# Processing logical states under transformations in logical subspaces

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An evolution of a logical state is well described by the superoperator formalism in the case of a logical subspace of an error-correction code is not a subject to various transformations. However, there is no general framework that applies to the situation when both logical state and its logical subspace change in a potentially non-unitary manner. In this work, we present a theoretical framework that offers the way to describe how a logical state evolves under the influence of simultaneous changes in both physical state and in the logical subspace structure. These results are applicable to a wide range of cases with the most prominent applications to defects movements in surface code. As we expect, our results of are interest for ongoing experiments on demonstrating quantum error correction with the use of existing generation of quantum processors, such as those based on superconducting circuits and Rydberg atom arrays.

## I. INTRODUCTION

Experiments with noisy intermediate-scale quantum (NISQ) devices demonstrate their power to solve certain computational problems [1–6], which are beyond the capabilities of existing supercomputers based on traditional principles. However, the full potential of quantum computing devices are believed to be uncovered with the appearance of fault-tolerant quantum computers, whereas the problem of efficient error correction in the quantum domain is highly challenging [7–10]. Principles of quantum error correction are under exploration in experiments with physical platforms of various nature, such as superconducting circuits [11–15], trapped ions [16–21], and Rydberg atom arrays [22, 23]. However, certain theoretical aspects of implementing quantum error corrections remain uncovered, in particular, the problem of describing logical states in the case of transformations of a logical subspace of an error-correction code. Such a problem appears, for example, in the case of defects movements in surface code [24–27]. Indeed, logical subspaces itself might be subject to transformations: for example, when a stabilizer $S$ of the code is changed, both logical subspace and the form of logical operators within this subspace are transformed [24–26]. Thus, one needs to close a gap in the theoretical description of the evolution of a logical state and logical measurement statistics that could take into account both changes in the logical subspace structure and an evolution of a physical state, which is not necessarily unitary.

In this work, we present a framework for describing how a logical state evolves under the influence of simultaneous changes in both physical state and in the logical subspace structure. The framework combines state- and operator-formalism, and allows to describe potentially non-unitary transformations. As an example, we demonstrate applications of our approach to the case of surface code, where we provide all the necessary details of our method. This paper is organized as follows. In Sec. II we provide a general background including the description of super-ket and super-bra notations [28] and a basic introduction to stabilizer codes. In Sec. III, we further build on these notations to introduce a general framework that incorporates processing logical states under changes in both logical subspaces and in physical state-vector that corresponds to a logical state. Additionally, we provide simple examples of how this framework can be used to analyse transformations of logical states. In Sec. IV we demonstrate the applicability of our approach to the problem of defining by-product operators for defects movements in surface code, as well as provide some additional technical results supporting the derivations. We conclude in Sec. V.

## II. BACKGROUND

### Super-Ket and Super-Bra Notations

This section represents a brief summary of super-ket and super-bra notations for the description of physical states, measurements and operations using Hilbert-Schmidt space.

Let $\mathcal{H}$ be a Hilbert space representing system of $n$ qubits, $d = 2^n$, and let $I$ be the identity matrix of size $d \times d$.

**Definition 1.** *Hilbert-Schmidt space $\mathcal{B}(\mathcal{H})$ is a linear space of $d \times d$ complex matrices equipped with an inner product*

$$\langle\langle A|B\rangle\rangle = \mathrm{tr}(A^\dagger B).$$

Hilbert-Schmidt space is a Hilbert space with respect to the given inner product. To represent states in this space, the notion of density matrix is used.

**Definition 2.** *Density matrix $\rho \in \mathcal{B}(\mathcal{H})$ is a positive semi-definite matrix with trace 1.*

To represent measurements, the notion of effects corresponding to some POVM is used.

**Definition 3.** *Effects $\{E_k\}_{k=1}^M$ are a set of positive semi-definite matrices from the dual space $\mathcal{B}(\mathcal{H})^*$ such that $\sum_{k=1}^M E_k = I$ and they represent all possible measurement outcomes $\{p_k\}_{k=1}^M$ for some POVM:*

$$p_k = \langle\langle E_k|\rho\rangle\rangle = \mathrm{tr}(E_k^\dagger \rho).$$

If we define a basis in $\mathcal{B}(\mathcal{H})$, we will be able to prescribe coordinates to elements from $\mathcal{B}(\mathcal{H})$ and from dual $\mathcal{B}(\mathcal{H})^*$. Given an orthonormal basis $\{B_i\}_{i=1}^{d^2}$, where

$$\langle\langle B_i|B_j\rangle\rangle = \mathrm{tr}(B_i B_j) = \delta_{ij} \ \ \forall i,j \in \{1,...,d^2\},$$

super-bra and super-ket notations are represented by the following coordinates' prescription to density matrices $\rho$ and effects $E_k$:

$$|\rho\rangle\rangle = \begin{pmatrix} \langle\langle B_1|\rho\rangle\rangle \\ \langle\langle B_2|\rho\rangle\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(B_1^\dagger \rho) \\ \mathrm{tr}(B_2^\dagger \rho) \\ \vdots \end{pmatrix},$$

$$\langle\langle E_k| = \begin{pmatrix} \langle\langle E_k|B_1\rangle\rangle & \langle\langle E_k|B_2\rangle\rangle & \dots \end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{tr}(E_k^\dagger B_1) & \mathrm{tr}(E_k^\dagger B_2) & \dots \end{pmatrix}.$$

The next set of definitions gives the notion of completely positive trace preserving (CPTP) superoperators. It is an important class of superoperators because they describe physically allowed operations.

**Definition 4.** *A superoperator $\Lambda : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is trace preserving (TP), if $\mathrm{tr}(\rho) = \mathrm{tr}(\Lambda[\rho]) \ \ \forall \rho \in \mathcal{B}(\mathcal{H})$.*

**Definition 5.** *A superoperator $\Lambda : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is positive, if for any positive semi-definite $\rho \in \mathcal{B}(\mathcal{H})$ the transformed $\Lambda[\rho]$ is positive semi-definite as well.*

**Definition 6.** *A superoperator $\Lambda : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is completely positive (CP), if $\Lambda \otimes Id_{\mathcal{A}} : \mathcal{B}(\mathcal{H}) \times \mathcal{H}_{\mathcal{A}} \mapsto \mathcal{B}(\mathcal{H}) \times \mathcal{H}_{\mathcal{A}}$ is positive for any auxiliary state space $\mathcal{A}$, where $Id_{\mathcal{A}} : \mathcal{H}_{\mathcal{A}} \mapsto \mathcal{H}_{\mathcal{A}}$ is the identity map.*

Now we introduce a very useful matrix representation of the action of CPTP superoperators.

**Definition 7.** *Given a CPTP superoperator $\Lambda : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ and a basis $\{B_i\}_{i=1}^{d^2}$, we can represent the superoperator by a transfer matrix $T_\Lambda$*

$$|\Lambda[\rho]\rangle\rangle = T_\Lambda|\rho\rangle\rangle.$$

When the superoperator $\Lambda$ acts on a state, the probabilities of outcomes corresponding to effects $\{E_k\}_{k=1}^M$ can be determined using transfer matrix $T_\Lambda$:

$$p_k' = \langle\langle E_k|\rho'\rangle\rangle = \langle\langle E_k|T_\Lambda|\rho\rangle\rangle.$$

Note that if we change a basis $\{B_i\}_{i=1}^{d^2}$ to some $\{B_i'\}_{i=1}^{d^2}$, while not changing $\{E_k\}_{k=1}^M$ and $\rho$, then probabilities expressed in a coordinate form will not change, since

$$p_k' = \langle\langle E_k|' \cdot |\rho\rangle\rangle' = \mathrm{tr}(E_k^\dagger \rho) = \langle\langle E_k|\rho\rangle\rangle = p_k$$

by definition of the Hilbert-Schmidt product.

### Stabilizer codes

In this section, We outline features of stabilizer codes that will be used in the context of density matrices. A more elaborate general explanation can be found in [10].

