Some Extraordinary Results, their Cooperative Genesis across Five or Six Separate Areas of Mathematics and Conclusions on Modern Research after Covid

Frank Uhlig *

Abstract

This is a mathematical and meta-mathematical paper that describes a recent cluster of extraordinary results in six distinct branches of mathematics that are connected in multiple ways. The first quoted paper in Section 1 deals with fractional ordinary differential equations and the proportional secting method for accelerating terminal value problems therein. The next section shows how a quantum physics problem can easily be solved by using the hermitean Johnson $\mathcal{F}(t)$ function of field of values computation to solve the unitary block-diagonalization problem for general square matrices, whose solution in quantum physics terms makes an assessment of our Chemical Element Tables finally possible after 100 years of not knowing. Section 3 studies the accurate and fast computation of field of values boundary curves, even for decomposable matrices by using the unitary block decomposition result and a discretized Zhang Neural Network method on the resulting Johnson block $\mathcal{F}_j(t)$ functions. Section 4 studies adapted ZNN methods for matrix flow $A(t)$ problems and finds nonsingular static matrix A symmetrizers with small condition numbers for the first time. Section 5 surveys Zhang Neural Networks and describes their seven step set-up process for the first time, giving ten matrix flow example derivations of this new process. Section 6 deals with elementary Linear Algebra education problems that have been festering since World War 2. It describes a set of new lesson plans that can replace our decades and centuries old way of teaching our young with modern uses of Matrix Theory and Matrix Computations in a coherent and complete way. The last Section, Section 7 is meta-mathematical in nature and tries to fathom how this extraordinary set of six papers could be conceived and brought to fruition by one author (and a few others) in our Covid and past–Covid times within 1 or 2 short years and what external forces helped to bring this about.

Keywords: math history, numerical analysis, fractional ODEs, shooting methods, proportional secting, Linear algebra, unitary block-decomposition, invariant subspace theory, quantum physics, time-varying matrix problem, Johnson matrix flow, neural network, zeroing neural network, discretized ZNN algorithm, field of values, matrix flows, time-varying numerical algorithm, ZNN set-up, predictive numerical method, matrix symmetrizer, math education, lesson plans, interactive teaching, meta-math

AMS : 65F10, 65F15, 65F30, 65J10, 65L10, 34A08, 35A08, 15A18, 15A21, 15A23, 15A60, 15B57, 81Q10, 97H60, 97N40

1 A New Result for Fractional Differential Equations

The recent paper [2] by Kai Diethelm and Frank Uhlig originated after a zoom talk by Kai on fractional differential equations on the Irish Numerical Analysis Forum INA zoom talk series in 2021. This paper's author, Frank, contacted Kai about the shooting start selections that were commonly used in this area.

Fractional derivatives occur naturally in DE systems when non-local time or space variable effects are present in the physical model. This may happen with viscoelastic materials whose mechanical properties exhibit memory dependent behavior. Such systems are usually set up as initial value problems. These are well understood theoretically, but they are often reformulated for computations in the form of terminal value problems. The numerical analysis and design of numerical methods for terminal value problems are, however, less well developed. Commonly such problems are solved by using shooting methods with their own problems that we can now mitigate via Proportional Secting.

There are a several efficient methods to solve initial value fractional differential ODEs, such as by Garrappa, by

*Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849-5310 (uhligfd@auburn.edu)

Ford et. al., and by Lubich. For details refer to the text of and to the References in [2]. These methods all use fractional versions of collocation methods, Adams-Bashforth ODE integrators, backward differentiation formulas or some other linear multistep methods and the latter are then used as building blocks in shooting methods for terminal value problems that are typically combined with the Classical Bisection method. Classical Bisection always halves the interval of consideration. Note that the first interval can be astronomically large and complicated as the classical estimation techniques for the bounds of this interval require to evaluate Mittag-Leffler functions with possibly very large positive arguments. These starting intervals with upper bound sizes of 10^7 units width or more can occur on one side of the 'correct' shooting start and with just 2 or 3 units width on the other side, making bisections run for dozens and dozens of iterations. The Proportional Secting method in [2], however, uses around 1/10th of the bisection method's iterations and gets done in around 1/10th of the standard shooting methods' CPU time.

