Staggered School Schedules for the morning commute problem - An MFD-based Optimization Approach

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Abstract— **This paper deals with the morning commute problem when two classes of commuters co-exist in an urban transportation network. While each school starts at the same time, traffic flow enters the network simultaneously, leading to the formation of a peak demand value that the network cannot fully accommodate. The task becomes more challenging when commuters head to their workplace after reaching their respective school. To tackle this issue, we propose the School Demand Allocation Paradigm (SDAP), which allows schools to have different starting times. Subsequently, we deploy an optimization framework, the target of which is to determine the optimal pair of the shifted school start time for each school that maintains operation under free-flow conditions. We utilize a macroscopic MFD-based traffic model, which can account for the coupling of the classes mentioned above. The effectiveness of the proposed approach is verified through macroscopic simulations.**

I. INTRODUCTION

The morning commute is a common challenge for many cities, with traffic congestion and long wait times causing frustration for commuters and reducing overall efficiency. In principle, during the morning traffic, many drivers first enter the network to reach the respective school their children belong to. After they complete this route, this portion of drivers heads towards their workplace. In a modern city center, the start time of school and this of work do not differ significantly; however, as each commuter wishes to arrive punctually at his workplace, he must start his daily traffic schedule at a time instant that will allow him to reach the respective school without delay before going to his work.

Typically, schools located within an area start at the same time. While schools start simultaneously, commuters who aim to drop off their children at school first before heading to work would be departing simultaneously from their homes. This would lead to synchronized flows entering the network at the same time instant, and as a result, the demand would exceed the road network capacity. This heavy overlap between the demands exacerbates the morning commute congestion problem. As an outcome, even though the commuters wish to arrive on time at their work, having first arrived at the school of interest, only a limited number of drivers will achieve their goal.

To overcome this issue, researchers have been trying to designate effective ways to curb traffic congestion by deconcentrating the temporal distribution of travel demand [1]. Shifting the work start times poses a promising research avenue aiming to tackle the shortcomings of peak demand formation. Li et al. (2022) examine linear staggered shifts, which replace the identical work start time in a single entry traffic corridor, accounting for social optimum conditions [2]. Yushimito et al. (2013) propose an optimization framework that aims to differentiate the employees' work schedules to improve network performance [3].

The emergence of congestion can be primarily attributed to the fact that schools in the city center start simultaneously. A branch of the morning commute problem that has remained under the radar is the differentiation of school start times when a macroscopic traffic model is devised for different classes of commuters present in the network. Several works consider the co-existence of different classes of commuters; however, they lack a concrete MFD-based mathematical model that efficiently captures the congestion propagation phenomena when these two classes are coupled [4], [5], [6]. An MFD-based model has also been proposed in conjunction with Vickrey's bottleneck model to formulate the user equilibrium as an ordinary differential equation (ODE), limited to a single class of commuters [7]. He et al. (2022) explore the effects of staggering work and school start times on the distribution of traffic congestion and social welfare; without utilizing an analytic model to capture the underlying traffic conditions [8]. Bertsimas and Delarue (2022) utilize a framework that assigns students to schools through an optimization framework but does not consider the congestion propagation aspect [9].

In this work, we consider two classes of commuters. Commuters of Class I first need to stop by the respective school their child is assigned to before going to work, whereas commuters of Class II head directly to work. In this context, the impetus of this paper is to differentiate the start time of schools through the scheme we propose entitled School Demand Allocation Paradigm (SDAP) to reduce the creation of synchronized flows. We utilize an MFD-based macroscopic model that accurately captures the dynamics of each class. The main contribution of this work consists of the formulation of an optimization framework that can be used to determine the optimal starting time of individual

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schools such that commuters are not heavily synchronized and congestion is avoided. A small-scale instance of the problem consisting of two classes of commuters, two origin areas, two schools, and two workplaces is considered and solved, demonstrating our approach's effectiveness.

The rest of the paper is organized as follows. Section II states the School Demand Allocation Paradigm. Section III illustrates the MFD-based macroscopic model for each class of commuters. Section IV describes the objective function of interest and provides the Problem Formulation. Section V shows the experimental setting we adopt in this paper, and Section VI concludes the paper and pinpoints future research directions.

II. Problem Statement

This section describes the School Demand Allocation Paradigm for two classes of commuters related to the morning commute problem.