Let $n$ and $k$ denote a number of physical and logical qubits, respectively. An $[[n,k]]$ code can be characterized using the notion of a code space that represents a subspace of the Hilbert space of states of a physical system of $n$ qubits. The code space contains all possible states of system of $k$ logical qubits and can itself be characterized by a stabilizer $S$.

Consider all $n$-fold tensor products of Pauli matrices with multiplicative factors $\pm 1, \pm i$ denoted by $G_n$. Generally, the stabilizer $S$ is an Abelian subgroup of $G_n$ that do not contain $-I$. It can be expressed as the set of all possible compositions of operators from a finite subset of $G_n$ called a generator. The generator's elements must be independent, and they should commute. For an $[[n,k]]$ code, there are $n-k$ elements in the generator of $S$.

Quantum states $|\psi\rangle$ from a Hilbert space for which $g|\psi\rangle = |\psi\rangle$ for some operator $g \in G_n$ are said to be stabilized by $g$.

The code space $C(S)$ represents a subspace within the composite system, where each state in the subspace is stabilized by any element from the stabilizer $S$.

## III. GENERAL FRAMEWORK FOR PROCESSING LOGICAL STATES UNDER CHANGES IN LOGICAL SUBSPACES

Consider a code with stabilizer $S$. Let $\mathcal{B}^S(\mathcal{H})$ be a subspace of $\mathcal{B}(\mathcal{H})$ containing density matrices $\rho$ constructed from states of the corresponding code space $C(S)$. Let $k$ denote the number of logical qubits, and $D = 2^k$. Note that the dimensionality of $\mathcal{B}^S(\mathcal{H})$ is $D^2$.

We can pick a basis set $\{B_{\gamma|\delta}\}$ within $\mathcal{B}^S(\mathcal{H})$, where $\gamma$ and $\delta$ are binary vectors of the size $k$ each, so that, for instance, the value $\gamma_j = 1$ corresponds to the logical operator $\bar{X}_j$, and the value $\delta_j = 1$ corresponds to the logical $\bar{Z}_j$. Note that the size of such a set is $D^2$, which is exactly the dimensionality of $\mathcal{B}^S(\mathcal{H})$.

Formally, let us define

$$B_{\gamma|\delta} = \frac{i^{\gamma\cdot\delta}}{\sqrt{D}} \prod_{j=1}^{k} (\bar{X}_j)^{\gamma_j} (\bar{Z}_j)^{\delta_j},$$

where $\bar{X}_j$ and $\bar{Z}_j$ are logical $X$- and $Z$-operators for $j$-th logical qubit, and $D = 2^k$. $\bar{X}_j$ and $\bar{Z}_j$ can be defined flexibly as physical operators, but we restrict this choice by the following set of commutation relations:

$$[\bar{X}_j, s] = [\bar{Z}_j, s] = 0 \;\forall j \;\forall s \in S,$$

$$[\bar{X}_j, \bar{Z}_k] = 0 \;\forall j \neq k,$$

$$\{\bar{X}_j, \bar{Z}_j\} = 0 \;\forall j.$$

In some cases it will be more convenient to use simpler indexation of the basis set. One possible choice could be a set of indexes $\{i\}_{i=1}^{D^2}$ such that binary index $(0, ..., 0)|(0, ..., 0)$ corresponds to $i = 1$, $(1, ..., 1)|(1, ..., 1)$ corresponds to $i = D^2$, and other indexes are ordered according to some binary representation of $\gamma|\delta$. Further in the text, when we write $\{B_i\}_{i=1}^{D^2}$, we mean exactly this type of indexation, so the commutation relations imposed on the basis remain.

We want to adopt notations from the previous section to describe how probabilities of particular measurements of logical qubits change, when there are 3 types of changes that happen simultaneously:

1. Stabilizer $S$ changes to $S'$.
2. Basis $\{B_{\gamma|\delta}\}$ of the subspace $\mathcal{B}^S(\mathcal{H})$ changes to $\{B'_{\gamma|\delta}\}$ of the subspace $\mathcal{B}^{S'}(\mathcal{H})$.
3. Density matrix $\rho$ corresponding to the set of states stabilized by $S$ changes to density matrix $\rho'$ corresponding to the set of states stabilized by $S'$.

For brevity, we will use a notion of configuration $\mathcal{K}$ that binds together stabilizer and a basis corresponding to this stabilizer:

**Definition 8.** *Configuration is a tuple consisting of stabilizer of a code $S$, a set of logical $X$-operators $\{\bar{X}_j\}_{j=1}^k$, and a set of logical $Z$-operators $\{\bar{Z}_j\}_{j=1}^k$*

$$\mathcal{K} = (S, \{\bar{X}_j\}_{j=1}^k, \{\bar{Z}_j\}_{j=1}^k).$$

Logical $X$- and $Z$- operators from the configuration $\mathcal{K}$ determine basis elements $B_{\gamma|\delta}$ of $\mathcal{B}^S(\mathcal{H})$.

Thus, all the 3 types of changes can be described by two transitions:

$$\mathcal{K} \mapsto \mathcal{K}', \quad \rho \mapsto \rho'.$$

Let us focus on coordinate description of objects associated with a configuration $\mathcal{K}$. If the amount of logical qubits in the code is $k$, then to every logical state with a density matrix $\rho$ from $\mathcal{B}^S(\mathcal{H})$ we can prescribe $D^2 = (2^k)^2$ coordinates using the basis determined from the configuration $\mathcal{K}$. Let us use

$$|\rho\rangle\!\rangle_{\mathcal{K}} = \begin{pmatrix} \langle\!\langle B_1|\rho\rangle\!\rangle \\ \vdots \\ \langle\!\langle B_{D^2}|\rho\rangle\!\rangle \end{pmatrix} = \begin{pmatrix} \mathrm{tr}({B_1}^\dagger \rho) \\ \vdots \\ \mathrm{tr}({B_{D^2}}^\dagger \rho) \end{pmatrix}.$$

Note the important detail: although the density matrix $\rho$ is from the physical space, the amount of coordinates in the vector $|\rho\rangle\!\rangle_{\mathcal{K}}$ corresponds to the logical space. That is, the size of $\rho$ is $d^2 = (2^n)^2$, where $n$ is the amount of physical qubits, but the size of $|\rho\rangle\!\rangle_{\mathcal{K}}$ is $D^2 = (2^k)^2$.

We can pick a POVM consisting of $M$ effects $\{E_k\}_{k=1}^M$ from $\mathcal{B}^S(\mathcal{H})^*$. This POVM will correspond to measurements of logical qubits. $D^2$ coordinates in a configuration $\mathcal{K}$ can be prescribed to effects in the same way:

$$_{\mathcal{K}}\langle\!\langle E_k| = \left(\langle\!\langle E_k|B_1\rangle\!\rangle \;\ldots\; \langle\!\langle E_k|B_{D^2}\rangle\!\rangle\right) =$$

$$= \left(\mathrm{tr}({E_k}^\dagger B_1) \;\ldots\; ({E_k}^\dagger B_{D^2})\right).$$

If some CPTP logical superoperator $\Lambda_L : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ acts on a state $\rho$ from a physical space, we can represent this action in a configuration $\mathcal{K}$ basis by a $D \times D$ matrix $T^{\mathcal{K}}_{\Lambda_L}$

$$|\Lambda_L[\rho]\rangle\!\rangle_{\mathcal{K}} = T^{\mathcal{K}}_{\Lambda_L} |\rho\rangle\!\rangle_{\mathcal{K}}.$$

Under the action of $\Lambda_L$, probabilities of logical measurements are transformed using $T^{\mathcal{K}}_{\Lambda_L}$ in the following way:

$$p'_k = {}_{\mathcal{K}}\langle\!\langle E_k|\rho'\rangle\!\rangle_{\mathcal{K}} = {}_{\mathcal{K}}\langle\!\langle E_k|T^{\mathcal{K}}_{\Lambda_L}|\rho\rangle\!\rangle_{\mathcal{K}}.$$

So far we have been only restricting the notations from the previous section to the subspace $\mathcal{B}^S(\mathcal{H})$. Now we need to build on this apparatus to be able to work with transitions that change not only the density matrix $\rho$, but the configuration $\mathcal{K}$ as well.

