Deriving Heisenberg's operator algebra from a classical model of stochastic mechanics

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A feature of quantum mechanics that distinguishes it from classical Newtonian mechanics is that momentum and position are described by non-commuting operators on a Hilbert space. It has long been known that certain diffusion theories provide stochastic models for Schrödinger's equation, together with a natural way to understand this non-commutative structure. This theory is revisited here. It suggests that the origins of quantum theory might be found in an algebraic extension of the geometry of space-time to complex numbers. It is a possible stepping stone to an emergent explanation for quantum mechanics and to a unification of classical and quantum physics.

I. INTRODUCTION[1]

Heisenberg introduced the concept of a time-dependent and non-commuting algebra for dynamic variables like position and momentum in his landmark 1925 paper [1]. Stochastic mechanics (SM) provides a stochastic interpretation for quantum mechanics [2–4]. It also provides a toolkit of rigorous mathematical theorems in stochastic processes for proving theorems about quantum mechanics.

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[1] PREPRINT V. 1 UNDER REVIEW FOR A WORLD SCIENTIFIC CENTENARY VOLUME

This is particularly relevant to the path-integral approach. Another interesting reason for studying it is that it can help in the search for an emergent explanation for quantum mechanics, as envisioned by Einstein [5] and many others [6–10]. Although stochastic mechanics has focused much more on the wave function formalism of Schrödinger than on the operator formalism of Heisenberg, there is an operator approach to generalized stochastic mechanics (GSM) that is closely related to the Heisenberg operator formalism in quantum mechanics [11, 12]. Whereas SM provides a diffusion model for quantum mechanics with a unique value for the diffusion constant, GSM provides many diffusion models that all have different diffusion constants [13, 14]. Jaekel and Pignon generalized this result even further by allowing the diffusion constant to depend on position and time [15]. GSM was based on the mathematical framework of SM [2–4, 16], and it depends on a nonlinear transformation of Schrödinger's equation and a modification of Nelson's force equation in stochastic mechanics [12, 13]. The empirical justification for a multitude of possible diffusion constants is based on the wave function collapse or reduction of the wave packet in quantum mechanics [17].

It is fair to say that GSM has been somewhat ignored by mathematicians in favor of a unique value for the diffusion constant. One reason for this is the action principle of Nelson [3, 18] that many have argued supports the claim of uniqueness of stochastic mechanics with the diffusion constant of $\hbar/2m$. But actually, it's possible to generalize the action principle to allow for a variable diffusion constant as is shown below.

Heisenberg introduced the wave-function collapse in 1927 [19]. The desire to have maximum mathematical rigor in stochastic mechanics made it difficult to incorporate it, so it was mostly ignored in the literature. The measurement problem is still a fundamental problem of quantum theory [20]. Petroni and Morato [21] remarked on the deficient lack of attention to this topic in the SM literature. The physical interpretation of SM or GSM can be taken to be similar to Bohmian mechanics, and good sources for this are [22–24]. Both theories are essentially nonlocal when multi-particle states are considered. Wave function collapse simply doesn't easily fit neatly into rigorous mathematical stochastic analysis. But nature and experiments require it. At the very core of quantum theory is this mysterious collapse, and it suggests that the diffusion parameter cannot be measured in an experiment. There have been many efforts to demystify it. Decoherence and spontaneous collapse are prime examples, but none of these efforts have achieved unanimous acceptance by physicists. Bohr's Copenhagen interpretation is still favored by most physicists, as is shown by many surveys.

In this paper the non-commutative approach is reviewed, and it is shown how the usual Heisenberg algebra can be constructed. The analytic continuation to complex values for the diffusion constant can be interpreted as a diffusion in complex space (or space and time in a relativistic framework). This adds to a growing list of reasons that we should take the arena of space-time to be a complex 4D manifold, where the real subspace plays a prominent role at the macroscopic level, but the complex embedding plays a large role when quantum mechanics is necessary.

Stochastic mechanics, like Bohmian mechanics, is a nonlocal theory because of the reduction of the wave packet. There are good arguments that quantum mechanics is inherently nonlocal, and this doesn't only apply to hidden variable interpretations [25–27]. Nelson eventually became very worried about this and recommended concentrating on field theory instead of wave function analysis of particle motion [28]. The nonlocality problems of stochastic mechanics are no worse than those faced by Bohmian mechanics, and since a fair number of physicists accept Bohmian mechanics as a viable interpretation, they would probably find stochastic mechanics equally acceptable. See, for example, [23, 29]. But there's no way around it, quantum theory is very paradoxical. One possible way out is if underlying the stochastic behavior is a hidden fully-deterministic theory. An example of this is 't Hooft's Planck scale cellular automata theory [30]. The stochastic motion of particles in stochastic mechanics might be caused by such a chaotic deterministic theory at the Planck scale that eliminates free-will in the analysis of Bell type experiments. This can avoid the nonlocality. Another possibility is that we are living in a complex space-time manifold that appears real to us because the events that we are aware of are happening on or near to a real subspace [31]. Generalized stochastic mechanics on complex manifolds provides a continuous path to the Heisenberg algebra by analytic continuation to imaginary diffusion constant.

In this paper, we shall mainly use the Markov transition function, the forward and backward differential equations for it, the Chapman-Kolmogorov equations, and the multiple-time densities of generalized Brownian motion. The mathematics needed is therefore familiar to most physicists, and is sufficient to do calcculations. We treat only the 1 dimensional case here for simplicity, but we use the ∇_x and $\triangle_x$ operator symbols to represent the first and second derivatives in 1D.