We consider an urban traffic network consisting of four homogeneous regions, $\mathcal{R} = \{1, \ldots, 4\}$. The network consists of a set of origin regions $\mathcal{O} \subseteq \mathcal{R}$ from where vehicular flows start their journey and a set of destination regions $\mathcal{D} \subseteq \mathcal{R}$ where vehicular flows finish their journey. Furthermore, the transportation network is associated with schools $s \in \mathcal{S}$ and workplaces $w \in \mathcal{V}$ alike. Set $\mathcal{P} \subseteq \mathcal{R}$ denotes the set comprising the regions associated with school s. $\mathcal{S}$ denotes the set that contains each school s and $\mathcal{V}$ represents the set that includes each workplace w. We assume that workplaces $w \in \mathcal{V}$ can only belong to destination region $d \in \mathcal{D}$. The time horizon is quantized into time slots of duration T_s, while L_r and l_r denote the total network length and the average trip length of vehicles inside region $r \in \mathcal{R}$, respectively. In this study, we distinguish two classes of commuters, i) *Class I* and ii) *Class II*. For simplicity, we will use the symbol (H), standing for the commuters of Class I, and (W) for the commuters of Class II. In Class I, a demand denoted by $d^H_{o,s}(k)$ (veh) requests to enter the network from origin $o \in \mathcal{O}$ towards school $s \in \mathcal{S}$ and later to destination region $d \in \mathcal{D}$. In Class II, a different portion of demand of vehicles denoted by $d^W_{o,w}(k)$ (veh) requests to enter the network from origin $o \in \mathcal{O}$ towards workplace $w \in \mathcal{V}$ that is subsequently located at destination region $d \in D$. In the considered network architecture illustrated in Fig. 2 only one destination exists, which coincides with region 4 ($d = 4$). The initial starting time of school $s \in \mathcal{S}$ is τ_s, while the initial starting time of workplace $w \in \mathcal{V}$ is t_w.

A. School Demand Allocation Paradigm

We assume that the initial pattern of demand $d^H_{o,s}$ for commuters of Class I is known beforehand. We also state that the respective demand profile for commuters of Class II, $d^W_{o,w}$, is also known. Typically, in the morning traffic, a class of commuters (Class I) first needs to stop at the respective school $s \in \mathcal{S}$ their child is assigned to (intermediate destination) before going to their work $w \in \mathcal{V}$ (final destination). In their effort to head to the school s, before the initial school start time τ_s, the demand of commuters belonging to Class I, $d^H_{o,s}$ intersects with the demand corresponding to the commuters of Class II, $d^W_{o,w}(k)$. Without a proper way to distribute the demand of commuters belonging to Class I over the entire morning traffic, the formation of congestion is inevitable. At this point, we need to mention that shifting the distribution of demand concerning commuters of Class II is out of the scope of the paper.

The purpose of the SDAP proposed in this study is to determine the optimal start time of each school s, $\tilde{\boldsymbol{\tau}}_s \in \mathbb{R}^{1\times M'}, s \in \mathcal{S}$. $\tilde{\boldsymbol{\tau}}_s$ is a vector comprising all the permissible possible values that can be associated with the shifted school start time of school s. With M', we mean the maximum number of permissible shifting (measured in time-steps) that we wish to assign to each school $s \in \mathcal{S}$. To accomplish this, we aim to control the time instant in which the demand with respect to commuters of Class I, i.e., $d^H_{o,s}$, will enter the network through origin $o \in \mathcal{O}$. First, we provide the network outline that will be of interest in this study. In Fig. 2, we depict an urban transportation network where schools $s \in \mathcal{S}$ are located in region 3, while workplaces $w \in \mathcal{V}$ are situated in destination region 4. For the sake of brevity, we will consider that the green arrows represent vehicular flows with respect to commuters of Class I. In contrast, the red arrows dictate the vehicular flows of commuters of Class II. We introduce index $m \in \mathcal{M} \subseteq \mathbb{Z}^+$ signifying the projected candidate time instants, which we allow the initial distribution of demand $d^H_{o,s}$ to be shifted to (see Fig. 1). Along with $\tilde{\boldsymbol{\tau}}_s$ the initial demand distribution regarding the commuters of Class I, $d^H_{o,s}$ is also shifted by m time-steps forward to the time spectrum, illustrated in variable $d^{H,*}_{o,s}$. Assume $\mathcal{M} = \{0, 1, \ldots, M'\}$ represents the possible values for the shifting index m. Let $\mathcal{K} = \{1, 2, \ldots, K'\}$ denote the set comprising the time steps for the morning commute period. The relationship between the optimal demand $d^{H,*}_{o,s}$, distribution for commuters of Class I with the actual school start time $\tilde{\boldsymbol{\tau}}_s \in \mathbb{R}^{1\times M'}$ is reflected in Eq. (1) $\forall k \in \mathcal{K}, o \in \mathcal{O}, s \in \mathcal{S}$

$$d^{H,*}_{o,s}(k) = \sum_{m \in \mathcal{M}} d^H_{o,s}(k-m) \cdot \xi_{m,s}, \tag{1}$$

where binary variable $\xi_{m,s}$ states whether the actual start time of school s would take the value $\tau_s + m \cdot T_s$. The distribution of demand $d^W_{o,w}$ for commuters of Class II from each origin o to each work w is independent of the shifting index $m \in \mathcal{M}$.

Assumption 1: The shape of the distribution of demand for both classes of commuters remains unchanged under free-flow conditions (see Fig. 1).

The following section introduces the traffic dynamics based on Assumption 1.