This is an extraordinary advance. Our Proportional Secting Method is simple to implement. We never aim at inclusion of the 'correct' starting position as bisection always does. Our iterates are computed without any logical "if" conditions and only for the second iterate do we rely on Mittag-Leffler estimates. Thereafter we always use Proportional Secting for the two iterates (out of the latest three known approximates) that are closest to the solution.

At the same time, some of the earlier fractional ODE shooters have been weighting the effects of using low accuracy solution at the beginning of the bisection process and other such questions. These could reduce the total iterations count by 1 or 2 steps in some cases or increase the count by similar small numbers in other cases. A uniform reason for this behavior has never been found, but often explored and surmised upon.

We explore this behavior of fractional ODE experts (and others) in Section 7 in a Meta-Mathematical Analysis.

Next we move to an open question in Quantum Physics or Quantum Chemistry about the Genesis of higher atomic weight elements from gases in the geological creation of our Universe. This problem had remained unsolved mathematically for almost 100 years. It was just solved via Matrix Geometry, Invariant Subspace Theory and a Matrix Algorithm in [16]; for our computational method, see Section 3 below. This new result might affect our Chemical Tables, particularly in the post Uranium chemical elements region if the physical data underlying the 1920s Quantum research could be found or regenerated anew today, see [9] and the references there.

2 Quantum Physics/Quantum Chemistry Completed after 95 Years; or the Unitary Block Decomposition of General Square Matrices

This result is the subject of [16] by the author. This work is based on his recent papers [15] and [13] and the papers [5] and [9] from 90+ years ago. It uses Bendixon's early work from 1910 [1], as well as Johnson's more recent paper [6] on drawing the Field of Values and on more recent work [13] by Uhlig and by Luca Dieci et al on eigencurve crossings. Eigencurve path following methods were introduced to this area of research in [8] and by Luca Dieci et al (and many others, see the quotes in [13]) .

The Quantum Physics premise (an unproved hypothesis since the 1920s) states that certain square matrices, derived from Physics and Chemistry models for the creation of our Chemical Elements, cannot be unitarily block diagonalized into several blocks. And the contrapositive should likewise hold for the geological creation of the Earth.

An algorithm to decide whether a given square matrix is unitarily block diagonalizable or not has been developed, programmed and widely tested in [16]. It was found by bypassing the computational methods of matrix analysis and the related eigencurve studies that were quoted earlier. Instead we found a way to use the intrinsic geometry of eigencurve crossings and the hyperbolic avoidance of such crossing that was discovered 95+ years ago by von Neumann, Hund, and Wigner. Our new ideas in [16] are based on Johnson's and Bendixon's idea of using the auxiliary hermitean matrix flow

$$\mathcal{F}(t) = \cos(t)H + \sin(t)K \ (= (\mathcal{F}(t))^* \ ) \text{ for } H = (A + A^*)/2 = H^* \text{ and } K = (A - A^*)/(2i) = K^* \quad (1)$$

for $0 \leq t \leq 2\pi$, see [15, p. 2], and invariant subspace theory. Our constructive proof in [16] is geometric in nature and benefits from simple combinatoric principles for the eigendata of the hermitean matrices $\mathcal{F}(t)$. This matrix additive decomposition can be used for all general real or complex square matrices $A = H + iK$ with their hermitean part H and skew part K. In fact, the normalized eigenvectors $x(t)$ and $y(t) \in \mathbb{R}^n$ for the extreme eigenvalues of $\mathcal{F}(t)$ determine two Field of Values boundary points when the points $x^*(t)Ax(t)$ and $y^*(t)Ay(t) \in \mathbb{C} \equiv \mathbb{R}^2$ are rotated by t degrees around the origin of $\mathbb{R}^2$. Charlie Johnson [6] constructed the FoV boundary curve of a matrix A from multiple hermitean matrix eigen-analyses of the associated hermitean matrices $\mathcal{F}(t_j)$ for discrete $0 \leq t_j \leq 2\pi$ and by using subsequent Bendixon [1] rotations.

Now we can describe the invariant subspace based ideas of the proof in [16].