II. ON MEASURING OR PLACING AN EXPERIMENTAL BOUND ON THE DIFFUSION CONSTANT OR THE DRIFT TERM IN STOCHASTIC MECHANICS

This section is a response to an argument that the diffusion constant cannot be larger than $\hbar/2m$ ([3], §14, p. 66). One important conclusion derived from wave function collapse is that the diffusion constant of a quantum particle cannot be measured, at least given our current understanding of quantum mechanics. It is not an observable. The reason why is quite simple, but worth presenting here. Consider a non-relativistic particle without spin. The diffusion constant (in 1D for simplicity) is defined by the stochastic expectation

$$\nu \;=\; \frac{1}{2} \underset{\epsilon \to 0}{lim} \left(\frac{(E(x(t+\epsilon) - x(t))^2)}{\epsilon} \right) \tag{1}$$

Now let's ask how we can measure this. We obviously must measure the particle first at time t and then later at time $t+\epsilon$. We take that the particle is described by a wave function $\psi(x,t)$ undergoing unitary time evolution up until the first measurement is performed, and this can be described as a diffusion process up until that point defined by the standard form of stochastic differential equation as

$$dx = b(x,t,\nu)dt \;+\; dw(t,\nu) \tag{2}$$

where $b(\nu,x,t)$ is a sufficiently well-behaved function called the drift term, $w(t,\nu)$ is a Wiener function with diffusion constant ν, and the second term is called the diffusion term [32]. The drift term depends on the wave function, and it also depends on what value of ν we are considering. If we precisely measure the particle position at time t, then instantly the wave function collapses, and the drift term will become huge after this measurement because of the Heisenberg uncertainty principle. The collapsed ψ will be localized at a point x, but it will have a large spread in momentum, and so the drift term will dominate the diffusion term even during the short time delay ϵ to the next measurement. This situation persists no matter how small we make ϵ, and consequently, we can never measure the diffusion constant. This was pointed out, for example, in [15]. In [3] it was argued that for large values of ν, the diffusion term would dominate and could thereby be ruled out with a measurement, but this assumes that the b term doesn't grow with increasing values of ν, and this is exactly what happens in GSM, as the probability density in GSM is independent of ν. After measuring the position at time t, the wave function collapses and evolves to time t+ε through the Schrödinger equation, which is independent of υ. Therefore, the argument against very large values of ν does not apply to GSM. One might try to get around this situation with some sort of weak measurement [33] that might indirectly determine the diffusion constant without disturbing the wave function so much, but thus far there's no known way to do this. One can't even place a bound on the magnitude of ν. GSM is possible because of this inability. Nelson ([3], §, p.117) agrees that strictly speaking, the diffusion constant cannot be measured, but he suggests that a reasonable assumption is that the forward velocity b right after a position measurement should be no bigger than p/m, where p here denotes the uncertainty in the momentum of the particle after measurement as required by the Heisenberg uncertainty principle. This assumption is not satisfied in GSM, where b would also depend on the diffusion constant after a measurement because of (22), and for a large diffusion constant, it would be bigger than Nelson's suggested upper bound. So in this paper, we abandon the "usual assumptions" used to dismiss large diffusion constants as metaphysical. Thus, we are free to analytically continue the stochastic functions of GSM into the entire complex ν plane.

So one can construct a stochastic model for Schrödinger's equation with any positive diffusion constant whatsoever. Negative values of the diffusion constant simply correspond to a time reversal. If you use the SM value, $\nu = \hbar/2m$, you get SM. If you take the limit $\nu \to 0$ you get Bohmian mechanics, a result first shown by Shucker [34]. If you analytically continue ν to imaginary values, you get the Heisenberg algebra when $\nu = \pm i\hbar/2m$. This requires complex space-time which we discuss below. From a strictly empirical point of view all of these theories are equivalent. This non-uniqueness of stochastic mechanics reflects a kind of nonlinear gauge invariance of Schrödinger's equation, as proved in [13]. Modern physics is happy with gauge invariance, and so striving to fix an unmeasurable gauge with an arbitrary mathematical bias might not be the wisest strategy. If any value of the diffusion constant is preferred, it is a purely imaginary value of $\pm i\hbar/2m$, for with this value, the Heisenberg algebra is perfectly mirrored in GSM.

Similar arguments can be applied to the forward velocity b and the backward velocity b_*. They cannot be measured directly, either. It is remarkable how effectively quantum mechanics manages to cover its tracks in this way.

III. A SHORT REVIEW OF STOCHASTIC MECHANICS WITH ARBITRARY DIFFUSION CONSTANT

Most of this can be found in [32]. We have two stochastic differential equations: a forward and a time-reversed equation in 1D for simplicity, although the higher dimensions are straightforward.

$$dx(t,\nu) = b(x(t),t,\nu)dt + dw(t,\nu) \tag{3}$$

$$dx_*(t,\nu) = b_*(x_*(t),t,\nu)dt + dw_*(t,\nu) \tag{4}$$

The existence of this pair as both Markov processes is assured by a theorem by Doob (see [32], §13, p. 85). In Appendix A a list of formulas that are useful in stochastic mechanics are presented. The parameter ν is the time independent diffusion parameter defined by

$$\nu = \frac{1}{2}E\left((dw(t,\nu))^2\right)/dt \tag{5}$$

and $w(t,\nu)$ is a Wiener process. Although b and b_* depend on ν, we shall not include it in the argument list of these and other functions to simplify the notation.

Central to our discussion is the Markov transition density function. We follow the convention used in [11] such that the earlier time is to the right of the semicolon which is opposite to the convention in [3], so we have

$$P(x,t;y,s) = \lim_{dx\to 0}\frac{1}{dx}P(x(t)\in dx \mid x(s)=y), \quad t>s \tag{6}$$

The notation $P(A|B)$ means the probability of A conditioned by B. This transition function satisfies the Chapman-Kolmogorov equation (unless otherwise specified, the domain of integration is taken to be $-\infty$ to $+\infty$)

$$P(x,t;y,r) = \int P(x,t;z,s)P(z,s;y,r)dz,\ for\, times\, t > s > r \tag{7}$$

Continuity of paths leads to the limiting behavior

$$\lim_{t \downarrow s} P(x,t;y,s) \ = \ \delta(x-y) \tag{8}$$

where δ denotes the Dirac delta function-distribution. The transition function is normalized.

$$\int P(x,t;y,s)dx = 1 \tag{9}$$

The time reversed process has a different transition density function P_*, but it is simply related to P provided that the density function is never zero by the formula from ([3], eqn. 6.2, p 35)

$$P_*(y,t;x,s) = P(y,t;x,s)\frac{\rho(x,s)}{\rho(y,t)},\ t > s \tag{10}$$

which also satisfies a Chapman-Kolmogorov equation. We have for the probability density ρ:

$$\rho(x,t) = \int P(x,t;y,s)\rho(y,s)dy,\ t > s \tag{11}$$

$$\rho(y,s) = \int P_*(x,t;y,s)\rho(x,t)dx,\ t > s \tag{12}$$

In general this does not describe a stationary process, and therefore the time reversed process is different from the forward process, even though the Schrödinger equation is invariant under time reversal. The Markov transition function plays a role similar to the time evolution operator in quantum mechanics. We can calculate the probability densities for multiple times very easily by forming products of Markov transition functions ([3], eqn. 6.3, p 35). See also (A20).