III. Traffic Dynamics

In this section, we will present the mathematical framework that will account for the traffic dynamics for each class of commuters considered in this note.

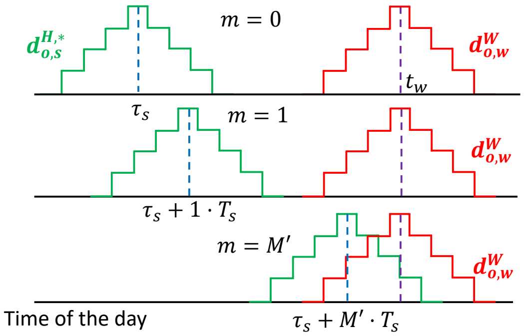


Fig. 1: Projection of distribution of demand $d_{o,s}^{H}$ for commuters of Class I (green) based on the shifted school start time $\tilde{\tau}_s = \tau_s + m \cdot T_s$ (T_s denoting the time-step), while the distribution of demand $d_{o,w}^{W}$ for commuters of Class II (red) remains the same for each different value of m being centered around work start time t_w.

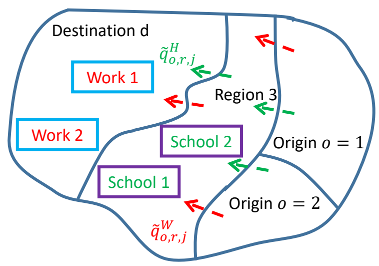


Fig. 2: Network Outline.

A. Macroscopic Traffic Flow Model

Each region $r \in \mathcal{R}$ in the transportation network is equipped with some macroscopic properties, the *jam density*, ρ_r^J, the critical density, ρ_r^C, the free-flow speed, u_r^f and the capacity, $q_r^C = \rho_r^C u_r^f$. The macroscopic properties are complemented by the fundamental diagram relationship, which describes the relationship of *intended outflow*[1] $q_r(\rho_r(k))$ (veh/h) with the density $\rho_r(k)$ (veh/km) of region r in a unified way. According to the macroscopic modeling, $q_r(\rho_r(k))$ is equal to the product of density $\rho_r(k)$ and speed $u_r(k)$ (km/h) of region r at time-step k, such that

$$q_r(\rho_r(k)) = \frac{\rho_r(k) u_r(k) L_r}{l_r}. \tag{2}$$

The *intended outflow* $q_r(\rho_r(k))$ implies the total flow that r can transmit to the outside world (i.e., completed trips) and its neighbouring regions when the interchanged flows between regions are not restricted from their inter-boundary capacity limitations. It is important to state that each region $r \in \mathcal{R}$ can operate in two traffic regimes, interchangeably, i) the free-flow regime, ii) the congested regime. In the first case, the region's density does not exceed the critical density, i.e., $\rho_r < \rho_r^C$. In the latter case, the region's density is above its critical density, $\rho_r > \rho_r^C$. The *intended outflow* of a region acquires a branching relation through the unimodal triangular MFD defined as

[1]Note that intended outflow is directly related with the production $P_r(n_r(k))$ (veh· km/h) which is a function of the number of vehicles $n_r(k)$ that are currently in the region $r \in \mathcal{R}$ such that $n_r(k) = \rho_r(k) L_r$. In this way, the fundamental relation of traffic can also be expressed as $P_r(n_r(k)) = \frac{v_r(n_r(k)) n_r(k)}{l_r}$

$$q_r(\rho_r(k)) = \begin{cases} \dfrac{q_r^C L_r}{\rho_r^C l_r} \rho_r(k), & \text{if } 0 \le \rho_r(k) \le \rho_r^C \\ \dfrac{q_r^C L_r}{l_r(\rho_r^J - \rho_r^C)} (\rho_r^J - \rho_r(k)), & \text{otherwise.} \end{cases} \tag{3}$$

Let $\mathcal{J}_r \subseteq \mathcal{R}$ denote the set of neighbouring regions directly accessible from region $r \in \mathcal{R}$. Let variable $q_{o,r,j}(k)$ denote the flow in region $r \in \mathcal{R}$ originating from $o \in \mathcal{O}$ destined to region 4 that passes through neighbouring region $j \in \mathcal{J}_r$ respectively. Then $C_{r,j}(\rho_j(k))$ stands for the inter-boundary capacity from region r to neighbouring region $j \in \mathcal{J}_r$. The inter-boundary capacity specifies the maximum flow that can be exchanged between the two neighbouring regions

$$C_{r,j}(\rho_j(k)) = \begin{cases} C_{r,j}^{\text{MAX}}, & \text{if } \rho_j(k) \le \alpha_{r,j} \rho_j^J \\ \dfrac{C_{r,j}^{\text{MAX}}}{1 - \alpha_{r,j}} \left(1 - \dfrac{\rho_j(k)}{\rho_j^J}\right), & \text{otherwise,} \end{cases} \tag{4}$$

where $C_{r,j}^{\text{MAX}}$ is the maximum inter-boundary capacity and $\alpha_{r,j} \rho_j^J$ is the point where the inter-boundary capacity starts to decrease with $0 < \alpha_{r,j} < 1, r \in \mathcal{R}, j \in \mathcal{J}_r$, similarly to [10]. Hence, the actual transfer flow from $r \in \mathcal{R}$ to $j \in \mathcal{J}_r$, $\tilde{q}_{o,r,j}(k)$ depends on the remaining storage capacity of region $j \in \mathcal{J}_r$

$$\tilde{q}_{o,r,j}(k) = \min\Big(q_{o,r,j}(k), C_{r,j}(\rho_j(k))\Big). \tag{5}$$