**Definition 9.** *A CPTP superoperator regards $\mathcal{K} \mapsto \mathcal{K}'$, if it transforms any density matrix from the subspace $\mathcal{B}^S(\mathcal{H})$ to the subspace $\mathcal{B}^{S'}(\mathcal{H})$, where $S$ corresponds to $\mathcal{K}$, and $S'$ corresponds to $\mathcal{K}'$.*

**Definition 10.** *Consider the transformation*

$$\mathcal{K} \mapsto \mathcal{K}', \quad \rho \mapsto \rho' = \Lambda[\rho],$$

*where $\rho$ is stabilized by $S$ from $\mathcal{K}$ and $\rho'$ is stabilized by $S'$ from $\mathcal{K}'$, and $\Lambda$ is a CPTP superoperator that regards $\mathcal{K} \mapsto \mathcal{K}'$. Logical transfer matrix $T^{\mathcal{K}\mapsto\mathcal{K}'}_{\Lambda}$ is a matrix such that for every proper $\rho$*

$$|\rho'\rangle\!\rangle_{\mathcal{K}'} = T^{\mathcal{K}\mapsto\mathcal{K}'}_{\Lambda} |\rho\rangle\!\rangle_{\mathcal{K}}.$$

Such a matrix exists because the transformation from $|\rho\rangle\rangle_{\mathcal{K}}$ to $|\rho'\rangle\rangle_{\mathcal{K}'}$ is a linear map from coordinate space $\mathbb{C}^{D^2}$ to coordinate space $\mathbb{C}^{D^2}$, as shown in Supplemental Materials.

Note that the superoperator $\Lambda$ and the transformation of configuration $\mathcal{K} \mapsto \mathcal{K}'$ fully define the logical transfer matrix $T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}$. Additionally, the transfer matrix $T_{\Lambda_L}^{\mathcal{K}}$ can be represented as a special case of $T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}$, when configuration doesn't change.

As we will see later, $T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}$ allows to incorporate action in Heisenberg picture corresponding to $\mathcal{K} \mapsto \mathcal{K}'$ and action in Schrodinger picture corresponding to $\rho \mapsto \rho'$.

To define how probabilities of logical measurements change under the transformation from Def. 10, we need to define how effects change. Let us require that under $\mathcal{K} \mapsto \mathcal{K}'$, coordinates of effects remain the same:

$$_{\mathcal{K}}\langle\langle E_k| =_{\mathcal{K}'} \langle\langle E_k'|.$$

This is a well-defined transformation from $E_k$ to $E_k'$, since the effect after transformation $E_k'$ will be simply defined by coordinates in the basis from $\mathcal{K}'$

The motivation for such a requirement is the following: in the Background section we showed that a basis change in $\mathcal{B}(\mathcal{H})$ preserves measurements' probability vector, if $\rho$ is fixed. This doesn't allow for the usage of Heisenberg representation, since these probabilities must change to represent some physical transformation. This leads us to a conclusion that for Heisenberg picture not a basis but rather effects should be redefined to represent operations under fixed $\rho$. In this case, the interpretation of what is $X$-operation and what is $Z$-operation will be defined by transformation of effects, representing Heisenberg picture's logic. When $\mathcal{K} \mapsto \mathcal{K}'$, there is no way to avoid change in basis, so we are "binding" changes in effects to this change in basis.

We are now ready to describe how a probability vector changes. Consider some POVM for logical measurements composed of effects $\{E_k\}_{k=1}^{M}$. These effects can be characterized using basis $\{B_{\gamma|\delta}\}$ defined by a configuration $\mathcal{K}$. Let the probability vector of logical measurements corresponding to the POVM for density matrix of a logical state $\rho$ in a configuration $\mathcal{K}$ be defined as

$$p_k =_{\mathcal{K}} \langle\langle E_k|\rho\rangle\rangle_{\mathcal{K}}.$$

Then, under the transformation

$$|\rho\rangle\rangle_{\mathcal{K}} \mapsto |\rho'\rangle\rangle_{\mathcal{K}'} = T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\rangle_{\mathcal{K}},$$

probabilities will change in the following way:

$$p_k' =_{\mathcal{K}'} \langle\langle E_k'|\rho'\rangle\rangle_{\mathcal{K}'} =_{\mathcal{K}} \langle\langle E_k|T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\rangle_{\mathcal{K}}.$$

Change in operators will be represented by possibility to pick any allowable physical operator to be a representative of the basis set. This particular choice of physical form of $\bar{X}_j$ and $\bar{Z}_j$ will be a key to incorporate the Heisenberg picture.

**Example - Heisenberg Picture**

It is clear that if only basis of a configuration changes, then logical transfer matrix $T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}$ will not be equal to identity, although density matrix didn't change. This will lead to a change in a measurement statistics that correspond to Heisenberg representation. The next example illustrates this idea.

Here we want to use our framework to represent an action by logical operator $\bar{X}$ in the Heisenberg picture. Let $\mathcal{H}$ be a Hilbert space for 1 physical qubit and let stabilizer $S$ be equal to identity $I$. Then $\mathcal{B}^S(\mathcal{H}) = \mathcal{B}^{\mathcal{I}}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. Logical $\bar{X}$ and $\bar{Z}$ are simply physical $X$ and $Z$, so the basis in the initial configuration $\mathcal{K} = (I, \{X\}, \{Z\})$ is defined as $\{B_i\} = \{\frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{Z}{\sqrt{2}}, \frac{iY}{\sqrt{2}}\}$. Let us only change the logical operators so that

$$\mathcal{K} \mapsto \mathcal{K}' = (S', \{X\}, \{-Z\}).$$

The basis after the transformation of the configuration will be $B_i' = \{\frac{I}{\sqrt{2}}, \frac{X}{\sqrt{2}}, \frac{-Z}{\sqrt{2}}, \frac{-iY}{\sqrt{2}}\})$. Although $\rho$ physically didn't change, its coordinates in the configuration $\mathcal{K}'$ are changed, so the transformation matrix $T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}$ will not be equal to identity:

$$|\rho\rangle\rangle_{\mathcal{K}'} = T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\rangle_{\mathcal{K}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} |\rho\rangle\rangle_{\mathcal{K}}.$$

Let us pick effects corresponding to a measurement defined by observable $Z$:

$$_{\mathcal{K}}\langle\langle E_0| = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$_{\mathcal{K}}\langle\langle E_1| = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

According to our convention, coordinates of the effects do not change:

$$_{\mathcal{K}}\langle\langle E_k| =_{\mathcal{K}'} \langle\langle E_k'|.$$

Let $\rho$ represent the state $|0\rangle$. Its coordinates are

$$|\rho\rangle\rangle_{\mathcal{K}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Probability vector for the chosen effects before the transformation is

$$p = \begin{pmatrix} _{\mathcal{K}}\langle\langle E_0|\rho\rangle\rangle_{\mathcal{K}} \\ _{\mathcal{K}}\langle\langle E_1|\rho\rangle\rangle_{\mathcal{K}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Probability vector after the transformation is

$$p = \begin{pmatrix} _{\mathcal{K}'}\langle\langle E_0'|\rho'\rangle\rangle_{\mathcal{K}'} \\ _{\mathcal{K}'}\langle\langle E_1'|\rho'\rangle\rangle_{\mathcal{K}'} \end{pmatrix} = \begin{pmatrix} _{\mathcal{K}}\langle\langle E_0|T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\rangle_{\mathcal{K}} \\ _{\mathcal{K}}\langle\langle E_1|T_{\Lambda}^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\rangle_{\mathcal{K}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

exactly as expected after the action of $\bar{X}$.

## IV. PROCESSING DEFECTS MOVEMENTS IN SURFACE CODE

This section is devoted to explicit application of the described framework. It is divided into four parts.