It is customary to write the probability density as

$$\rho(x,t) = exp(2R(x,t)) \tag{13}$$

We define S_N up to a function independent of x by

$$2\nu\frac{\partial}{\partial x}S_N(x,t,\nu) \equiv \frac{b(x,t,\nu)+b_*(x,t,\nu)}{2} \tag{14}$$

and similarly ([32], §15)

$$2\nu\frac{\partial}{\partial x}R(x,t,\nu) \equiv \frac{b(x,t,\nu)-b_*(x,t,\nu)}{2} \tag{15}$$

and it follows that (in 3D this assumes that $curl(b) = 0$)

$$b(x,t) = 2\nu\nabla\left(R+S_N\right) \tag{16}$$

$$b_*(x,t) = 2\nu\nabla\left(-R+S_N\right) \tag{17}$$

The wave function will be written as

$$\psi(x,t) = exp(R(x,t)+iS_Q(x,t)) \tag{18}$$

The wave function will be independent ν in GSM.

The following two equations are equivalent, as shown for real-valued constant z in [13]

$$\left(-\frac{\hbar^2}{2m}\triangle + V\right) exp(R + iS_Q) = i\hbar\frac{\partial}{\partial t}exp(R + iS_Q) \tag{19}$$

$$\left(-\frac{(z\hbar)^2}{2m}\triangle + \left(V + \frac{\hbar^2}{2m}\left(z^2-1\right)\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right)\right) exp(R{+}iS_Q/z) = i\left(z\hbar\right)\frac{\partial}{\partial t}exp(R{+}iS_Q/z) \tag{20}$$

and where $\rho = exp(2R)$. This equivalence is also true for all complex values of z by simple analytic continuation from the real values. The extra nonlinear term in the potential in (20) is just Bohm's quantum potential up to a multiplicative constant. The second equation looks like a modified Schrödinger equation with the replacements $\hbar \to z\hbar$ and $V \to \left(V + \frac{\hbar^2}{2m}\left(z^2-1\right)\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right)$, so we can immediately, model this with a stochastic process of the form (2) with

$$\nu = z\frac{\hbar}{2m} \tag{21}$$

$$b(x,t,\nu) = 2\nu\nabla\left(R(x,t) + S_Q(x,t)/z\right) = \nabla\left(2\nu R(x,t) + \frac{\hbar}{m}S_Q(x,t)\right) \tag{22}$$

This then defines the generalized stochastic process for any value of the diffusion constant. The generalized "equation of motion" is given in (31) along with (32).

IV. AN ACTION PRINCIPAL COMPATIBLE WITH ARBITRARY DIFFUSION CONSTANT

Here we utilize the theorem and technique of Yasue [35], but with a modified Lagrangian. Given an action of the form

$$I = E\left[\int_a^b L(x(t), Dx(t), D_*x(t))dt\right] \tag{23}$$

where the operators D and D_* are defined in (A10) and (A11). He obtained the extended Euler-Lagrange equation

$$D\left(\frac{\partial L}{\partial D_*x(t)}\right) + D_*\left(\frac{\partial L}{\partial Dx(t)}\right) - \frac{\partial L}{\partial x(t)} = 0 \tag{24}$$

(see also Zambrini, equation 25 in [36]). The operators D and D_* are the forward and backward time derivatives of [32]. Yasue showed that the following time reversible Lagrangian

$$L_N = \frac{m}{2}\left(\frac{(Dx(t))^2 + (D_*x(t))^2}{2}\right) - V(x) \tag{25}$$

leads to the Euler-Lagrange equation if $Dx(t)$ and $D_*x(t)$ are varied independently

$$m\frac{DD_* + D_*D}{2}x(t) = -\frac{\partial V(x(t))}{\partial x(t)} \tag{26}$$

and this is the same equation of motion that Nelson found which gives the Schrödinger equation provided the diffusion constant is given by

$$\nu_N = \frac{\hbar}{2m} \tag{27}$$

and where we have chosen units so that $\hbar = 1$. The following acceleration is called the mean acceleration

$$a_N = \frac{DD_* + D_* D}{2} x(t) \tag{28}$$

Now consider modifying the Lagrangian to read

$$L_\nu = A\left(L_N + V(x)\right) + B\left(Dx(t)\right)\left(D_* x(t)\right) - V(x) \tag{29}$$

where A and B are constants. Note that this modified Lagrangian is still invariant under time reversal. The Euler-Lagrange equations for this Lagrangian are

$$A\left(m\frac{DD_* + D_* D}{2} x(t)\right) + B\left(m\frac{DD + D_* D_*}{2} x(t)\right) = -\frac{\partial V(x(t))}{\partial x(t)} \tag{30}$$

It was shown in [13] that the following "equation of motion" also leads to Schrödinger's equation

$$\left(m\frac{DD_* + D_* D}{2} x(t)\right) + m\frac{\beta}{8}\left(D - D_*\right)^2 x(t) = -\frac{\partial V(x(t))}{\partial x(t)} \tag{31}$$

provided that

$$\nu = \frac{\hbar}{2m}\frac{1}{\sqrt{1 - \beta/2}} \tag{32}$$

or equivalently

$$\beta = 2\left(1 - \left(\frac{\hbar}{2m\nu}\right)^2\right) \tag{33}$$

and therefore, we get Schrödinger's equation provided that

$$A = 1 - \frac{\beta}{4},\ and\ B = \frac{\beta}{4} \tag{34}$$

and therefore we can have an action principal for any value of the diffusion constant. Nelson has argued in [3] that although we cannot physically measure the diffusion constant according to existing quantum theory, maybe someday a means will be found to get around this, which would be a violation of our current understanding of what is possible. The action chosen in [3] for simplicity is divergent, but when the infinite part is subtracted off, one gets a remainder that gives a unique diffusion constant that is Nelson's original value of $\hbar/2m$. It is argued that the infinite part of the action depends only on the diffusion constant, and therefore does not enter into the variation. Since the diffusion constant cannot be measured, this argument is metaphysical, but it could nevertheless be the way that nature works. Jaekel has also given a rendition of this result [18]. For the purpose of tying together the Heisenberg operator approach to stochastic mechanics, we consider the variable diffusion constant general theory of GSM in this paper.