Any square matrix flow $B(t)$ of hermitean matrices with a fixed domain $[t_o, t_f] \subset \mathbb{R}$ allows us to diagonalize each flow matrix $B(t_a)$ with $t_a \in [t_o, t_f]$ via a unitary similarity transformation $V(t_a)$ so that $B(t_a) \cdot V(t_a) = V(t_a) \cdot D(t_a)$ and $D(t_a)$ is real diagonal for each such t_a. For each $t_a \in [t_o, t_f]$ the transforming unitary matrix $V(t_a)$ contains the normalized eigenvectors of $B(t_a)$ in its columns and the real eigenvalues of $B(t_a)$ appear in descending order on the diagonal of $D(t_a)$ if we use MATLAB's built-in `eig` function to compute them. Assuming that $B(t)$ is block diagonalizable into $\ell > 1$ diagonal blocks, then $B(t_a) = C^{-1} \cdot \text{blockdiag}(B_1(t_a), ..., B_\ell(t_a)) \cdot C$ for some nonsingular fixed entry matrix C. Note that each eigenvector in the columns of $V(t_a)$ is associated with an eigenvalue of some $B_k(t_a)$ block for $1 \leq k \leq \ell$ and the eigenvector columns of $V(t_a)$ that belong to the eigenvalues of one diagonal block $B_k(t_a)$ form an orthonormal basis for an invariant subspace of $B(t)$ of which there are $\ell > 1$ by assumption..

For any two parameter values $t_b \neq t_a \in [to, tf]$ for which the matrices $B(t_a)$ and $B(t_b)$ are linearly independent, $B(t_a)$ and $B(t_b)$ have the same block structure that conforms with that of $B(t)$ and is indicated by the relevant block $B_k(t_a)$ of $B(t)$. Therefore the related matrix

$$\tilde{B}_{t_a}(t_b) = (\tilde{V}(t_a))^* \cdot B(t_b) \cdot \tilde{V}(t_a) \tag{2}$$

will be block-diagonal with the same common block structure as soon as we have re-arranged the eigenvector rows or columns of $V(t_a) = (V(t_a))^*$ in $\tilde{V}(t_a)$ into ℓ groups that individually generate equal zero and non-zero patterns in the columns of $\tilde{B}_{t_a}(t_b)$. The re-arrangement of the columns of $V(t_a)$ can be achieved numerically when the spread of entry magnitudes in $\tilde{B}_{t_a}(t_b)$ has increased sufficiently by looking at the logic 0-1 spy matrix $B_{logic}(t_b)$ of $\tilde{B}_{t_a}(t_b)$ for a certain threshold in MATLAB. The actual setting of thresholds is described and tested in detail in [16, Sect. 3]. When we re-sort the columns of $V(t_a)$ so that logic 0-1 row or column vectors of $B_{logic}(t_b)$ (and thus of $\tilde{B}_{t_a}(t_b)$) fall into ℓ distinct groups according to the location of their almost-zero and their well above zero magnitude entries, we obtain $\tilde{V} = V(t_a)$ which block-diagonalizes every flow matrix $B(t)$ for $t \in [t_0, t_f]$.

All of the above theoretic considerations are essential in part for understanding the unitary block decomposition of general matrix flows $A(t)$ and of single static entry matrices $A_{n,n}$. But they are not sufficient to prove the unitary block-decomposition theorem outright. We needed to employ simple matrix flow ideas to the Johnson-Bendixon auxiliary $\mathcal{F}(t)$ function from the field of values computation area here.

In 2020 an early version of this paper was rejected by the Editorial Board of one Linear Algebra journal quickly finding the result too simple and lacking enough explanations. They advised "that the results in this manuscript should appear as sections of another wider paper where applications and more results are presented together". I had originally only included the $\mathcal{F}(t)$ and $\mathcal{F}_{logic}(t)$ based algorithm then. At that time the Linear Algebra refereeing pool was empty here after 90 years of not knowing and subsequent neglect. This turned out to be good advise for me because I had fallen into the unfortunate Chinese ZNN paper writing trap and mode of just describing a given model and then use ZNN iterations to compute the solution without ever explaining the discretized ZNN matrix process. I had to think about the meta-mathematical realm and explain my underlying understandings of the old block-diagonalization problem which actually involved modern invariant subspace theory and current matrix computational practices, see Section 3 below.