The first part contains general description of a simple approach to analyse defects movements in surface code with the help of the logical transfer matrix $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ from Def. 10. In Lemma 1 we argue that if $|\psi\rangle, |\psi'\rangle$ correspond to physical state-vectors, and $\{B_i\}_{i=1}^{D^2}, \{B'_i\}_{i=1}^{D^2}$ are basis sets before and after the transformation, then the values $\langle\psi'|B'_i|\psi'\rangle$ and $\langle\psi|B_j|\psi\rangle$ are related via the logical transfer matrix $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$. In addition, we provide a useful technical result in the form of Lemma 2.

The rest three parts describe how to express values $\langle\psi'|B'_i|\psi'\rangle$ as linear combinations of $\langle\psi|B_j|\psi\rangle$. This helps reconstruct logical transfer matrix $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ in three different scenarios:

- One-cell defect movement
- Multi-cell defect movement
- Braiding for logical CNOT

### General approach

We want to prove that some physical transformation represents an action of a logical superoperator $\Lambda_L$. To do this, we do the following steps:

1. Define a candidate physical transformation
$$\mathcal{K} \mapsto \mathcal{K}', \quad \rho \mapsto \rho' = \Lambda[\rho],$$
where $\rho$ is stabilized by $S$ from $\mathcal{K}$ and $\rho'$ is stabilized by $S'$ from $\mathcal{K}'$, $\Lambda$ is a superoperator that regards $\mathcal{K} \mapsto \mathcal{K}'$, and $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ is a logical transfer matrix
$$|\rho'\rangle\!\rangle_{\mathcal{K}'} = T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\!\rangle_{\mathcal{K}}.$$

2. Find a transfer matrix $T_{\Lambda_L}$ corresponding to
$$|\Lambda_L[\rho_L]\rangle\!\rangle = T_{\Lambda_L}|\rho_L\rangle\!\rangle,$$
where $|\rho\rangle\!\rangle$ is a super-bra vector in an abstract logical space, not related to the actual physical space linked to configurations $\mathcal{K}, \mathcal{K}'$, and potentially non-unitary physical superoperator $\Lambda$ from the previous step.

3. Prove that
$$T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'} = T_{\Lambda_L}.$$

If we do these steps, we will essentially prove that probabilities of particular measurements will change as they would change after acting on a density matrix of logical state by the logical superoperator $\Lambda_L$. We will consider a practically relevant type of $\Lambda$, such that $\Lambda[\rho] = \Lambda\rho\Lambda^\dagger$. This transformation, nevertheless, can be defined by randomized parameters and can be projective.

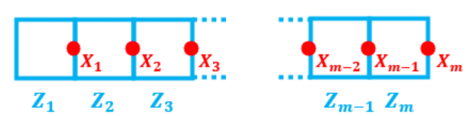
The last step is the most cumbersome. The lemma below states that to perform this step, one needs to find the explicit relation between $\langle\psi'|B'_i|\psi'\rangle$ and $\langle\psi|B_j|\psi\rangle$.

**Lemma 1.** *Let $|\psi\rangle, |\psi'\rangle$ correspond to physical state-vectors, and $\Lambda$ is a superoperator that describes transformation of the pure state $|\psi\rangle$ to the pure state $|\psi'\rangle$. Let $\{B_i\}_{i=1}^{D^2}, \{B'_i\}_{i=1}^{D^2}$ are basis sets from the configurations $\mathcal{K}, \mathcal{K}'$ before and after the transformation. Let $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ be corresponding logical transfer matrix, so that*
$$|\rho'\rangle\!\rangle_{\mathcal{K}'} = T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\!\rangle_{\mathcal{K}}.$$
*Then the values $\langle\psi'|B'_i|\psi'\rangle$ and $\langle\psi|B_j|\psi\rangle$ are related in the following way:*
$$\langle\psi'|B'_i|\psi'\rangle = \sum_j (T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'})_{ij}\langle\psi|B_j|\psi\rangle \ \ \forall i.$$

**Proof.** See Supplemental Materials.

Another important result that will be used in all the parts is the following lemma about relations between projectors during defects movements:

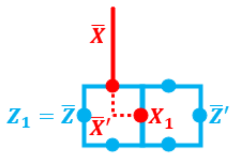
**Lemma 2.** *Consider the set of operators $\{X_i\}_{i=1}^m$ and the set of operators $\{Z_i\}_{i=1}^m$, where $m$ is the number of cells in a defect movement. Let $\alpha_{X_i}, \alpha_{Z_i}$ be the eigenvalues corresponding to measurements defined by $X_i, Z_i$, respectively.*




*Let the corresponding projectors be defined as*
$$P_{\alpha_{X_i}} = \frac{I + (-1)^{\alpha_{X_i}} X_i}{2},$$
$$P_{\alpha_{Z_i}} = \frac{I + (-1)^{\alpha_{Z_i}} Z_i}{2}.$$
*Let $\{\beta_i\}$ be some set of numbers that equal to either 0 or 1. Then the following relation holds:*
$$\prod_{i=1}^m (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$
$$\prod_{k=1}^m \left((-1)^{\alpha_{Z_k}\sum_{j=k}^m \beta_j}(Z_k)^{\sum_{j=k}^m \beta_j}\right) \prod_{k=1}^m (P_{\alpha_{X_i}}).$$

**Proof.** See Supplemental Materials.

**One-cell movement**




Thus, we can express the state after the one-cell movement as

$$|\psi'\rangle = \frac{\bar{X}'^{d_Z}\bar{Z}'^{d_X} P_{\alpha_{Z_1}} P_{\alpha_{X_1}}|\psi\rangle}{\sqrt{N}},$$

where $\bar{X}'^{d_Z}$ and $\bar{Z}'^{d_X}$ are by-product operators, and the constant $\sqrt{N}$ is the normalization factor.

$$N = \langle\psi|P_{\alpha_{X_1}} P_{\alpha_{Z_1}} P_{\alpha_{Z_1}} P_{\alpha_{X_1}}|\psi\rangle =$$

$$= \langle\psi|P_{\alpha_{X_1}} P_{\alpha_{Z_1}} P_{\alpha_{X_1}}|\psi\rangle.$$

Although a guess for the form of $d_X, d_Z$ is given in [27], we will keep this variables till the end of the derivation.

Now let us consider the general approach. For the step 1, we need to define a candidate physical transformation. In our case, elements taking part in this transformation are

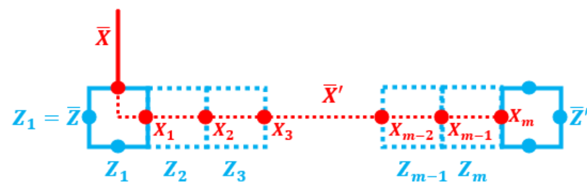
$$\mathcal{K} = \big(S, \{\bar{X}\}, \{\bar{Z}\}\big) \mapsto \big(S', \{\bar{X}'\}, \{\bar{Z}'\}\big) = \mathcal{K}',$$

$$\rho = |\psi\rangle\langle\psi| \mapsto |\psi'\rangle\langle\psi'| = \rho'.$$

On the step 2 we define $T_{\Lambda_L}$, given $\Lambda_L$. In our case, we want $\Lambda_L$ to be equal to $I$, so $T_{\Lambda_L}$ should be an identity.

Finally, we will make use of Lemma 1 to find elements of $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ and then show that $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'} = T_{\Lambda_L}$. Since $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ should be identity, we want to show that $\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle$ is equal to $\langle\psi|B_{\gamma|\delta}|\psi\rangle$.

**Multi-cell defects movements**




The case of multi-cell movement is a straight-forward generalization of one-cell movement, so we put in this paragraph a description of the general approach only. For rigorous proof of the step 3, see Supplemental Materials.