V. DERIVING THE HEISENBERG OPERATOR FORMALISM

The simplest way to illustrate or derive the Heisenberg operator formalism in the framework of stochastic mechanics is to use diffusion in a complex space or space-time. Here we start with real ν and use results from [11, 12], but we alert the reader that the definition of ν in these two papers differ by a factor of 2. Here in (5) we use the definition in [11] which coincides with ([32], §13). The basic idea of the derivation of non-commuting operators in stochastic mechanics is that the order of operations at equal times is simply the infinitesimal time-ordering of certain expectation values in the limit where the difference between the times tends to zero. Consider the random variable $x(t)$ in SM setting. The time derivative does not exist for these functions as is well known. However, if we consider the two-point function $E(x(t)x(s))$,then this function is typically differentiable with respect to either time if the forward velocity $b(x,t)$ is sufficiently well-behaved which we assume. Now to get an idea of where the noncommutativity comes from in this framework, consider the following limiting procedure with $t > s$

$$Commutator = \lim_{t \uparrow s} \left[\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right) E(x(t)x(s)) \right] \tag{35}$$

If the sample paths were smooth, this difference would vanish. It is a difference between the two time orderings of a velocity times a position. So It's like a commutator. It can be evaluated by means of the forward and backward derivatives

$$Commutator = E\left(\left(b_*(x,s) - b(x,s) \right) x(s) \right) \tag{36}$$

Now we use (A6) to obtain by using integration by parts

$$Commutator = -\int 2\nu \frac{grad\, \rho(x,t)}{\rho(x,t)} x\rho(x,t)dx = 2\nu \tag{37}$$

This simple result suggests that we can use the microscopic time ordering of expectations to define a non-commuting operator algebra. If we consider Nelson's value of $\nu = \hbar/2m$, and that the momentum is m times the velocity, it looks like the commutator of p and x is simply $\hbar$ up to a sign at least, since we could have chosen the opposite ordering. Thus the time-ordering of expectations of this type induce a non commutative structure as was described in [11, 12]. In order to get the usual Heisenberg commutation relations, we must analytically continue to imaginary values of ν and set

$$\nu = -i\frac{\hbar}{2m} \tag{38}$$

In order to develop the Heisenberg algebra further, we follow the approach in [11]. In the Heisenberg picture the coordinates and momenta operators are time-dependent, but the wave function is fixed, as opposed to the Schrödinger picture where the wave function is time-dependent and the coordinates and momenta operators are time independent. Towards this end we introduce a base time s which will be fixed, and we derive time-dependent operators from this point using the Markov transition functions. first let's introduce a Hilbert space $\mathcal{H}_s$ with the following inner product:

$$(f, g) = E\left(f^*(x(s), s)g(x(s), s)\right) = \int \rho(x, s)f^*(x, s)g(x, s)dx \tag{39}$$

The functions f and g that we consider are assumed to be differentiable to all orders in both independent variables. Now let us define operators on these functions. The operator for x we denote by simple multiplication, and since the base time s is fixed, we drop it from the argument of $\hat{x}$

$$\hat{x}f(x,s) \equiv xf(x,s) \tag{40}$$

Next we define an operator for $\dot{x}$ that acts on elements of the Hilbert space $\mathcal{H}_s$ and is defined as the following limit of a derivative of a conditional expectation

$$\hat{\dot{x}}f(x,s) \equiv \lim_{t\downarrow u} \lim_{u\downarrow s} \frac{\partial}{\partial u} E\left(x(u)f(x(t),t)|x(s)=x\right),\ t>u>s,\ f\in\mathcal{H}_s \tag{41}$$

Now it's convenient to write this expectation in terms of the Markov transition function densities because we can then use the forward and backward equations to simplify it. Using (A20) we can write the probability density for three times, where $t_1 < t_2 < t_3$, as

$$\rho(x_3,t_3;x_2,t_2;x_1,t_1) = P(x_3,t_3;x_2,t_2)P(x_2,t_2;x_1,t_1)\rho(x_1,t_1) \tag{42}$$

and this approach can be extended to any number of different times. To get the conditional expectation we simply drop the term $\rho(x_1,t_1)$. So we rewrite (41) as

$$\hat{\dot{x}}f(x,s) \equiv \lim_{t\downarrow u} \lim_{u\downarrow s} \frac{\partial}{\partial u} \int f(x_t,t)x_u P(x_t,t;x_u,u)P(x_u,u;x,s)dx_t dx_u,\ t>u>s \tag{43}$$

We assume that we can bring the $\frac{\partial}{\partial u}$ inside the integral.

$$\hat{\dot{x}}f(x,s) = \lim_{t\downarrow u} \lim_{u\downarrow s} \int f(x_t,t)\left[\int x_u \frac{\partial}{\partial u}\left(P(x_t,t;x_u,u)P(x_u,u;x,s)\right)dx_u\right]dx_t,\ t>u>s \tag{44}$$

$$\hat{\dot{x}}f(x,s) = \lim_{t\downarrow u}\ \lim_{u\downarrow s}\int f(x_t,t)\times$$
$$\times\left[\int x_u\left(\frac{\partial P(x_t,t;x_u,u)}{\partial u}P(x_u,u;x,s)+P(x_t,t;x_u,u)\frac{P(x_u,u;x,s)}{\partial u}\right)dx_u\right]dx_t,\ t>u>s \tag{45}$$

Using the backward equation of Kolmogorov A18 (using ∇ and $\triangle$ symbols for derivatives in 1D) we ha

$$\frac{\partial P(x_t,t;x_u,u)}{\partial u} = -\left(b(x_u,t)\cdot\nabla_{x_u}+\nu\triangle_{x_u}\right)P(x_t,t;x_u,u),\ t>u \tag{46}$$