B. Coupling of Two Classes of Commuters

Previously, we presented the relevant framework that describes several traffic quantities of interest at the regional level. Let ρ_r^H and ρ_r^W denote the density in region r that corresponds to the commuters of Class I and Class II, respectively, and let q_r^H and q_r^W represent the intended flow concerning commuters of Class I and Class II, respectively. Then

$$\rho_r(k) = \rho_r^W(k) + \rho_r^H(k), \tag{6}$$

$$q_r(k) = q_r^W(k) + q_r^H(k). \tag{7}$$

In this paper, the dynamics for the two classes of commuters considered use the network architecture illustrated in Fig. 2. More specifically, origins 1,2 transmit flow to Region 3, $\mathcal{J}_1 = \mathcal{J}_2 = \{3\}$. Region 3 (which contains schools $s \in \mathcal{S}$, $\mathcal{P} = \{3\}$) can send flow to Region 4, $\mathcal{J}_3 = \{4\}$. Lastly, when flow leaves Region 4 (where works $w \in \mathcal{V}$ are located), we consider that both classes of commuters have completed their trip, $\mathcal{J}_4 = \varnothing$. Note that in case that $r = o$, then variable $q_{o,o,j}(k)$ denotes the flow that originates from origin o and is at time step k in region o heading towards neighbouring region $j \in \mathcal{J}_o$.

We introduce variable $q_{o,r}^H(k)$ (veh/h), which denotes the aggregated intended transfer flow of *Class I* commuters, originating from $o \in \mathcal{O}$ that is in region $r \in \mathcal{R}$ at time step k heading to each school $s \in \mathcal{S}$. $q_{o,r,j}^H(k)$ (veh/h) signifies the aggregated *intended flow* of *Class I* commuters in region $r \in \mathcal{R}$ originating from $o \in \mathcal{O}$ that passes through neighbouring region $j \in \mathcal{J}_r$. At the same time, we account for the actual transfer flow of *Class I* commuters, $\tilde{q}_{o,r,j}^H(k)$.

Variable $q^W_{o,r}(k)$ (veh/h) denotes the aggregated intended transfer flow of *Class II* commuters originating from $o \in \mathcal{O}$ that is in region $r \in \mathcal{R}$ at time step k heading to each work $w \in \mathcal{V}$. In addition, $q^W_{o,r,j}(k)$ (veh/h) denotes the *intended flow* of Class II commuters in region $r \in \mathcal{R}$ originating from $o \in \mathcal{O}$ that crosses neighbouring region $j \in \mathcal{J}_r$. The actual transfer flow of commuters belonging to Class II is reflected in $\tilde{q}^W_{o,r,j}(k)$. Eqs. (8) - (9) conclude the coupling mechanism with $j \in \mathcal{J}_r, o \in \mathcal{O}, r \in \mathcal{R}$

$$q_{o,r,j}(k) = q^W_{o,r,j}(k) + q^H_{o,r,j}(k), \tag{8}$$

$$\tilde{q}_{o,r,j}(k) = \tilde{q}^W_{o,r,j}(k) + \tilde{q}^H_{o,r,j}(k). \tag{9}$$

C. Traffic Dynamics Representation for each class of commuters

Let index $y \in \{H, W\}$ correspond to the traffic dynamics of Class I when index H is considered and to the traffic dynamics of Class II when index W is considered. Thus,

$$q^y_{o,r}(k) = \sum_{j \in \mathcal{J}_r} q^y_{o,r,j}(k), \tag{10}$$

$$\tilde{q}^y_{o,r,j}(k) = \min\left(q^y_{o,r,j}(k), C_{r,j}(\rho_j(k))\frac{q^y_{o,r,j}(k)}{q_{o,r,j}(k)}\right). \tag{11}$$

The quantity that appears in the denominator of Eq. (11) denotes the flow in region $r \in \mathcal{R}$ originating from $o \in \mathcal{O}$ that passes through the neighbouring region $j \in \mathcal{J}_r$ and is pinpointed through Eq. (8). Let the variable $\rho^y_{o,r}(k)$ indicate the density with respect to class $y \in \{H, W\}$ that is in region $r \in \mathcal{R}$ at time step k originating from region $o \in \mathcal{O}$ (veh/km)

$$q^y_{o,r}(k) = \frac{\rho^y_{o,r}(k)L_r}{\rho_r(k)l_r} q_r(k) = \frac{u_r(k)\rho^y_{o,r}(k)L_r}{l_r}. \tag{12}$$