We can express the state after the multi-cell movement as

$$|\psi'\rangle = \frac{\bar{X}'^{d_Z}\bar{Z}'^{d_X}\prod_{i=1}^m P_{\alpha_{Z_i}}\prod_{i=1}^m P_{\alpha_{X_i}}|\psi\rangle}{\sqrt{N}},$$

where $\bar{X}'^{d_Z}$ and $\bar{Z}'^{d_X}$ are by-product operators, and the constant $\sqrt{N}$ is the normalization factor.

$$N = \langle\psi|\prod_{i=1}^m P_{\alpha_{X_i}}\prod_{i=1}^m P_{\alpha_{Z_i}}\prod_{i=1}^m P_{\alpha_{Z_i}}\prod_{i=1}^m P_{\alpha_{X_i}}|\psi\rangle =$$

$$= \langle\psi|\prod_{i=1}^m P_{\alpha_{X_i}}\prod_{i=1}^m P_{\alpha_{Z_i}}\prod_{i=1}^m P_{\alpha_{X_i}}|\psi\rangle.$$

Now let us consider the general approach. For the step 1, we need to define a candidate physical transformation. In our case, elements taking part in this transformation are

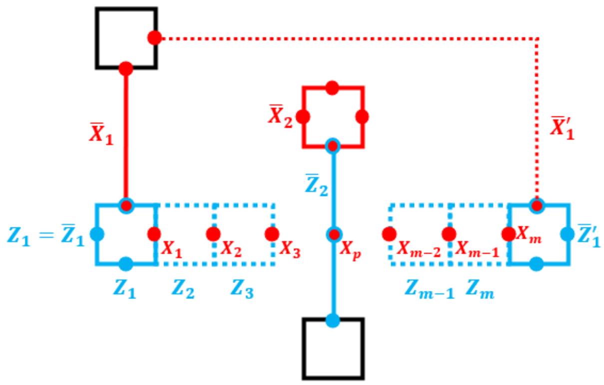
$$\mathcal{K} = \big(S, \{\bar{X}\}, \{\bar{Z}\}\big) \mapsto \big(S', \{\bar{X}'\}, \{\bar{Z}'\}\big) = \mathcal{K}',$$

$$\rho = |\psi\rangle\langle\psi| \mapsto |\psi'\rangle\langle\psi'| = \rho'.$$

On the step 2 we define $T_{\Lambda_L}$, given $\Lambda_L$. In our case, we want $\Lambda_L$ to be equal to $I$, so $T_{\Lambda_L}$ should be an identity.

Finally, we will make use of Lemma 1 to find elements of $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ and then show that $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'} = T_{\Lambda_L}$. Since $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ should be identity, we want to show that $\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle$ is equal to $\langle\psi|B_{\gamma|\delta}|\psi\rangle$. For proof, see Supplemental Materials.

### Defects movements for logical CNOT




We put in this paragraph a description of the general approach only. For rigorous proof of the step 3, see Supplemental Materials.

We can express the state after the multi-cell movements as

$$|\psi'\rangle = \frac{\bar{X}_2'^{d_{Z_2}}\,\bar{Z}_2'^{d_{X_2}}\,\bar{X}_1'^{d_{Z_1}}\,\bar{Z}_1'^{d_{X_1}}\prod_{i=1}^{m}P_{\alpha_{Z_i}}\prod_{i=1}^{m}P_{\alpha_{X_i}}|\psi\rangle}{\sqrt{N}},$$

where $\bar{X}_2'^{d_{Z_2}}$, $\bar{Z}_2'^{d_{X_2}}$, $\bar{X}_1'^{d_{Z_1}}$, and $\bar{Z}_1'^{d_{X_1}}$ are by-product operators, and the constant $\sqrt{N}$ is the normalization factor.

$$N = \langle\psi|\prod_{i=1}^{m}P_{\alpha_{X_i}}\prod_{i=1}^{m}P_{\alpha_{Z_i}}\prod_{i=1}^{m}P_{\alpha_{Z_i}}\prod_{i=1}^{m}P_{\alpha_{X_i}}|\psi\rangle =$$

$$= \langle\psi|\prod_{i=1}^{m}P_{\alpha_{X_i}}\prod_{i=1}^{m}P_{\alpha_{Z_i}}\prod_{i=1}^{m}P_{\alpha_{X_i}}|\psi\rangle.$$

Now let us consider the general approach. For the step 1, we need to define a candidate physical transformation. In our case, elements taking part in this transformation are

$$\mathcal{K} = \left(S, \{\bar{X}_i\}_{i=1}^2, \{\bar{Z}_i\}_{i=1}^2\right) \mapsto$$

$$\mapsto \left(S', \{\bar{X}'_i\}_{i=1}^2, \{\bar{Z}'_i\}_{i=1}^2\right) = \mathcal{K}',$$

$$\rho = |\psi\rangle\langle\psi| \mapsto |\psi'\rangle\langle\psi'| = \rho'.$$

On the step 2 we define $T_{\Lambda_L}$, given $\Lambda_L$. In our case, we want $\Lambda_L$ to be equal to represent logical CNOT-operation.

Finally, we will make use of Lemma 1 to find elements of $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ and then show that $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'} = T_{\Lambda_L}$. Since $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ should be equal to the transfer matrix of logical CNOT, we want to show that $\langle\psi'|B'_{(\gamma_1\oplus\gamma_2,\gamma_2)|(\delta_1,\delta_1\oplus\delta_2)}|\psi'\rangle$ is equal to $\langle\psi|B_{(\gamma_1,\gamma_2)|(\delta_1,\delta_2)}|\psi\rangle$. For proof, see Supplemental Materials.

## V. CONCLUSION

We presented a series of theoretical clarifications that combine state- and operator-formalism to describe the evolution of logical states under the influence of simultaneous transformations in both physical state and logical subspace structure. We demonstrated the application of the framework for processing defect movements in surface code by deriving powers of byproduct operators. We expect that our results will prove valuable for ongoing experiments attempting to demonstrate quantum error correction using the existing generation of quantum processors.

## VI. ACKNOWLEDGEMENTS

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## SUPPLEMENTAL MATERIALS

### Proof of the Lemma 1.

Let us expand super-ket notations for density matrices:

$$|\rho\rangle\!\rangle_{\mathcal{K}} = \begin{pmatrix} \langle\!\langle B_1|\rho\rangle\!\rangle \\ \vdots \\ \langle\!\langle B_{D^2}|\rho\rangle\!\rangle \end{pmatrix} = \begin{pmatrix} \mathrm{tr}({B_1}^\dagger \rho) \\ \vdots \\ \mathrm{tr}({B_{D^2}}^\dagger \rho) \end{pmatrix} = \begin{pmatrix} \langle\psi|B_1|\psi\rangle \\ \vdots \\ \langle\psi|B_{D^2}|\psi\rangle \end{pmatrix},$$

$$|\rho'\rangle\!\rangle_{\mathcal{K}'} = \begin{pmatrix} \langle\!\langle B'_1|\rho'\rangle\!\rangle \\ \vdots \\ \langle\!\langle B'_{D^2}|\rho'\rangle\!\rangle \end{pmatrix} = \begin{pmatrix} \mathrm{tr}({B'_1}^\dagger \rho') \\ \vdots \\ \mathrm{tr}({B'_{D^2}}^\dagger \rho') \end{pmatrix} = \begin{pmatrix} \langle\psi'|B'_1|\psi'\rangle \\ \vdots \\ \langle\psi'|B'_{D^2}|\psi'\rangle \end{pmatrix}.$$

If we write elementwise the relation $|\rho'\rangle\!\rangle_{\mathcal{K}'} = T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}|\rho\rangle\!\rangle_{\mathcal{K}}$, we get

$$\langle\psi'|B'_i|\psi'\rangle = \sum_j (T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'})_{ij}\langle\psi|B_j|\psi\rangle \ \ \forall i,$$

and this completes the proof.