And then the paper's acceptance (elsewhere) went quickly in [16]. Since then I have written to and were approached by several Quantum physicists whom I encouraged by stating that the paper's connection to fundamental geological physics and chemistry might actually gain them the big Prize in Stockholm. Soon I was invited to the Quantum Days in Paris last spring for an invited talk with free room and no conference fees. But I had to decline since I had already committed to continue research with the indigenous on the Osa peninsula in Costa Rica at exactly the same time.

3 Fast and Accurate Field of Values Computations for Unitarily Decomposable Matrices

The actual block decomposition of matrix flows or of general square matrices has never been solved by algebraic means in 100 + years and was possibly never even tried.

On the other hand, our new way of establishing the unitary decomposability of a matrix flow or of a static matrix hinges entirely on a numerical algorithm that decides this issue depending on the lay of near zero entries and of the sizable ones in the Johnson matrix flow $\mathcal{F}(t) = \cos(t)H + \sin(t)K$ for $0 \leq t \leq 2\pi$. WOLG we assume that

$$\mathcal{F}(t_1) \neq O_{n,n} \text{ and } \mathcal{F}(t_2) \neq O_{n,n} \text{ are linearly independent as matrices .} \tag{3}$$

Then we form the 0-1 logic matrices $\mathcal{F}_{logic}(t_1)$ and $\mathcal{F}_{logic}(t_2)$ and sort the normalized eigenvectors of $\mathcal{F}(t_1)$ in $\tilde{V}(t_1)$ so that adjacent rows in $\mathcal{F}_{logic}(t_1)$ have equal 0 and 1 patterns throughout. And we arrange the rows of $\mathcal{F}_{logic}(t_2)$ in the same pattern as we have done for $\mathcal{F}_{logic}(t_1)$. This uses combinatorial algorithm that establishes a joint unitary block decomposition of both logic 0-1 pattern matrices if such exists. Repeating this process from the top block on down eventually leads us to a joint block diagonal structure for all $\mathcal{F}(t)$ with $t \in [0, 2\pi]$. Since $\mathcal{F}(0) = H$ and $\mathcal{F}(\pi/2) = K$ the general matrix A has the same unitary block structure as the hermitean matrices $\mathcal{F}(t_1)$ and $\mathcal{F}(t_2)$ and our algorithm is complete.

However, this algorithm depends on the near zero entries and on the far from zero entries judgement in $\mathcal{F}(t_i)$. How to judge computed entries to be 'zero' is a difficult question. After many experiments we have set the 'zero' threshold im our algorithm generally to $||A|| \cdot 10^{-13} \approx ||A|| \cdot 500\ eps$ when working in double precision in Matlab with machine constant $eps = 2.2... \cdot 10^{-16}$. Further details, many tests and experiments with matrices $A_{n,n}$ and matrix flows $A(t)$ until sizes $n \approx 1,000$ can be found in [16].

Our next task is to study and establish the connection of our previously described new results on unitary block decompositions of matrix flows with single parameter matrix problems via Zhang Neural Networks. All of these recent developments are inter-connected and they all have contributed to the author's understanding of the connected extraordinary cluster of results surveyed in this paper.

4 Adapted AZNN Methods for Time–varying Matrix Flow Problems

Zhang Neural Networks [21] were invented twenty years ago as extensions of Hopfield's single dimensional networks [4] to solve multidimensional dynamic parameter-varying matrix and vector problems. Zhang Neural Networks are designed to reduce the computed error exponentially over time for a multitude of time-varying matrix and vector problems. If $e(t)$ is the error function

$$e(t) = \text{ the difference between the computed solution and the (unknown) exact solution at time } t \tag{4}$$

of a given time-varying matrix and vector problem, the exponential error decay is achieved by stipulating

$$\dot{e}(t) = -\eta \cdot e(t) \text{ for } \eta > 0. \tag{5}$$

Zhang Neural networks never try to solve the differential equation (5) and - though akin to prediction-correction ODE multistep solvers - they are very different in their behavior, easily achieving minuscule errors of 10^{-18} over time, see Section 5 below for further explanations and examples. Discretized ZNN relies on convergent look-ahead finite difference formulas that were developed in [11].