The evolution of traffic dynamics associated with commuters of Class I can be seen in Eq. (13)

$$\rho^H_{o,r}(k+1) = \rho^H_{o,r}(k) + \frac{1}{L_r}\sum_{s \in \mathcal{S}} d^{H,*}_{o,s}(k) + \frac{T_s}{L_r}\sum_{j \in \mathcal{J}_r}\left(\tilde{q}^H_{o,j,r}(k) - \tilde{q}^H_{o,r,j}(k)\right), \tag{13}$$

where $d^{H,*}_{r,s} \neq 0, r \in \mathcal{O}$. The evolution of traffic dynamics associated with commuters of Class II can be in Eq. (14)

$$\rho^W_{o,r}(k+1) = \rho^W_{o,r}(k) + \frac{1}{L_r}\sum_{w \in \mathcal{V}} d^W_{o,w}(k) + \frac{T_s}{L_r}\sum_{j \in \mathcal{J}_r}\left(\tilde{q}^W_{o,j,r}(k) - \tilde{q}^W_{o,r,j}(k)\right), \tag{14}$$

where $d^W_{r,w} \neq 0, r \in \mathcal{O}$. We introduce the following quantities that conclude the description of the dynamics for class $y \in \{H, W\}$

$$\rho^y_r(k) = \sum_{o \in \mathcal{O}} \rho^y_{o,r}(k), \tag{15}$$

$$q^y_r(k) = \sum_{o \in \mathcal{O}} q^y_{o,r}(k). \tag{16}$$

IV. Problem Formulation

The School Demand Allocation Paradigm aims to designate the actual school start time of each school $s, \tilde{\boldsymbol{\tau}}_s$. The actual school start time must be chosen in such a way that the demand pattern $d^{H,*}_{o,s}(k)$ that accompanies the chosen school time leads to the minimization of the Total Travel Time metric, J_{TTT} (veh·h) $\forall k \in \mathcal{K}$. To define our objective function, we introduce variables $S_a(k)$ and $S_b(k)$, which denote the cumulative number of vehicles admitted to the network, and successfully arrived at their destination region 4, respectively, such that

$$S_a(k+1) = S_a(k) + \sum_{o \in \mathcal{O}}\sum_{s \in \mathcal{S}} d^{H,*}_{o,s}(k) + \sum_{o \in \mathcal{O}}\sum_{w \in \mathcal{V}} d^W_{o,w}(k), \quad S_a(0) = 0, k \in \mathcal{K}, \tag{17}$$

$$S_b(k+1) = S_b(k) + T_s\sum_{o \in \mathcal{O}}\left(\tilde{q}^H_{o,4,4}(k) + \tilde{q}^W_{o,4,4}(k)\right), \quad S_b(0) = 0, k \in \mathcal{K}. \tag{18}$$

When $r = j = 4$, where region 4 serves as the destination region in our network architecture, then variable $\tilde{q}^H_{o,4,4}(k)$ determines the number of vehicular flows corresponding to the class of Class I commuters that exit the network at time step k. Likewise, $\tilde{q}^W_{o,4,4}(k)$ determines the number of vehicular flows corresponding to Class II that exit the network at time step k. The Total Travel Time objective, J_{TTT} (veh · h), is defined as the sum of the difference between variables $S_a(k)$ and $S_b(k)$ over all time steps, such that

$$J_{\text{TTT}} = T_s \cdot \sum_{k \in \mathcal{K}}(S_a(k) - S_b(k)). \tag{19}$$

The mathematical formulation of the problem is given in (20) below

$$(P_1) \quad \min \quad T_s \cdot \sum_{k \in \mathcal{K}}(S_a(k) - S_b(k)) \tag{20a}$$

$$\text{s.t.} \quad \text{SDAP (1)},$$

$$\text{Traffic Dynamics } (2) - (18),$$

$$\xi_{m,s} = \begin{cases} 1, & \text{if school s starts at } \tau_s + m \cdot T_s \\ 0, & \text{otherwise,} \end{cases} \tag{20b}$$

$$\sum_{m \in \mathcal{M}} \xi_{m,s} = 1, \quad s \in \mathcal{S}, \tag{20c}$$

$$0 \leq \rho_r(k) \leq \rho^J_r, \quad r \in \mathcal{R}, \tag{20d}$$

$$\tilde{\boldsymbol{\tau}}_s \in \mathbb{R}^{1 \times M'}, s \in \mathcal{S}. \tag{20e}$$

In problem P_1, constraint (20c) ensures that only one shifting index $m \in \mathcal{M}$ related to the actual school start time can be assigned to each school s. Constraint (20d) sustains the density of each region within its physical limits. The proposed optimization problem P_1 is a nonconvex nonlinear problem.