### Proof of the Lemma 2.

To prove the Lemma 2, we iteratively substitute the expression

$$P_{\alpha_{X_i}+\beta_i} = \frac{I + (-1)^{\alpha_{X_i}+\beta_i}X_i}{2}$$

into the expression

$$\prod_{i=1}^m (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}),$$

and use commutation relations between operators on every step. Let us start from $i = m$.

$$\prod_{i=1}^m (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$

$$= \frac{1}{2}\prod_{i=1}^{m-1} (P_{\alpha_{X_i}+\beta_i}) \Bigg( I \cdot \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) +$$

$$+ (-1)^{\alpha_{X_m}+\beta_m} X_m \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) \Bigg).$$

Note that

$$X_m \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$

$$= \prod_{i=1}^{m-1} (P_{\alpha_{Z_i}}) P_{\alpha_{Z_m}+1} X_m \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$

$$= (-1)^{\alpha_{X_i}} \prod_{i=1}^{m-1} (P_{\alpha_{Z_i}}) P_{\alpha_{Z_m}+1} \prod_{i=1}^m (P_{\alpha_{X_i}}).$$

Thus,

$$\prod_{i=1}^m (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$

$$= \frac{1}{2}\prod_{i=1}^{m-1} (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^{m-1} (P_{\alpha_{Z_i}}) \cdot$$

$$\cdot \left(P_{\alpha_{Z_m}} + (-1)^{\beta_m}(P_{\alpha_{Z_m}+1})\right) \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$

$$= \prod_{i=1}^{m-1} (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^{m-1} (P_{\alpha_{Z_i}}) \left((-1)^{\alpha_{Z_m}\beta_m} Z_m^{\beta_m}\right) \prod_{i=1}^m (P_{\alpha_{X_i}}).$$

Note that the product $\prod_{i=1}^m (P_{\alpha_{X_i}})$ in the end of the initial expression remains unchanged, while the part

$$\prod_{i=1}^m (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^m (P_{\alpha_{Z_i}})$$

transforms to

$$\prod_{i=1}^{m-1} (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^{m-1} (P_{\alpha_{Z_i}}) \left((-1)^{\alpha_{Z_m}\beta_m} Z_m^{\beta_m}\right)$$

after making transformations for $i = m$. Thus, doing the same set of transformations for remaining $i$ in descending order, we arrive at the conclusion that

$$\prod_{i=1}^m (P_{\alpha_{X_i}+\beta_i}) \prod_{i=1}^m (P_{\alpha_{Z_i}}) \prod_{i=1}^m (P_{\alpha_{X_i}}) =$$

$$\prod_{k=1}^m \left((-1)^{\alpha_{Z_k}\sum_{j=k}^m \beta_j} (Z_k)^{\sum_{j=k}^m \beta_j}\right) \prod_{k=1}^m (P_{\alpha_{X_i}}).$$

**Proof of existence of a logical transfer matrix.**

We want to prove that a logical transfer matrix $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ given in Def. 10 exists. We will show that $T_\Lambda^{\mathcal{K}\mapsto\mathcal{K}'}$ corresponds to a linear map $\mathbb{C}^{D^2}\mapsto\mathbb{C}^{D^2}$.

Note that $\mathcal{K}$ contains particular basis ${B_i}_1^{D^2}$ and $\mathcal{K}'$ contains basis ${B'_i}_1^{D^2}$. This means that $\rho$ and $\rho'$ have coordinate representations in $\mathbb{C}^{D^2}$, so it remains to prove that the transformation $|\rho\rangle\!\rangle_{\mathcal{K}} \mapsto |\rho'\rangle\!\rangle_{\mathcal{K}'} = |\Lambda\rho\Lambda^\dagger\rangle\!\rangle_{\mathcal{K}'}$ is linear, which is evident from the coordinate representation:

$$|\Lambda\rho\Lambda^\dagger\rangle\!\rangle_{\mathcal{K}'} = \begin{pmatrix} \langle\!\langle B'_1|\Lambda\rho\Lambda^\dagger\rangle\!\rangle \\ \vdots \\ \langle\!\langle B'_{D^2}|\Lambda\rho\Lambda^\dagger\rangle\!\rangle \end{pmatrix} = \begin{pmatrix} \mathrm{tr}({B'_1}^\dagger\Lambda\rho\Lambda^\dagger) \\ \vdots \\ \mathrm{tr}({B'_{D^2}}^\dagger\Lambda\rho\Lambda^\dagger) \end{pmatrix}.$$

Indeed, since trace is linear operator, we have

$$|(\rho_1+\rho_2)'\rangle\!\rangle_{\mathcal{K}'} = |\Lambda(\rho_1+\rho_2)\Lambda^\dagger\rangle\!\rangle_{\mathcal{K}'} =$$

$$= |\Lambda\rho_1\Lambda^\dagger\rangle\!\rangle_{\mathcal{K}'} + |\Lambda\rho_2\Lambda^\dagger\rangle\!\rangle_{\mathcal{K}'} = |\rho'_1\rangle\!\rangle_{\mathcal{K}'} + |\rho'_2\rangle\!\rangle_{\mathcal{K}'},$$

and this completes the proof.

**One-cell defect movement - step 3.**

Let us first find $N$. Using Lemma 2 for $\beta = 0$, we get

$$N = \langle\psi|P_{\alpha_{X_1}}|\psi\rangle = \frac{1}{2}\left(\langle\psi|\psi\rangle + (-1)^{\alpha_{X_1}}\langle\psi|X_1|\psi\rangle\right).$$

Note that the operator $\bar{Z}'$ is in the stabilizer $S$ of the state $|\psi\rangle$, and this operator anti-commutes with $X_1$. Let's denote eigenvalue for $\bar{Z}'$ as $(-1)^{\alpha^S_{Z'}}$, then

$$\bar{Z}'|\psi\rangle = (-1)^{\alpha^S_{Z'}}|\psi\rangle,$$

$$\langle\psi|X_1|\psi\rangle = (-1)^{\alpha^S_{Z'}}\langle\psi|X_1\bar{Z}'|\psi\rangle = -\langle\psi|X_1|\psi\rangle.$$

Thus, $\langle\psi|X_1|\psi\rangle = 0$, and $N$ transforms to $N = \frac{1}{2}$.
Now we are ready to perform the step 3.

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma d_X + \delta d_Z}\frac{1}{N}\langle\psi|P_{\alpha_{X_1}}P_{\alpha_{Z_1}}B'_{\gamma|\delta}P_{\alpha_{Z_1}}P_{\alpha_{X_1}}|\psi\rangle =$$

$$= (-1)^{\gamma d_X + \delta d_Z}\frac{1}{N}\langle\psi|B'_{\gamma|\delta}P_{\alpha_{X_1}+\delta}P_{\alpha_{Z_1}}P_{\alpha_{X_1}}|\psi\rangle.$$

We can use Lemma 2 with $m = 1$, $\beta_1 = \delta$ to get

$$P_{\alpha_{X_1}+\delta}P_{\alpha_{Z_1}}P_{\alpha_{X_1}} = (-1)^{\delta\alpha_{Z_1}}{Z_1}^\delta P_{\alpha_{X_1}}.$$

Thus,

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma d_X + \delta(d_Z+\alpha_{Z_1})}\frac{1}{N}\langle\psi|B'_{\gamma|\delta}{Z_1}^\delta P_{\alpha_{X_1}}|\psi\rangle.$$

Note that $Z_1 = \bar{Z}$, $\bar{X}' = \bar{X}X_1$, and $\bar{Z}' \in S$, where $S$ is the stabilizer for the initial state $|\psi\rangle$.
Consider two cases:
**Case 1.** $\gamma = 0$.

$$\langle\psi'|B'_{0|\delta}|\psi'\rangle =$$

$$= (-1)^{\delta(d_Z+\alpha_{Z_1})}\frac{1}{N}\langle\psi|B'_{0|\delta}{Z_1}^\delta P_{\alpha_{X_1}}|\psi\rangle =$$

$$(-1)^{\delta(d_Z+\alpha_{Z_1}+\alpha^S_{Z'})}\frac{1}{N}\frac{1}{\sqrt{2}}\cdot$$

$$\cdot\frac{1}{2}\big(\langle\psi|\bar{Z}^\delta|\psi\rangle + (-1)^{\alpha_{X_1}}\langle\psi|\bar{Z}^\delta X_1|\psi\rangle\big).$$