The adapted ZNN method of [17] studies the effect of varying the decay constant η in the ODE (5) over time for the error function $e(t)$ and also changing the ZNN finite difference formula used in different stages of the iterations. To smooth out some observed error function wiggles in the middle of the iteration process we have experimented with astronomical eta values there and have also used the non-convergent Euler method at start-up for a small number of steps – with improved results for the time-varying matrix eigendata problem, for matrix square roots and matrix symmetrizer computations. With an appropriately adapted AZNN method we were able to compute nonsingular matrix symmetrizers with low condition numbers where all other known methods - see [3] - have failed miserably to find invertible symmetrizers. Of special interest are matrices with involved Jordan structures or with (previously unknown) block diagonalizeabilities. To our surprise, about one quarter of the matrices in Matlab's 'gallery' with difficult eigen-structures turned out to be block diagonalizable and will benefit computationally from our adapted AZNN method. With AZNN we can now compute nonsingular symmetrizers for static matrices quickly with symmetrizing errors generally less than Matlab's `eps` and with very small condition numbers, see [3, Section 4].

The above specialized results are extraordinary by themselves. They lead us directly into the discovery and numerical analysis of predictive time-varying matrix methods like ZNN. ZNN methods have been used extensively, with over 400 papers in engineering journals and 5 books published by authors mainly from the East. But how they work and how to transform a physical or engineering model into a specific ZNN ready form is unfortunately never mentioned.

The mathematical unraveling of ZNN is the subject of the next section, followed by a look at our modern math teaching challenges and a separate meta-mathematical analysis at the end.

5 The Structure and the Seven Set-up Steps of Discretized Zhang Neural Networks for Time–varying Matrix Flows

Given a physical, chemical, statistics etc or engineering model that needs to be solved, every Zhang Neural Network starts from the error function $e(t)$ in equation (5) that should ideally be zero for the unknown variable $x(t)$ in all of its entries if the solution were known. Next form the exponential decay ODE (2) and solve it algebraically for the derivative $\dot{x}(t)$ if possible. If not, alter the model or error equations and try again, see [18] for examples of changed model or error equations to be able to extract $\dot{x}(t)$ algebraically from equation (2). And set the exponential decay constant η in ODE (2) experimentally to lie between 0.4 and 8 at first. This setting can later be changed as needed.

After the above initial set-up steps, select a look-ahead convergent finite difference formula for the desired local truncation error order $O(\tau^{j+2})$ that expresses $\dot{x}(t_k)$ in terms of $x(t_{k+1}), x(t_k), ..., x(t_{k-(j+s)+2})$ for $j+s$ known data points from the table of known convergent look-ahead finite difference formulas of type `j_s` in [11] and [12] or use the rational coefficient convergent and look-ahead finite difference formulas of [20] . Here $\tau = t_{k+1} - t_k = const$ for all k is the sampling gap of the discretization. From here on out, we deal only with discretized ZNN methods.

Next equate the $\dot{x}(t_k)$ derivative terms from the start-up phase and the recent finite difference equation and thereby dispose of $\dot{x}(t_k)$ in the error function ODE problem (2) altogether from further considerations. We then solve this solution-free linear equation for $x(t_{k+1})$ predictively and iterate from $k+1$ to $k+2$ till the end of our time interval.

The discretized ZNN process was explained in detail in seven steps in [18]. ZNN can be likewise be employed for continuous time-varying matrix based problems and their predictive computations.

We finally come to our most urgent task today, the sixth venture: to help modernize our early College teachings in Linear Algebra, to enliven the area that has helped us so much and so extraordinally with the previous five new mathematical results.

How do we teach linear algebra today; what are the subjects taught? Using modern Matrix Theory based ideas or classical algebraically taught?

6 Math Education; Modernizing our First Linear Algebra Courses via Coherent Matrix Theory Based Lesson Plans has just Become Possible

This section is based on [19] and we look at how and what is standardly taught in our first College or University Linear Algebra courses in the US, in Europe and around the globe.