Using the forward equation of Kolmogorov A17 we have:

$$\frac{P(x_u,u;x_s,s)}{\partial u} = -\left(\nabla_{x_u}\cdot b(x_u,u)+b(x_u,u)\cdot\nabla_{x_u}-\nu\triangle_{x_u}\right)P(x_u,u;x_s,s),\ u>s \tag{47}$$

and integrating this by parts, results in :

$$\int x_u\left(\frac{\partial P(x_t,t;x_u,u)}{\partial u}P(x_u,u;x,s)+P(x_t,t;x_u,u)\frac{P(x_u,u;x,s)}{\partial u}\right)dx_u$$
$$=\int P(x_t,t;x_u,u)\left(-x_u\left(\nabla_{x_u}\cdot b(x_u,u)+b(x_u,u)\cdot\nabla_{x_u}-\nu\triangle_{x_u}\right)\right)P(x_u,u;x,s)dx_u+$$
$$\int P(x_t,t;x_u,u)\left(\left(\nabla_{x_u}\cdot b(x_u,u)+b(x_u,u)\cdot\nabla_{x_u}-\nu\triangle_{x_u}\right)x_u\right)P(x_u,u;x,s)dx_u \tag{48}$$

This can be expressed more compactly in terms of a commutator

$$=\int P(x_t,t;x_u,u)\left[\left(\nabla_{x_u}\cdot b(x_u,u)+b(x_u,u)\cdot\nabla_{x_u}-\nu\triangle_{x_u}\right),x_u\right]P(x_u,u;x,s)dx_u \tag{49}$$

but $\left[\left(\nabla_{x_u}\cdot b(x_u,u)\right),x_u\right]=0$ and so

$$=\int P(x_t,t;x_u,u)\left[\left(b(x_u,u)\cdot\nabla_{x_u}-\nu\triangle_{x_u}\right),x_u\right]P(x_u,u;x,s)dx_u \tag{50}$$

and we have $\left[\left(b(x_u,u)\cdot\nabla_{x_u}-\nu\triangle_{x_u}\right),x_u\right]=\left(b(x_u,u)-2\nu\nabla_{x_u}\right)$ (51)

Now we can take the limits

$$\lim_{t\downarrow u}\ \lim_{u\downarrow s} P(x_t,t;x_u,u) = \delta(x_t - x_u) \tag{52}$$

$$\lim_{t\downarrow u}\ \lim_{u\downarrow s} P(x_u,u;x_s,s) = \delta(x_u - x_s) \tag{53}$$

$$\hat{\dot{x}} f(x,s) = \lim_{t\downarrow u}\ \lim_{u\downarrow s} \int f(x_t,t)\delta(x_t - x_u)\left(b(x_u,u) - 2\nu\nabla_{x_u}\right)\delta(x_u - x_s)dx_t dx_u,\ t > u > s \tag{54}$$

now we integrate over x_t

$$\hat{\dot{x}} f(x,s) = \lim_{t\downarrow u}\ \lim_{u\downarrow s} \int f(x_u,t)\left(b(x_u,u) - 2\nu\nabla_{x_u}\right)\delta(x_u - x_s)dx_u,\ t > u > s \tag{55}$$

now we integrate by parts once again

$$\hat{\dot{x}} f(x,s) = \lim_{t\downarrow u}\ \lim_{u\downarrow s} \int \delta(x_u - x_s)\left(b(x_u,u) + 2\nu\nabla_{x_u}\right)f(x_u,t)dx_u,\ t > u > s \tag{56}$$

$$\hat{\dot{x}} f(x,s) = \left(b(x,s) + 2\nu\nabla_x\right)f(x,s) \tag{57}$$

$$\hat{\dot{x}} = \left(b(x,s) + 2\nu\nabla_x\right) \tag{58}$$

It follows then that a commutator relation exists

$$\left[\hat{\dot{x}}(s), \hat{x}(s)\right] = 2\upsilon \tag{59}$$

and this is very similar to the Heisenberg commutation rule, except that its value is real and not imaginary. Since $\hat{x}$ is just a simple multiplication operator, we will drop the hat over this symbol for simplicity. To proceed further, we note that we can write

$$\hat{\dot{x}} = [(b(x,s) + \nu\nabla_x) \cdot \nabla_x, x] \tag{60}$$

For a smooth ordered operator $\hat{j}(x,s)$ let us define an operator corresponding to a time derivative by a commutation rule derived from this as

$$\hat{\dot{j}}(x,s) = \left[(b(x,s) + \nu\nabla_x) \cdot \nabla_x, \hat{j}(x,s)\right] + \frac{\partial \hat{j}(x,s)}{\partial s} \tag{61}$$

Next we define recursively derived higher order derivative operators by the formula

$$\hat{j}(x,s,n+1) = \left[(b(x,s) + \nu\nabla_x) \cdot \nabla_x, \hat{j}(x,s,n)\right] + \frac{\partial \hat{j}(x,s,n)}{\partial s} \tag{62}$$

$$\hat{j}(x,s,0) = x,\ \hat{j}(x,s,1) = (b(x,s) + 2\nu\nabla_x) \tag{63}$$

For the acceleration defined this way we find

$$\hat{\ddot{x}} = \hat{j}(x,s,2) = [(b(x,s) + \nu\nabla_x) \cdot \nabla_x, (b(x,s) + 2\nu\nabla_x)] + \frac{\partial\,(b(x,s) + 2\nu\nabla_x)}{\partial s} \tag{64}$$

$$\hat{\ddot{x}} = [(b(x,s) \cdot \nabla_x + \nu\triangle_x), (b(x,s) + 2\nu\nabla_x)] + \frac{\partial b(x,s)}{\partial s} \tag{65}$$

$$\hat{\ddot{x}} = [b(x,s) \cdot \nabla_x, b(x,s)] + [b(x,s) \cdot \nabla_x, 2\nu\nabla_x] + [\nu\triangle_x, b(x,s)] + \frac{\partial b(x,s)}{\partial s} \tag{66}$$

$$\hat{\ddot{x}} = b(x,s)\cdot\nabla_x b(x,s) - \cancel{2\nu\left(\nabla_x b(x,s)\right)}\nabla_x + \nu\left(\left(\triangle_x b(x,s)\right) + \cancel{2\left(\nabla_x b(x,s)\right)}\nabla_x\right) + \frac{\partial b(x,s)}{\partial s} \tag{67}$$