Note that

$$\langle\psi|\bar{Z}^\delta X_1|\psi\rangle = (-1)^{\alpha^S_{Z_1}}\langle\psi|\bar{Z}^\delta X_1\bar{Z}'|\psi\rangle =$$

$$= (-1)^{\alpha^S_{Z_1}}\langle\psi|\bar{Z}^\delta(-\bar{Z}'X_1)|\psi\rangle =$$

$$= -\langle\psi|\bar{Z}^\delta X_1|\psi\rangle \Rightarrow \langle\psi|\bar{Z}^\delta X_1|\psi\rangle = 0.$$

Thus, for $\gamma = 0$,

$$\langle\psi'|B'_{0|\delta}|\psi'\rangle = (-1)^{\delta(d_Z+\alpha_{Z_1}+\alpha^S_{Z_1})}\langle\psi|B_{0|\delta}|\psi\rangle.$$

**Case 2.** $\gamma = 1$.

$$\langle\psi'|B'_{1|\delta}|\psi'\rangle =$$

$$= (-1)^{d_X+\delta(d_Z+\alpha_{Z_1})}\frac{1}{N}\langle\psi|B'_{1|\delta}{Z_1}^\delta P_{\alpha_{X_1}}|\psi\rangle =$$

$$= (-1)^{d_X+\delta(d_Z+\alpha_{Z_1})}\frac{1}{N}\frac{i^{1\cdot\delta}}{\sqrt{2}}\langle\psi|(\bar{X}'L)^1(\bar{Z}')^\delta Z_1^\delta P_{\alpha_{X_1}}|\psi\rangle =$$

$$= (-1)^{d_X+\delta(d_Z+\alpha_{Z_1})}\frac{1}{N}\frac{i^{1\cdot\delta}}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta\left(I + (-1)^{\alpha_{X_1}}X_1\right)|\psi\rangle.$$

Note that in the last expression the term $\langle\psi|(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta|\psi\rangle$ is equal to 0, since the operator $\bar{Z}' \in S$ anti-commutes with $(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta$. Thus,

$$\langle\psi'|B'_{1|\delta}|\psi'\rangle =$$

$$= (-1)^{(d_X+\alpha_{X_1})+\delta(d_Z+\alpha_{Z_1})} \frac{1}{N} \frac{i^{1\cdot\delta}}{\sqrt{2}} \cdot$$

$$\cdot \langle\psi|(\bar{X}')^1 (\bar{Z}')^\delta Z_1^\delta X_1 |\psi\rangle =$$

$$= (-1)^{(d_X+\alpha_{X_1})+\delta(d_Z+\alpha_{Z_1})} \frac{1}{N} \frac{i^{1\cdot\delta}}{\sqrt{2}} \cdot$$

$$\cdot \langle\psi|(\bar{X}' X_1)(\bar{Z}')^\delta Z_1^\delta |\psi\rangle =$$

$$= (-1)^{(d_X+\alpha_{X_1})+\delta(d_Z+\alpha_{Z_1}+\alpha^S_{Z'})} \frac{1}{N} \frac{i^{1\cdot\delta}}{\sqrt{2}} \cdot$$

$$\cdot \langle\psi|\bar{X} Z_1^\delta |\psi\rangle.$$

Combining these two cases together, we get

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma(d_X+\alpha_{X_1})+\delta(d_Z+\alpha_{Z_1}+\alpha^S_{Z'})} \cdot$$

$$\cdot \langle\psi|B_{\gamma|\delta}|\psi\rangle.$$

If we substitute

$$d_X = \alpha_{X_1},$$

$$d_Z = \alpha_{Z_1} + \alpha^S_{Z'},$$

then we will get the expression we wanted to prove:

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle = \langle\psi|B_{\gamma|\delta}|\psi\rangle.$$

**Multi-cell defects movements - step 3.**

Let us first find $N$. Using Lemma 2 for $\beta = (0, ..., 0)$,

$$N = \langle\psi| \prod_{i=1}^{m} P_{\alpha_{X_i}} |\psi\rangle = \frac{1}{2^m} \left(\langle\psi|\psi\rangle + 0 + ... + 0\right) = \frac{1}{2^m}.$$

Now we are ready to perform the step 3.

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma d_X + \delta d_Z} \frac{1}{N} \cdot$$

$$\cdot \langle\psi| \prod_i P_{\alpha_{X_i}} \prod_i P_{\alpha_{Z_i}} (B'_{\gamma|\delta}) \prod_i P_{\alpha_{Z_i}} \prod_i P_{\alpha_{X_i}} |\psi\rangle =$$

$$= (-1)^{\gamma d_X + \delta d_Z} \frac{1}{N} \cdot$$

$$\cdot \langle\psi|(B'_{\gamma|\delta}) \left(\prod_{i=1}^{m-1} P_{\alpha_{X_i}}\right) P_{\alpha_{X_m}+\delta} \prod_i P_{\alpha_{Z_i}} \cdot$$

$$\cdot \prod_i P_{\alpha_{Z_i}} \prod_i P_{\alpha_{X_i}} |\psi\rangle =$$

$$= (-1)^{\gamma d_X + \delta d_Z} \frac{1}{N} \cdot$$

$$\cdot \langle\psi|(B'_{\gamma|\delta}) \left(\prod_{i=1}^{m-1} P_{\alpha_{X_i}}\right) P_{\alpha_{X_m}+\delta} \prod_i P_{\alpha_{Z_i}} \prod_i P_{\alpha_{X_i}} |\psi\rangle.$$

Note that we can use Lemma 2 to transform the product of projectors, since in our case $\beta = (0\ 0\ ...\ 0\ \delta)$. After the transformation we get

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma d_X + \delta(d_Z + \sum_i \alpha_{Z_i})} \frac{1}{N} \cdot$$

$$\cdot \langle\psi|B'_{\gamma|\delta} \prod_{i=1}^{m} Z_i^\delta \prod_i P_{\alpha_{X_i}} |\psi\rangle.$$

Note that

$$\bar{Z}' \in S,$$

$$Z_i \in S \quad \forall i \in \{2, ..., m\},$$

$$[Z_i, B'_{\gamma|\delta}] = 0 \quad \forall i \in \{2, ..., m\}.$$

Let us denote the corresponding eigenvalues of $Z_i$ as $(-1)^{\alpha^S_{Z_i}}$ for $i \in \{2, ..., m\}$, and let the eigenvalue of $\bar{Z}'$ be $(-1)^{\alpha^S_{Z'}}$. Then we can write

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma d_X + \delta(d_Z + \sum_i \alpha_{Z_i} + \sum_{i=2}^{m} \alpha^S_{Z_i})} \frac{1}{N} \cdot$$

$$\cdot \langle\psi|B'_{\gamma|\delta} Z_1^\delta \prod_i P_{\alpha_{X_i}} |\psi\rangle.$$

Consider 2 cases:
**Case 1.** $\gamma = 0$.

$$\langle\psi'|B'_{0|\delta}|\psi'\rangle =$$

$$= (-1)^{\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S)} \frac{1}{N}\frac{1}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|(\bar{Z}')^\delta Z_1^\delta \prod_i P_{\alpha_{X_i}}|\psi\rangle =$$

$$= (-1)^{\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S)} \frac{1}{N}\frac{1}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|Z_1^\delta\left(\frac{I+...+(-1)^{\sum_{i=1}^m \alpha_{X_i}}\prod_{i=1}^m X_i}{2^m}\right)|\psi\rangle.$$

Note that in the last expression all terms of the type $\langle\psi|Z_1^\delta\prod_{j\in J}X_j|\psi\rangle$ are equal to 0 for any non-empty set of indexes $J$. This follows from the fact, that for any such set $J$ there exists operator $Z_k$ in the stabilizer $S$ of the state $|\psi\rangle$ that anti-commutes with $\prod_{j\in J}X_j$:

$$\langle\psi|Z_1^\delta\prod_{j\in J}X_j|\psi\rangle = (-1)^{\alpha_{Z_k}^S}\langle\psi|Z_1^\delta\prod_{j\in J}X_jZ_k|\psi\rangle =$$

$$= (-1)^{\alpha_{Z_k}^S+1}\langle\psi|Z_kZ_1^\delta\prod_{j\in J}X_j|\psi\rangle = -\langle\psi|Z_1^\delta\prod_{j\in J}X_j|\psi\rangle.$$