Historically Linear Algebra has had two periods of great expansions. One was in the 19th century and another is going on right now as this area of mathematics has gained great importance and is being rejuvenated by deep changes in the subject matter and our understandings thereof since the 1950s.

With the advent of online teaching, it is easy to get a feel how elementary Linear Algebra courses are generally taught world wide now. Just search any university website for relevant video resources and you will find that the majority of our subject's first College courses are given in 'Vorlesungs' style. Here the English word of 'lecture' is replaced by its German equivalent term of 'Vorlesung" which literately means a 'reading' out of a script or book. The 'readers' are mostly young Ph. Ds. or postdocs and young faculty who most often lecture top-down online, with little interactive inverse methods of teaching. The subject selection in these courses is congruent with the majority of our elementary Linear Algebra textbooks: Linear Equations, row reduction and a little bit on Linear Transformations. Then a foray into linear (In-)dependence, packaged in logic terms for Linear Combinations of the zero vector. Followed by Subspace studies, Determinants and Polynomial Root Finding and some applications, and - if time allows - special matrices, such as normal, hermitean, orthogonal, or positive ones.

Since the early 1970s, elementary Linear Algebra teaching has become overall more algebraized and more abstract. This and the limitation to subjects from the 1920 and before is very detrimental for our students whose cell-phoning and life depends on modern matrix knowledge, yet they are taught nothing modern or useful for their lives. Modern Matrix Theory is used in search engines, in AI and modern engineering applications and so forth.

This defect has to be overcome quickly, like yesterday. It requires alternate methods of teaching and modified subject lists and development methods that are congruent with the modern subjects of Linear Algebra.

Can we teach Linear Algebra in a new way? What should that way be? Will it succeed to modernize our first Linear Algebra courses soon enough and teach our students the same subject matter, but in a modern, useful and a usable way?

We must adopt different ways to teach; the times are ripe for online teaching. But 'Vorlesungs' top-down lecturing, even online, is out. Matrix Theory can best be taught via interactive teaching and learning about matrices. Online teaching can help us here. Ideally I can see an online source of Lessons that are read at home by students who then come to class to discuss and deepen each Lesson's contents with the teacher at hand, in small groups (5 - 8), on several black- or whiteboards life, in one classroom on campus. An inverse teaching approach generally works wonders. Usually my students become coaches and tutors after a few weeks for students in parallel elementary Linear Algebra classes that are taught classical subjects textbook style, online or off.

A set of Lesson plans for interactive teaching and that is suitable for inverse teaching has been developed and is available at `https://la-education.oucreate.com/teaching-resources/lesson-plans/franks-lesson-plans/`. My Lesson Plans were designed to help students and teachers understand and appreciate the subjects of our classical algebraic first Linear Algebra curriculum intuitively from the action of matrix $\times$ vector multiplication. They exemplify the role of matrices and linear transformations in a fully and completely matrix theoretical approach. A holistic analysis of our first Linear Algebra course dilemma is given in [19]. There we also include two lists of Linear Algebra notions, one with results that originated in the 19th century and before, and a second list with 20th and twenty-first century matrix innovations. It is well worth the readers' time to study [19]

The Lesson Plans at `https://la-education.oucreate.com/` start with Riesz's Representation Theorem for the standard matrix representation of linear transforms $f : \mathbb{R}^n \to \mathbb{R}^m$. Students are asked and guided to set up a proof and actually develop the standard matrix representation of linear transformation in course class three. In Lesson Plans 2 and 3 they learn about row reduction and the Row Echelon Form of matrices and study their invariants, i.e. the number and lay of pivots and of free variable columns. This information is used to solve Linear Equations that are solvable and unsolvable. In these computations they encounter small entries that must not be used as pivots or the results will become nearly meaningless and thereby they learn of computational pitfalls and cation early on. We also suggest that students develop their own codes for row reduction in order to have a simple software tool that can help them decide about subspaces, their bases, bases for matrix nullspaces and their ranges, the Dimension Theorem (without any mention of quotient spaces, homomorphisms and commutative diagrams as needed in abstract, algebraized courses). This sets up easy decisions on linear (in-)dependence questions that are given concretely in vector form and avoids the 'fog that sets in' when students are trying to pars out the logic based definition of linear (in-)dependence in abstract vector space language. Lesson Plan 4 then delves into coordinate vectors, general basis changes via matrix similarity, and applications for diagonalizable matrices and ordinary differential matrix equations. It ends with a bit of Linear Algebra history and Cauchy's idea from around 1820 to replace the n^2 data in $A_{n,n}$ by the polynomial root problem $\det(A - \lambda I_n) = 0$ and the associated characteristic equation. This n-fold data reduction for finding matrix eigenvalues λ of A has had no computational solution for