A. Solution Approach

Problem P_1 poses a challenging optimization problem. More explicitly, Eqs. (2), (4), (5), (11), (12) are nonlinear and nonconvex. We provide a tractable reformulation of P_1

that offers a feasible solution for problem P_1 by relaxing the nonlinear and nonconvex constraints mentioned above acquiring their linear counterparts. To achieve this, we must ensure that each region operates under free-flow conditions. Similarly to [11] and starting with constraint (3), operation under free-flow conditions is accomplished if we enforce the constraint

$$0 \leq \rho_r(k) \leq \rho_r^C, \quad r \in \mathcal{R}, \tag{21}$$

which ensures the acquisition of density values for each region that do not exceed the region's critical density. Then Eq. (2) can be simplified to

$$q_r(k) = \frac{q_r^C L_r}{\rho_r^C l_r} \rho_r(k) = \frac{u_r^f \rho_r(k) L_r}{l_r}, \quad r \in \mathcal{R}. \tag{22}$$

Second, let us consider constraint (12), which involves the product of two variables for each class $y \in \{H, W\}$. Since $u_r(k) = u_r^f$ (we operate in the free-flow regime), the constraint (12) $\forall o \in \mathcal{O}, \forall r \in \mathcal{R}$ can be relaxed into

$$q_{o,r}^y(k) = u_r^f \rho_{o,r}^y(k) \frac{L_r}{l_r}. \tag{23}$$

To linearise the nonconvex constraint (4), we enforce the inter-boundary capacity to consistently maintain its maximum value (i.e., $C_{r,j}^{\text{MAX}}, \forall j \in \mathcal{J}_r$). To achieve this and given constraint (21), we need to impose the following constraint

$$0 \leq \rho_r(k) \leq \alpha_{r,j} \rho_r^J, \quad r \in \mathcal{R}, j \in \mathcal{J}_r, \tag{24}$$

i.e., the region's density should never exceed the critical density and also the point of density where its region's inter-boundary capacity starts to decrease. As a result, constraint (4) is simplified to

$$C_{r,j}(\rho_j(k)) = C_{r,j}^{\text{MAX}}, \quad r \in \mathcal{R}, j \in \mathcal{J}_r. \tag{25}$$

Finally, given the already defined constraints, the nonconvex constraint (5) $\forall r \in \mathcal{R}, \forall j \in \mathcal{J}_r, o \in \mathcal{O}$ can be relaxed into the following two inequalities

$$\tilde{q}_{o,r,j}(k) = q_{o,r,j}(k), \tag{26}$$

$$C_{r,j}^{\text{MAX}} \geq \sum_{o \in \mathcal{O}} \tilde{q}_{o,r,j}(k). \tag{27}$$

As it was performed for Eq. (5), constraint (11) (for commuters of Class $y \in \{H, W\}$) can be relaxed into the following two inequalities, $\forall o \in \mathcal{O}, r \in \mathcal{R}, j \in \mathcal{J}_r$:

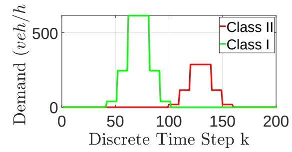
$$\tilde{q}_{o,r,j}^y(k) = q_{o,r,j}^y(k), \tag{28}$$

$$C_{r,j}^{\text{MAX}} \geq \sum_{o \in \mathcal{O}} \tilde{q}_{o,r,j}^y(k). \tag{29}$$

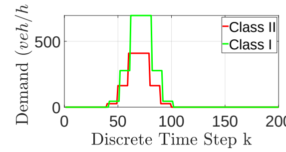
The relaxations performed in this section yield the LP formulation (30).

$$\begin{aligned} (P_2) \quad \min \quad & T_s \cdot \sum_{k \in \mathcal{K}} (S_a(k) - S_b(k)) \\ \text{s.t.} \quad & \text{SDAP (1)}, \\ & \text{Traffic Dynamics: } (2) - (3), (6) - (10), \\ & (13) - (18), (20b) - (20c), (20e), (21) - (29). \end{aligned} \tag{30}$$

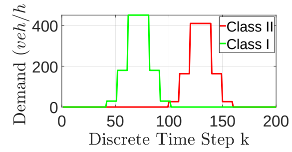
Formulation (30) is a Mixed-Integer Linear Program (MILP) that minimizes the Total Travel Time within the traffic system under the enforcement of free-flow conditions through Eq. (21). Therefore, the solution obtained from P_2 is subsequently feasible to Problem P_1 [11].



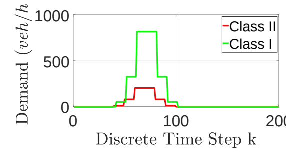
(a) Demand entering from Region 1 heading to Work 1 and School 1



(b) Demand entering from Region 1 heading to Work 2 and School 2



(c) Demand entering from Region 2 heading to Work 1 and School 1



(d) Demand entering from Region 2 heading to Work 2 and School 2

Fig. 3: Initial Demand Profile for commuters of Class I entering at $(\tau_1, \tau_2) = $ (7:40,7:40) AM and commuters of class II that enter at $(t_1, t_2) = $ (8:10,7:40) AM.