This simplifies to

$$\hat{\ddot{x}} = \frac{\partial b(x,s)}{\partial s} + \nu\triangle_x b(x,s) + \frac{1}{2}\nabla_x\left(b^2(x,s)\right) \tag{68}$$

This can be conveniently rewritten in the following form [11] as

$$\hat{\ddot{x}} = \nabla_x\left(exp(-R-S_N)\left(2\nu\frac{\partial}{\partial t} + 2\nu^2\triangle_x\right)exp(R+S_N)\right), b = 2\nu\nabla(R+S_N) \tag{69}$$

Lets compare (68) with the swapped version by interchanging b and b_* which is

$$\hat{\ddot{x}}_* = \frac{\partial b_*(x,s)}{\partial s} + \nu\triangle_x b_*(x,s) + \frac{1}{2}\nabla_x\left(b_*^2(x,s)\right) \tag{70}$$

Now consider the difference

$$\frac{\hat{\ddot{x}} - \hat{\ddot{x}}_*}{2} = \frac{\partial u(x,s)}{\partial s} + \nu\triangle_x u(x,s) + \nabla_x\left(u(x,s)v(x,s)\right) \tag{71}$$

Comparing this with the kinematic equation 13.5 in [32] which is

$$\frac{\partial u(x,s)}{\partial s} + \nu\triangle_x v(x,s) + \nabla_x\left(u(x,s)v(x,s)\right) = 0 \tag{72}$$

We see that

$$\frac{\hat{\ddot{x}} - \hat{\ddot{x}}_*}{2} = \nu\triangle_x u(x,s) - \nu\triangle_x v(x,s) \tag{73}$$

Kinematic here means that this result does not depend on the particular equation of motion chosen. Now we can write

$$\ddot{\hat{x}} = \frac{\left(\ddot{\hat{x}} + \ddot{\hat{x}}_*\right) + \left(\ddot{\hat{x}} - \ddot{\hat{x}}_*\right)}{2} \tag{74}$$

$$= \left(\frac{\partial v(x,s)}{\partial s} + \cancel{\nu \triangle_x v(x,s)} + \frac{1}{4}\nabla_x \left(b^2(x,s) + b_*^2(x,s)\right)\right) + \left(\nu \triangle_x u(x,s) - \cancel{\nu \triangle_x v(x,s)}\right) \tag{75}$$

$$= \frac{\partial v(x,s)}{\partial s} + \frac{1}{4}\nabla_x \left(b^2(x,s) + b_*^2(x,s)\right) + \nu \triangle_x u(x,s) \tag{76}$$

Let's compare this with Nelson's mean acceleration in equation ([2] eq. 13.6) in

$$a_N = \frac{\partial v(x,s)}{\partial s} - u \cdot \nabla_x u + v \cdot \nabla_x v - \nu \triangle_x u \tag{77}$$

$$a_N = \frac{\partial v(x,s)}{\partial s} + \frac{1}{2}\nabla_x \left(v(x,s)^2 - u(x,s)^2\right) - \nu \triangle_x u(x,s) \tag{78}$$

Let's now find the difference

$$\ddot{\hat{x}} - a_N = \left(\frac{1}{4}\nabla_x \left(b^2 + b_*^2\right) + \nu \triangle_x u(x,s)\right) - \left(\frac{1}{2}\nabla_x \left(v(x,s)^2 - u(x,s)^2\right) - \nu \triangle_x u(x,s)\right) \tag{79}$$

$$\ddot{\hat{x}} - a_N = 2\nu \triangle_x u(x,s) + \frac{1}{4}\nabla_x \left(b(x,s)^2 + b_*(x,s)^2 - 2b(x,s)b_*(x,s)\right) \tag{80}$$

$$\ddot{\hat{x}} - a_N = 2\nu \triangle_x u(x,s) + \nabla_x \left(u(x,s)^2\right) \tag{81}$$

But $u = 2\nu\nabla_x R$, and therefore

$$\ddot{\hat{x}} - a_N = 2\nu \triangle_x \left(2\nu \nabla_x R\right) + \nabla_x \left(\left(2\nu \nabla_x R\right)^2\right) \tag{82}$$

$$\hat{\ddot{x}} - a_N = (2\nu)^2 \nabla_x \left(\triangle_x R + (\nabla_x R)^2 \right) \tag{83}$$

$$\hat{\ddot{x}} - a_N = (2\nu)^2 \nabla_x \left(\frac{\triangle_x \sqrt{\rho}}{\sqrt{\rho}} \right) \tag{84}$$

It follows from the GSM equation of motion (31) that

$$m a_N = -\nabla_x V(x,t) - m\frac{\beta}{8} (D - D_*)^2 x(t) \tag{85}$$

$$m\frac{\beta}{8} (D - D_*)^2 x(t) = \nabla_x \left(m\beta\nu^2 \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{86}$$

$$m a_N = -\nabla_x \left(V(x,t) + m\beta\nu^2 \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{87}$$

$$m a_N = -\nabla_x \left(V(x,t) + m \left(2 \left(\nu^2 - \left(\frac{\hbar}{2m} \right)^2 \right) \right) \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{88}$$

Therefore

$$\hat{\ddot{x}} = a_N + (2\nu)^2 \nabla_x \left(\frac{\triangle_x \sqrt{\rho}}{\sqrt{\rho}} \right) \tag{89}$$

$$m\hat{\ddot{x}} = -\nabla_x \left(V(x,t) + m \left(2 \left(\nu^2 - \left(\frac{\hbar}{2m} \right)^2 \right) \right) \frac{\triangle\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) + m (2\nu)^2 \nabla_x \left(\frac{\triangle_x \sqrt{\rho}}{\sqrt{\rho}} \right) \tag{90}$$

$$m\hat{\ddot{x}} = -\nabla_x \left(V(x,t) + m \left(2 \left(-\nu^2 - \left(\frac{\hbar}{2m} \right)^2 \right) \right) \frac{\triangle \sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \right) \tag{91}$$

$$m\hat{\ddot{x}} = -\nabla_x V(x,t) + \nabla_x \left(2m\nu^2 + \frac{\hbar^2}{2m} \right) \frac{\triangle_x \sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \tag{92}$$