200 years now, but it is the only definition that deals with matrix eigenvalues that is taught to most of our students today quite needlessly.

All of these topics cover the first half (or a bit more than half) of a Lesson Plan based course - and by default of a classical course as well. All beginning notions are introduced concretely. Proofs are created and given on the students' level by the students themselves. Understandings and misunderstandings are constantly discussed and accepted.

This an interactive class; the modern subject requires it. And our students deserve it, rather than mostly being taught Linear Algebra on the 1820s to 1920s level.

Lesson Plan 5 directs students into simple extensions of the basic Krylov iteration method [7] of 1931, preferably on their own. It is designed to lead students deeper into computer software and computations and encourages mathematical trial and error investigations where simple matrix computations may fail, but a bit of Matrix Theory can show students and teachers alike how simple erroneous computations can repair themselves in the matrix realm. Lesson Plan 6 studies orthogonal vectors in space and is designed to be taught highly interactively. Lesson Plan 7 investigates orthogonal matrices geometrically such as Householder transforms, studies orthonormal bases and the least squares problem via Householders. It then applies all our matrix knowledge to the Krylov vector iteration process [7] to find invariant subspaces and matrix eigenvectors and their associated eigenvalues, now in reverse order of discovery. Students can then encouraged to work on final projects for Krylov eigen-computations.

7 Remarks on and Conclusions about the Genesis of these Six New and Extraordinary Results, after Covid Times

How did this cluster, how could such a cluster of extraordinary results that span across six distinct areas of Math Research occur within one mind (and that of the coauthors) in four years and come to fruition mostly in one year, in 2023?

This provocative question was not posed for boasting, not at all; but it needs to be understood in a wider than 'personal' perspective because this is an unusual sequence of events that cannot be attributed to one person alone. There were outside forces.

Over spans of millennia, human capabilities and achievements have grown and developed in jumps and spurts, forward and back. With the advent of Covid a few years ago, the human condition on Earth has apparently been changed. This has affected our individual and collective behavior, our communications, our social web and society as a whole. Our teaching methods and contents and our research capabilities have changed and, taken as a whole, our global consciousness as a species is altered today, for better or worse, one cannot tell.

The new shooting method [2] for terminal value problems might revise the return trajectory of spacecraft significantly. The standard NASA procedure of returning spacecraft, now like skipping flat stones across smooth ponds, might make this reentry process much safer and more accurate by replacing the current look-in-the-rearview-mirror return-from-space approach with incorporating discretized look-ahead ZNN instead and thereby even become predictive thanks to the shortening (up to 80 %) of shooting method computing times in [2].

The six main results of this paper are all interconnected; not one could have been found by itself (one lay dormant and unsolved for a century before [16]). Throughout this work and time, I needed the constant advice and help from friends across the globe. This points us to be more cooperative such as through zoom talks, through giving lectures, traveling and just 'asking about' which lead to [3] and eventually to [15] and then to [17] that cleared up much of [18] and thus brought Zhang Neural Networks to the West from China.

In our Linear Algebra classes we need to teach interactively now and give students a fair chance to learn what we all know now – but were never taught: modern Matrix Theory and Matrix Computations,

rather than centuries old 'stuff'. The times, they are changing.

An Outlook

Where will this all lead to, where will it end?

The universe, Earth and those who live on Earth, humans and animals, plants and clouds, climate and societies, nations and social interactions are constantly adapting and changing and they have been changing for billions of years, forwards and back. The universe is continually evolving and so are we. That is life.