V. Simulation Results

To evaluate the proposed framework, we consider an urban network in which a triangular MFD governs each region. The network architecture we consider can be seen in Fig. 2. In this study, we will examine the effectiveness of our optimization algorithm when two schools and two works are incorporated in regions 3 and 4, accordingly, i.e., $\mathcal{S} = \{1, 2\}, \mathcal{V} = \{1, 2\}$. The total length L_r and average trip length l_r of vehicles are set to $L_r = 1$ km, $l_r = 1$ km for each region. In addition, the free-flow speed is set to $u_r^f = 60$ km/h for each region $r \in \mathcal{R}$. The simulation horizon is set to 90 min from [7:00 - 8:30] AM, and the simulation time step is set equal to $T_s = 30$ s. The traffic parameters associated with each region $r = 1, \ldots, 4$ are illustrated in Table I. We consider that for the network architecture that we employ in the simulation stage depicted in Fig. 2, the initial school start time is set equal to $\tau_s = $ 7:40 AM, $s \in \mathcal{S}$, while work 1 starts at $t_1 = $8:10 AM and work 2 begins at $t_2 = $ 7:40 AM. We consider the following range of 9 candidate school start values ($M' = 9$) for each of the two schools we consider in this study. Each index value m corresponds to a 5-minute shift in the initial school start time τ_s.

$$\tilde{\boldsymbol{\tau}}_1 = [7{:}40,\ 7{:}45,\ \ldots,\ 8{:}15,\ 8{:}20] \text{ AM}$$
$$\tilde{\boldsymbol{\tau}}_2 = [7{:}40,\ 7{:}45,\ \ldots,\ 8{:}15,\ 8{:}20] \text{ AM}$$

Finally, the proposed optimization problem P_2 is solved using the Gurobi mathematical programming solver [12].

TABLE I: Traffic parameters for each region.

Region	ρ_r^C	q_r^C	ρ_r^J	$C_{r,j}$	$\alpha_{r,j}$
1	30	1800	140	1800	0.21
2	30	1800	140	1800	0.21
3	45	2700	160	2700	0.28
4	45	2700	160	2700	0.28

Fig. 4 depicts the Total Travel Time obtained using Eqs. (1) - (16). Each combination of $(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ leads to a value

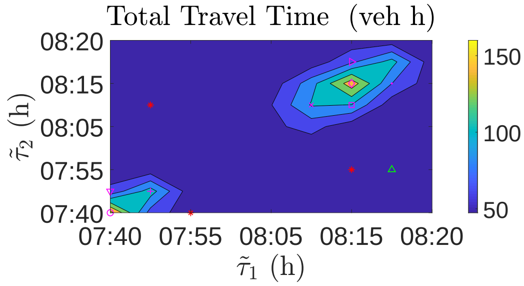


Fig. 4: Contour plot of the impact of shifted school start times (in AM format) to the Total Travel Time metric J_{TTT} obtained from the Exhaustive Search Procedure for each possible combination of shifted school start time $\tilde{\boldsymbol{\tau}}_s$. Some of the optimal pairs obtained from the Exhaustive Search are illustrated with the red $*$, while some pairs that result in a higher Total Travel Time value are shown with magenta. The optimal pair derived from the solution of the proposed optimization problem P_2 is illustrated with the green $\triangle$.

of Total Travel Time. We underline that as we shift the school start time $\tilde{\boldsymbol{\tau}}_s$ towards the right of the time domain, then the corresponding entry time of demand $d_{o,s}^{H,*}$ is shifted accordingly. It is worth reiterating that the distribution of demand with respect to Class II commuters, $d_{o,w}^{W}$ is independent of the shifting index $m \in \mathcal{M}$. We can see that for some combinations, we get a higher value of Total Travel Time (areas with yellow in the contour plot). In Fig. 4, we can see that we get an increased value of J_{TTT} in two areas.

TABLE II: Total Travel Time for different combinations of the school start time.

Pair of School Start Time	Symbol	Total Travel Time (veh h)
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (7:40,7:40)	O	167.66
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (7:40,7:45)	∇	71.37
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (7:45,7:45)	$+$	109.51
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (7:50,7:40)	$*$	**47.40**
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (7:45,8:05)	$*$	**47.40**
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:10,7:50)	$*$	**47.40**
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:05,8:05)	X	101.83
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:10,8:05)	$\square$	92.04
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:10,8:10)	$\diamond$	152.15
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:10,8:15)	$\triangleright$	80.23
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:15,7:50)	$\triangle$	**47.40**
$(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:15,8:10)	.	47.97