Thus, continuing our derivation and taking into account that $Z_1=\bar{Z}$,

$$\langle\psi'|B'_{0|\delta}|\psi'\rangle =$$

$$= (-1)^{\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S)} \frac{1}{N}\frac{1}{\sqrt{2}}\cdot$$

$$\cdot\frac{1}{2^m}\langle\psi|Z_1^\delta|\psi\rangle =$$

$$= (-1)^{\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S)}\langle\psi|B_{0|\delta}|\psi\rangle.$$

**Case 2.** $\gamma=1$.

$$\langle\psi'|B'_{1|\delta}|\psi'\rangle =$$

$$= (-1)^{d_X+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S)} \frac{1}{N}\frac{i^{1\cdot\delta}}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta \prod_i P_{\alpha_{X_i}}|\psi\rangle =$$

$$= (-1)^{d_X+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S)} \frac{1}{N}\frac{i^{1\cdot\delta}}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta\left(\frac{I+...+(-1)^{\sum_{i=1}^m \alpha_{X_i}}\prod_{i=1}^m X_i}{2^m}\right)|\psi\rangle.$$

Note that in the last expression all terms of the type $\langle\psi|(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta\prod_{j\in J}X_j|\psi\rangle$ are equal to 0 for any $J\neq\{1,...,m\}$, since we can find an operator from $S$ that anti-commutes with $(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta\prod_{j\in J}X_j$ for any such $J$, including $J=\emptyset$. Thus,

$$\langle\psi'|B'_{1|\delta}|\psi'\rangle =$$

$$= (-1)^{d_X+\sum_i \alpha_{X_i}+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S)}\cdot\frac{i^{1\cdot\delta}}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|(\bar{X}')^1(\bar{Z}')^\delta Z_1^\delta\prod_{i=1}^m X_i|\psi\rangle =$$

$$= (-1)^{d_X+\sum_i \alpha_{X_i}+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S)}\cdot\frac{i^{1\cdot\delta}}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|\left(\bar{X}'\prod_{i=1}^m X_i\right)(\bar{Z}')^\delta Z_1^\delta|\psi\rangle =$$

$$= (-1)^{d_X+\sum_i \alpha_{X_i}+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S)}\cdot\frac{i^{1\cdot\delta}}{\sqrt{2}}\cdot$$

$$\cdot\langle\psi|\bar{X}Z_1^\delta|\psi\rangle =$$

$$= (-1)^{1\cdot(d_X+\sum_i \alpha_{X_i})+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S)}\cdot$$

$$\cdot\langle\psi|B_{1|\delta}|\psi\rangle.$$

Combining these two cases together, we get

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle =$$

$$= (-1)^{\gamma(d_X+\sum_i \alpha_{X_i})+\delta(d_Z+\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S)}\cdot$$

$$\cdot\langle\psi|B_{\gamma|\delta}|\psi\rangle.$$

If we substitute

$$d_X=\sum_i \alpha_{X_i},$$

$$d_Z=\sum_i \alpha_{Z_i}+\sum_{i=2}^m \alpha_{Z_i}^S+\alpha_{Z'}^S,$$

then we will get the expression we wanted to prove:

$$\langle\psi'|B'_{\gamma|\delta}|\psi'\rangle = \langle\psi|B_{\gamma|\delta}|\psi\rangle.$$

**Logical CNOT - step 3.**

Let us first find $N$. Using Lemma 2 for $\beta = (0, ..., 0)$,

$$N = \langle\psi|\prod_{i=1}^{m} P_{\alpha_{X_i}}|\psi\rangle = \frac{1}{2^m}\left(\langle\psi|\psi\rangle + 0 + ... + 0\right) = \frac{1}{2^m}.$$

Now we are ready to perform the step 3. Let's denote $(\gamma'|\delta') = (\gamma_1, \gamma_1 + \gamma_2|\delta_1 + \delta_2, \delta_2)$.

$$\langle\psi'|B'_{\gamma'|\delta'}|\psi'\rangle =$$

$$= (-1)^{\gamma_1 d_{X_1} + (\gamma_1+\gamma_2)d_{X_2} + (\delta_1+\delta_2)d_{Z_1} + \delta_2 d_{Z_2}}\frac{1}{N}\cdot$$

$$\cdot\langle\psi|\prod_i P_{\alpha_{X_i}}\prod_i P_{\alpha_{Z_i}}(B'_{\gamma'|\delta'})\prod_i P_{\alpha_{Z_i}}\prod_i P_{\alpha_{X_i}}|\psi\rangle =$$

$$= (-1)^{\gamma_1 d_{X_1} + (\gamma_1+\gamma_2)d_{X_2} + (\delta_1+\delta_2)d_{Z_1} + \delta_2 d_{Z_2}}\frac{1}{N}\cdot$$

$$\cdot\langle\psi|(B'_{\gamma'|\delta'})\left(\prod_{i\setminus\{p,m\}} P_{\alpha_{X_i}}\right)P_{\alpha_{X_p}+\delta_2}P_{\alpha_{X_m}+(\delta_1+\delta_2)}\cdot$$

$$\cdot\prod_i P_{\alpha_{Z_i}}\prod_i P_{\alpha_{X_i}}|\psi\rangle.$$

Note that we can use Lemma 2 to transform the product of projectors, since in our case $\beta = (0\ ...\ 0\ \delta_2\ 0\ ...\ \delta_1 + \delta_2)$, where non-zero entries are on the $p$-th and $m$-th places. After the transformation we get

$$\langle\psi'|B'_{\gamma'|\delta'}|\psi'\rangle =$$

$$= (-1)^{\gamma_1 d_{X_1} + (\gamma_1+\gamma_2)d_{X_2} + (\delta_1+\delta_2)d_{Z_1} + \delta_2 d_{Z_2}}\cdot$$

$$\cdot(-1)^{\delta_1\sum_{i=1}^{m}\alpha_{Z_i} + \delta_2\sum_{i=p+1}^{m}\alpha_{Z_i}}\frac{1}{N}\cdot$$

$$\cdot\langle\psi|B'_{\gamma'|\delta'}\prod_{i=1}^{m} Z_i^{\delta_1}\prod_{i=p+1}^{m} Z_i^{\delta_2}\prod_i P_{\alpha_{X_i}}|\psi\rangle.$$

Note that

$$\bar{Z}' \in S,$$

$$Z_i \in S \quad \forall i \in \{2, ..., m\},$$

$$[Z_i, B'_{\gamma'|\delta'}] = 0 \quad \forall i \in \{2, ..., m\}.$$

Let us denote the corresponding eigenvalues of $Z_i$ as $(-1)^{\alpha^S_{Z_i}}$ for $i \in \{2, ..., m\}$, and let the eigenvalue of $Z'_{L_1}$ be $(-1)^{\alpha^S_{Z'_{L_1}}}$. Then we can write

$$\langle\psi'|B'_{\gamma'|\delta'}|\psi'\rangle =$$

$$= (-1)^{\gamma_1 d_{X_1} + (\gamma_1+\gamma_2)d_{X_2} + (\delta_1+\delta_2)d_{Z_1} + \delta_2 d_{Z_2}}\cdot$$

$$\cdot(-1)^{\delta_1\sum_{i=1}^{m}\alpha_{Z_i} + \delta_2\sum_{i=p+1}^{m}\alpha_{Z_i}}\cdot$$

$$\cdot(-1)^{\delta_1\sum_{i=2}^{m}\alpha^s_{Z_i} + \delta_2\sum_{i=p+1}^{m}\alpha^s_{Z_i}}\frac{1}{N}\cdot$$

$$\cdot\langle\psi|B'_{\gamma'|\delta'}Z_1^{\delta_1}\prod_i P_{\alpha_{X_i}}|\psi\rangle.$$

Considering 2 cases in a similar manner to Multi-cell movements case, we arrive at the desired relation.