197
0.1	0.900317	0.902005	0.0016884
0.5	0.532281	0.517755	0.0145256

Let us consider the case of applying the proposed formula to the calculation of the mathematical expectation of the functional, which depends on the entire trajectory of the process in the time period under study, namely

$$G[t, W_1(\cdot), W_2(\cdot)] = \sin \int_0^t (s+W_1(s)+W_2(s))ds.$$

The approximate formula for this functional has the following form:

$$J(G) = \frac{1}{4}\sum_{j=1}^{2} A_j \int_0^1\int_0^1\int_{-1}^1\int_{-1}^1 du_1du_2dv_1dv_2 \sin\left(\int_0^t \rho_{j1}(s,u_1) + \rho_{j2}(s,u_2) + \rho_1(s,v_1) + \rho_2(s,v2)ds\right).$$

The table 3 shows the results of the approximate calculations of $\mathbb{E}[G]$ by the last proposed formula.

Table 3: Results of approximate calculation of $\mathbb{E}[G]$.

t	$J(G)$
0.01	0.000000
0.1	0.00500167
0.2	0.0200199
0.3	0.045066
0.4	0.0784129
0.5	0.119885

4. Conclusion

The proposed formula allows one to calculate the mathematical expectations of functionals that combine not only the presence of multiple random components represented through a set of independent Wiener processes, but also take into account their nonlinear interaction. Based on the approach to constructing the formula, we are talking, first of all, about smooth functionals, which in one form or another can be represented through the sum of monomials of various orders.

The resulting formula can be applied to estimate the mathematical expectations of solutions to systems of stochastic differential equations.

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