This happens because there is a heavy overlap between the demand corresponding to each of the two classes we consider in this note. In Figs. 3a - 3d we give the demand profile stemming from origin o towards school s for Commuters of Class I and the respective demand from o to work w for Commuters of Class II with respect to the initial school start time τ_s and the work start time t_w. It turns out that multiple combinations lead to an optimal value $J_{\mathrm{TTT}}^{*,\mathrm{ES}} = 47.41$ veh·h, in which ES stands for Exhaustive Search procedure. In Table II we take some specific pairs of shifted school start time, $(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ that correspond to certain symbols in Fig. 4. For instance, symbol $\triangleright$ signifies the pair $(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:10, 8:15) AM. For each pair illustrated in Table II, keeping the work start times t_1 = 8:10, t_2 = 7:40 AM for a simulation horizon that spans the period from [7:00 - 8:30] AM, we get a respective value of Total Travel Time. The three pairs (denoted with $*$ in Fig. 4) give us the optimal Total Travel Time, 47.40 veh h. We stress that among the three candidate pairs of school start time depicted in Table II, the one pair that takes into account the minimum discrepancy between the initial school start time τ_s and the actual start time $\tilde{\boldsymbol{\tau}}_s$ retaining the same value of $J_{\mathrm{TTT}}^{*,\mathrm{ES}} = 47.41$ veh·h is the pair $(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (7:50,7:40) AM denoted with the $*$ in contour plot with yellow. Next, the optimal pair designated from the solution of optimization problem P_2 is $(\tilde{\boldsymbol{\tau}}_1, \tilde{\boldsymbol{\tau}}_2)$ = (8:15,7:50) AM, which as can be verified from the Fig. 4 (shown with $\triangle$ in green colour) corresponds to free-flow conditions giving us a cost $J_{\mathrm{TTT}}^{*,\mathrm{OPT}} = J_{\mathrm{TTT}}^{*,\mathrm{ES}} = 47.41$ veh h. $J_{\mathrm{TTT}}^{*,\mathrm{OPT}}$ signifies the optimal solution that we obtain from the optimization problem P_2.

VI. CONCLUSIONS

This paper proposes the differentiation of schools' start times within an urban transportation network. The proposed scheme aims to shift the demand leading to minimum permissible overlap between the distribution of demands of the two classes of commuters. An optimization algorithm is devised to pinpoint the optimal pair of shifted school start times, leading to operation under free-flow conditions. Future research will include a more extensive network architecture incorporating more schools and workplaces.

REFERENCES

[1] M. Yildirimoglu, M. Ramezani, and M. Amirgholy, "Staggered work schedules for congestion mitigation: A morning commute problem," *Transportation Research Part C: Emerging Technologies*, vol. 132, p. 103391, 2021.

[2] C. Li, Y. Lu, G. Jiang, and Q. Sun, "Social optimum for linear staggered shifts in a single-entry traffic corridor with no late arrivals," *Transportmetrica B: Transport Dynamics*, vol. 11, no. 1, pp. 630–648, 2023.

[3] W. F. Yushimito, X. J. Ban, and J. Holguín-Veras, "A two-stage optimization model for staggered work hours," *Journal of Intelligent Transportation Systems*, vol. 18, no. 4, pp. 410–425, 2014.

[4] W. Liu, F. Zhang, and H. Yang, "Modeling and managing morning commute with both household and individual travels," *Transportation Research Part B: Methodological*, vol. 103, pp. 227–247, 2017, green Urban Transportation.

[5] F. Zhang, W. Liu, X. Wang, and H. Yang, "A new look at the morning commute with household shared-ride: How does school location play a role?" *Transportation Research Part E: Logistics and Transportation Review*, vol. 103, pp. 198–217, 2017.

[6] Z. Zhang and N. Zhang, "The morning commute problem with ridesharing when meet stochastic bottleneck," *Sustainability*, vol. 13, no. 11, 2021.

[7] M. Amirgholy and H. O. Gao, "Modeling the dynamics of congestion in large urban networks using the macroscopic fundamental diagram: User equilibrium, system optimum, and pricing strategies," *Transportation Research Part B: Methodological*, vol. 104, pp. 215–237, 2017.

[8] D. He, Y. Liu, Q. Zhong, and D. Z. Wang, "On the morning commute problem in a y-shaped network with individual and household travelers," *Operations Research*, 2022.

[9] D. Bertsimas and A. Delarue, "Policy analytics in public school operations," *Operations Research*, 2022.

[10] I. I. Sirmatel and N. Geroliminis, "Economic model predictive control of large-scale urban road networks via perimeter control and regional route guidance," *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 4, pp. 1112–1121, 2018.

[11] C. Menelaou, S. Timotheou, P. Kolios, and C. G. Panayiotou, "Joint route guidance and demand management for real-time control of multi-regional traffic networks," *IEEE Transactions on Intelligent Transportation Systems*, vol. 23, no. 7, pp. 8302–8315, 2022.

[12] Gurobi Optimization, LLC, "Gurobi Optimizer Reference Manual," 2023.