This is the same formula that was found in ([11], eq. 35). Here we have presented the tedious derivation which was ommitted there. We see that the simplest possibility occurs when $\nu = \pm i\hbar/2m$, for then we get the Heisenberg formula of quantum mechanics

$$m\hat{\ddot{x}} = -\nabla_x V(x,t),\ if\ \nu = \pm i\hbar/2m \tag{93}$$

In general we have

$$m\hat{\ddot{x}} = -\nabla_x U(x,t) \tag{94}$$

for some function $U(x,t)$, and in particular, for the GSM equations of motion, we further have

$$U(x,t) = V(x,t) - \left(2m\nu^2 + \frac{\hbar^2}{2m} \right) \frac{\triangle \sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}} \tag{95}$$

The zero diffusion limit of the stochastic process gives Bohmian mechanics [34].

$$\lim_{\nu \to 0} m\hat{\ddot{x}} = -\frac{\partial \left(V - \frac{\hbar^2}{2m} \frac{\triangle \sqrt{\rho}}{\rho} \right)}{\partial x} \tag{96}$$

Here V is the classical potential, and the extra term is called the quantum potential of Bohm [37]. The theoretical description of the Markov process is concisely summarized in the following two equations which were called "Markov wave equations" in [12].

$$\left[m\frac{(2\nu)^2}{2}\triangle_x + W\right] e^{R\pm S_N} = \mp m2\nu\frac{\partial}{\partial t}e^{R\pm S_N} \tag{97}$$

These are true for any given function $W(x,t)$, and not just for the GSM equations. These equations were also used in [3, 16], and were almost certainly found earlier than [12], but I don't know who found them first. Notice that $\hat{\dot{x}}$ in (58) can be put into canonical form by a similarity transformation which does not affect the commutation rules

$$\hat{\dot{x}}_{canonical} = e^{-R-S_N}\hat{\dot{x}}e^{R+S_N} = 2\nu\nabla_x \tag{98}$$

The operator $\hat{x}$ doesn't change under this transformation.

$$\hat{x}_{canonical} = \hat{x} \tag{99}$$

So we can have the exact Heisenberg momentum operator at an imaginary value for the diffusion constant

$$\hat{P}_{Heisenberg} = m\hat{\dot{x}}_{canonical} = -i\hbar\nabla_x,\ if\ \nu = -i\hbar/2m \tag{100}$$

$$\left[\hat{x}, \hat{P}_{Heisenberg}\right] = i\hbar \tag{101}$$

We can now define a Hamiltonian operator by

$$\hat{H} = \left[m\frac{(2\nu)^2}{2}\triangle + U \right] = \left[m\frac{(2\nu)^2}{2}\left(\hat{\dot{x}}_{canonical}\right)^2 + U \right] \tag{102}$$

$$\hat{\dot{x}}_{canonical} = \frac{1}{2m\nu}\left[H, \hat{x}\right] = 2\nu\nabla_x \tag{103}$$

$$\hat{\ddot{x}}_{canonical} = \frac{1}{2m\nu}\left[H, \hat{\dot{x}}\right] = -\frac{1}{m}\nabla_x U \tag{104}$$

Comparing this with (94) we find

$$\hat{\ddot{x}} = \hat{\ddot{x}}_{canonical} \tag{105}$$

We can construct higher derivatives of order n using the recursive formula

$$\hat{x}_n = \frac{1}{2m\nu}\left[H, \hat{x}_{n-1}\right] + \frac{\partial \hat{x}_{n-1}}{\partial t},\ \hat{x}_0 = \hat{x} \tag{106}$$

and in this way we can build up a Taylor's expansion for propagating in time, very similar to the time evolution operator in quantum mechanics. At the special imaginary value of $\nu = -i\hbar/2m$ this time evolution becomes exactly the usual time evolution operator. For other values, so long as the equation of motion is that of GSM, the probability density at different times propagated in this way is always the same as Schrödinger's equation gives, provided we calculate the density only on the real subspace of complex spacetime using Born's rule there.

VI. THE CASE FOR COMPLEX SPACE-TIME

We have seen that generalized stochastic mechanics takes a particularly simple form for a diffusion constant which is purely imaginary with a value $\nu = \pm i\hbar/2m$. In this case the commutation relations also become exactly the Heisenberg rules. Such a diffusion constant would require the particle's motion to be in a complex space and because of space-time mixing in special relativity, in a complex space-time. If there are any values of the diffusion constant which are uniquely associated with quantum mechanics it is these two imaginary ones. But if the diffusion constant is imaginary, then the position of the particle will take on complex values. There is a history of suggestions that the space-time manifold might be embedded in a complex one. Perhaps the most interesting suggestion along these lines was made by Einstein. He believed that quantum theory might be derivable from classical field theory [5, 38, 39]. This is what he meant by unified field theory, and not the modern interpretation of the expression. He explored the complexification of space-time in this effort [40–42]. He used a Hermitian metric so that the metric length between points would be real-valued. Many others have pursued this line of research [43–57] An entirely different application of complex space-time comes from a fascinating method of solving difficult problems in general relativity involving analytical continuation of electromagnetic charges to complex-valued coordinates [31, 57–59]. In quantum mechanics there have been a number of applications as well [31, 53, 60–69]

David Bohm introduced the concept of the Implicate Order to physics [70]. A mathematically tangible example was proposed based on higher order algebras by Frescura and Hiley [71]. The simplest algebraic extension of space-time is to replace the algebra of real numbers with that of complex numbers for the coordinates. The idea of the implicate order is that behind our classical perception of reality lies a hidden realm which interconnects non-locally events in our world. This is a manifestation of a web of quantum entanglement that permeates our universe. Freshly minted physicists tend to find the concepts of wholeness and implicate order to be too metaphysical to be of interest, but as one matures in the study of quantum foundations they tend to increase in perceived importance, as the other alternatives like the Many Worlds or 't Hooft determinism seem equally strange. In a complex space-time interpretation there are many Riemann sheets for the electromagnetic fields and for the particle motions. This Riemann sheet web embedded in a complex space-time manifold is what we would compare to Bohm's implicate order. This comparison resonates with recent ideas in quantum gravity interpreting quantum entanglement in terms of Einstein-Rosen bridges [72].