So what is new? Why did I compose this meta-math paper and write these recent math papers? I will now try to explain the genesis of this new interconnected and extraordinary cluster of mathematical thought.

Something rarer and more special has been going on and affecting the Earth and our lives in the last couple of years. Our minds and our individual and group consciousnesses have made leaps and have bounced forward. We have become more awake spiritually, within ourselves and of others, we try to think 'diversity, equity and inclusion' now.

At the same time we are confronting our own demise, threatened by a global epidemic, through overusing our resources and environmental pollution, by needless consumption 'rich in things and poor in spirit', affecting our own survival with man-made climate change. In wars, started for dubious reasons and with ill effects, and with genocide showing its face around the globe again. We are still stewards of the Earth; are we or not?
We are barely so.

Back to mathematics, how did this research cluster come into being, mathematically and meta-mathematically?

Each of the achievements in this cluster originated in cooperation with others, by looking outside our individual boxes, with friends and inspired by combining multiple unlikely mathematical areas and sources. There were outside forces that drew me in and lead me through this research process. I repeatedly felt like a passive passenger sometimes in this journey for knowledge and as an avid finder at others. I tried to stop its feverish pace sometimes, take a break and just then I nailed one new insight down somewhere else and advanced my understanding of a different math topic meta-mathematically. This process felt safe to me, yet it was at no time and nowhere predictable – until everything fell in place and was mathematically proved.

At that moment I realized that all of the matrix algorithm in this cluster were unstable, meaning that all could fail in their iterations; i.e., at the start or in the middle of an iteration run. But they were only conditionally stable. However, their potential but rare breakdowns could always be bypassed to give us near exact results in the end. These new numerical methods were not backward stable. But neither would they solve nearby problems accurately. They would not leave us with high inaccuracies in their 'backward stable' results. For example, the all-out effort to solve the matrix symmetrizer problem in [3] via the backward stable QR, SVD, steepest descend or Lanczos methods had failed dramatically to find non-singular matrix symmetrizers for many examples. But time-varying adapted Zhang Neural Networks compute well conditioned nonsingular symmetrizers for the static matrices that gave all backward stable methods a fit, albeit ZNN does so only with 'conditional stability'.

Why and how to understand what brought on this extraordinary cluster of results all at once and why they had not been found earlier is open for interpretation and beyond the scope of this paper and author.

I am not, I was not the sole creator here. Besides my co-authors and colleagues and Linear Algebra's history some other force was at work helping me, a spiritual or karmic entity, earthly or most likely otherwise. It drove me on, it sustained me, it pushed me onwards for a long time.

At first as a second year student, I was tasked to Gta a section of first year Linear Algebra students. Thus I began to learn and think about how to teach at university. I did gain my first experiences with interactive teaching vectors and matrices then at Cologne University. And Linear Algebra has followed me in all of my teachings and much of my research life. There was a natural, broad support for me from Dorothy, my wife, from my family, my friends, from Anthony, Mike, Rachel and Sepideh of the ILAS Education committee, and all of ILAS, and from the doctors who cured me of a deadly self-poisoning disease that affected me badly and could have killed me.

But I survived and am now beginning to be rehabilitated physically. And I still feel that something is missing from this description of extraordinary events, namely the spiritual force that guarded and guided me throughout my life. Let me admit and say clearly now that I was saved by my Angel who would not let me die with such great tasks ahead. The Lesson Plan task is finished and the Lesson Plans are coherent and complete now. All of this makes me very grateful. Now I can dedicate myself again to my other creative outlet, photographing the world as I see it in my inner eye and also continue with math – on rainy days – or when I am called back to Math work and to Teaching again.

May everyone find ways to listen to his or her mentors throughout their life and stay in touch with her or his Angel. Let us be encouraged to fulfill our karmic tasks and spiritual work on Earth humbly and wisely.

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[A corrected version (with four erroneous 'statements' on p. 5 and 6 about diagonalizability crossed out) is available at http://www.auburn.edu/~uhligfd/C--OnTheUnitaryBlock-decomposabil--Corrected.pdf .]

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November 29, 2023