There is literature from stochastic mechanics extending it to complex space and space-time. These techniques give precise mathematical meaning to an imaginary value for the diffusion constant. See for example Wang [73], Rosenbrock [74–77], and Kuipers [78]. These papers all provide a framework for analytically continuing the diffusion constant to complex space and time. Therefore it seems that the combination of Heisenberg algebra with stochastic mechanics strongly favors a universe which is embedded in a complex manifold. It therefore also suggests a possible connection between stochastic mechanics and the geometric Langlands program as applied to physics [79]. The paper by Kuipers also considers relativistic wave equations,

VII. A POSSIBLE CONNECTION TO ADLER'S TRACE DYNAMICS

A preliminary exploration of a possible connection between stochastic mechanics and Adler's theory of emergent quantum mechanics based on trace dynamics [9] was made in [80]. It was found that the non-commutative structure of stochastic mechanics might be used to provide an explanation for trace dynamics. However, it was also found that the result tended to lead to the thermal diffusion equation rather than the Schrödinger equation unless a nonlinear potential was added without any justification. However, only real-valued diffusion constants were considered in that work. If one revisits this mathematics with now an imaginary diffusion constant, it seems likely that the Schrödinger equation will arise in thermal equilibrium in the context of suitably defined trace dynamics. This is still another reason to consider complex space-time to be ontologically real.

VIII. CONCLUSION

Stochastic mechanics provides a thorough derivation of the non-commutative algebra of Heisenberg. The fact that an imaginary diffusion constant is preferred adds theoretical evidence that complex space-time is related to a deeper understanding of quantum mechanics. It's amazing that Einstein not only declared the need for a deeper theory but also that some of his last papers dealt with complex space-time pointing the way. Not only that, but he also invented the theory of Brownian motion, which is the basis of stochastic mechanics, and he made critical insights into quantum theory. It's also remarkable that Heisenberg's non-commuting operators should point to the same conclusion, and that his postulate of wave function collapse plays a key role in it. Nelson was perhaps correct that there is a preferred value for the diffusion constant, but it might not be $\hbar/2m$, but rather it could well be $\pm i\hbar/2m$. This leads to the proposition that we are possibly living in a complex space-time, but our normal sense perception has evolved to be aware of only the real subspace.

Appendix A: Compendium of useful formulae for stochastic mechanics

(We use the notation of [32], for the most part, in this appendix)

Fokker-Planck equation or Kolmogorov forward eqn.: $$\frac{\partial \rho}{\partial t} = -div(b\rho) + \nu \triangle \rho \tag{A1}$$

Schrödinger wave function: $$\psi = e^{R+iS_Q} \tag{A2}$$

Probabilty density: $$\rho = \psi^* \psi = e^{2R} \tag{A3}$$

Fokker-Planck equation time reversed or Kolmogorov backward eqn.: $$\frac{\partial \rho}{\partial t} = -div(b_* \rho) - \nu \triangle \rho \tag{A4}$$

Backward velocity: $$b_* = b - 2\nu(grad\, \rho)/\rho \tag{A5}$$

Osmotic velocity: $$u = \frac{b - b_*}{2} = \nu \frac{grad\, \rho}{\rho} = 2\nu grad(R) \tag{A6}$$

Current velocity: $$\mathsf{v} = \frac{b + b_*}{2} = 2\nu grad(S_N) \tag{A7}$$

Equation of continuity: $$\frac{\partial \rho}{\partial t} = -div(\mathsf{v}\rho) \tag{A8}$$

Time deriviative of osmotic velocity: $$\frac{\partial \mathsf{u}}{\partial \mathsf{t}} = -\nu grad\, div\, \mathsf{v} - grad\, \mathsf{v} \cdot u \tag{A9}$$

Forward time derivative formula: $$Df(x(t),t) = \left(\frac{\partial}{\partial t} + b \cdot \nabla + \nu \triangle \right) f(x(t),t) \tag{A10}$$

Backward time derivative formula: $D_* f(x(t),t) = \left(\frac{\partial}{\partial t} + b_* \cdot \nabla - \nu \triangle\right) f(x(t),t)$ (A11)

Mean acceleration of Nelson: $\mathsf{a}(t) = \frac{DD_* + D_* D}{2} x(t)$ (A12)

Markov transition function: $P(x,t;y,s) = \lim_{d^3x \to 0} \frac{1}{d^3x} P(x(t) \in d^3x \mid x(s) = y), \quad t > s$ (A13)

Chapman-Kolmogorov equation: $P(x,t;y,r) = \int P(x,t;z,s) P(z,s;y,r) d^3z, \; for\, times\, t > s > r$ (A14)

Continuity of paths: $\lim_{t \downarrow s} P(x,t;y,s) \;=\; \delta^3(x-y)$ (A15)

Time evolution of the density function: $\rho(x,t) = \int P(x,t;z,s)\rho(z,s) d^3z, \; t > s$ (A16)

Forward equation for Markov transition (§3, [81]):

$$\left[\frac{\partial}{\partial t} + \nabla_x \cdot b(x,t) + b(x,t) \cdot \nabla_x - \nu \triangle_x\right] P(x,t;y,s) = 0, \; t > s \qquad \text{(A17)}$$

Backward equation for Markov transition (§3, [81]):

$$\left[\frac{\partial}{\partial s} + b(y,t) \cdot \nabla_y + \nu \triangle_y\right] P(x,t;y,s) = 0, \; t > s \qquad \text{(A18)}$$

Integration by parts when justified: $\int_{-\infty}^{\infty} E(Df(x(t),t) g(x(t),t)) = -\int_{-\infty}^{\infty} E(f(x(t),t) D_* g(x(t),t))$ (A19)

Expectation value of a function of multiple positions at different times:

$$E(F(x(t_n),...,x(t_0))) = \int F(x_n,...,x_0) P(x_n,t_n;x_{n-1},t_{n-1}) \cdots P(x_1,t_1;x_0,t_0) \rho(x_0) dx_0 ... dx_n, \; t_0 \leq$$

(A20)

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